A MODEL NONLINEAR PROBLEM HAVING A CONTINUOUS LOCUS OF SINGULAR POINTS

BY

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1. Introduction

Numerous studies of the existence and the asymptotic behavior as $\varepsilon \to 0^+$ of solutions of the linear singular perturbation problem

$$(1.1)\quad \begin{cases} \varepsilon y'' + r(t)y' + s(t)y = 0, & -a < t < b, \quad a, \ b > 0, \\ y(-a,\varepsilon), y(b,\varepsilon) \text{ prescribed}, \end{cases}$$

have recently been published in the case that r has a zero in $(-a,b)$; see, for example, [22], [5], [19], [1], [25], [26], [16], [4], [15], [20], [7], [17], and [21]. In most of these works it is assumed that the singular point (or turning point) occurs at $t = 0$, i.e., $r(0) = 0$ and then, depending upon the signs of $r'(0)$ and $s(0)$, solutions of (1.1) are shown to exhibit a variety of behavior for small values of $\varepsilon > 0$, ranging from boundary and interior (shock) layer behavior to exponential unboundedness in ε. Some analogous results are also available for quasilinear problems of the form

$$(1.2)\quad \begin{gathered} \varepsilon y'' + r(t,y)y' + s(t,y) = 0, \quad -a < t < b, \ a, \ b > 0 \\ y(-a,\varepsilon), \ y(b,\varepsilon) \text{ prescribed}, \end{gathered}$$

when $r(t,y)$ vanishes, say at $t = 0$, for all values of y in a bounded domain; see, for example, [5], [7], [9] and [11].

The class of two-dimensional systems of the form

$$(1.3)\quad \begin{cases} u'' = f(t,u,v), \quad 0 < t < 1, \\ u(0) = u(1) = 0 \\ \varepsilon v'' = g(t,u,u')\, v' + c(t,u,u')v, \quad 0 < t < 1, \\ v(0) = v_0, \quad v(1) = v_1 , \end{cases}$$

has also been studied by one of us in the case that $g(t,u,u')$ has an interior zero along a certain curve in (t,u,u') -space [6], [7]. Finally a quadratically nonlinear problem of the form

$$(1.4)\quad \begin{cases} \varepsilon y'' = f(t,y,y'), \quad -1 < t < 1, \\ y(-1,\varepsilon), \quad y(1,\varepsilon) \text{ prescribed}, \end{cases}$$

has been considered in [12] under the assumptions that $f_{y'y'} = O(1)$ as $|y'| \to \infty$, while $f_{y'y'}[0] = 0$ and $f_{y'}[0] = 0$. Here $[0] = (0,u(0),u'(0))$ where $u=u(t)$ is a solution of $f(t,u,u') = 0$.

These results are all distinguished by the fact that the singular point (turning point) is isolated in the sense that the (linearized) coefficient of the first derivative appearing in equations (1.1), (1.2), (1.3) and (1.4) is zero only at a single point in the interval under consideration. It seems of interest to us to discuss briefly a simple example which has continuous branches of singular points. In other words, the (linearized) coefficient of the first derivative is zero along one or more prescribed arcs in the (t,y)-plane. As we shall see the existence of such "singular arcs" forces solutions of the full $(\varepsilon > 0)$ problem to behave in a nonobvious and interesting way depending upon the stability (or instability) of these arcs.

The model problem we consider here is

$$\varepsilon y'' = (y^2 - t^2)y', \quad -1 < t < 0. \tag{1.5}$$

$$y(-1,\varepsilon) = A, \quad y(0;\varepsilon) = B \tag{1.6}$$

and the main objective of this paper is to discuss the dependence of the limit of solutions (as $\varepsilon \to 0^+$) on the values of A and B. In particular, for certain choices of A and B a solution $y = y(t,\varepsilon)$ of (1.5), (1.6) satisfies

$$\lim_{\varepsilon \to 0^+} y(t,\varepsilon) = \pm \frac{1}{\sqrt{3}}, \quad -1 < t < 0.$$

The singular arcs are clearly the pair of straight lines $u(t) = \pm t$, and we further illustrate their influence on the behavior of solutions by considering this problem on the reflected interval (0,1), namely, the problem

$$\varepsilon y'' = (y^2 - t^2)y', \quad 0 < t < 1. \tag{1.7}$$

$$y(0,\varepsilon) = A, \quad y(1,\varepsilon) = B. \tag{1.8}$$

In the sense discussed in [11], the arcs $u(t) = \pm t$ are unstable in the case of (1.5), (1.6) and stable in the case of (1.7), (1.8).

Our results are established using methods developed in [7] for the study of related classes of problems and the results of [11] concerning stable versus unstable solutions of the reduced equation. For the convenience of the reader we isolate in the next section the mathematical preliminaries which are central to our approach before discussing the problems (1.5), (1.6) and (1.7), (1.8) in Sections 3 and 4.

2. Estimation Results

We first give an elementary proof of the existence of solutions of (1.5), (1.6) and (1.7), (1.8) for all $\varepsilon > 0$ together with a relatively crude estimate on the location of these solutions.

Lemma 2.1 *For each* $\varepsilon > 0$ *the boundary value problems* (1.5), (1.6) *and* (1.7), (1.8) *have a solution* $y = y(t,\varepsilon)$ *satisfying for* $t_1 = -1, t_2 = 0$ $(t_1=0, t_2=1)$ *in the case of* (1.5), (1.6) ((1.7), (1.8))

$$\min\{A,B\} \le y(t,\varepsilon) \le \max\{A,B\}, \quad t_1 \le t \le t_2.$$

Moreover,

(i) *if* $-1 \le A < 0$ *and* $B \geq 0$ $(0 \leqslant A < 1$ *and* $B \geqslant 1)$,
$\max\{A,t\} \leqslant y(t,\varepsilon) \leqslant B, \quad t_1 \leqslant t \leqslant t_2$;

(ii) *if* $0 < A \leqslant 1$ *and* $B \leqslant 0)$ $(-1 < A \leqslant 0$ *and* $B \leqslant -1)$,
$B \leqslant y(t,\varepsilon) \leqslant \min\{A,-t\}, \quad t_1 \le t \le t_2$;

(iii) *if* $A \geqslant 1$ *and* $0 \leqslant B < 1$ $(A \geqslant 0$ *and* $-1 \leqslant B < 0)$,
$\max\{B,-t\} \leqslant y(t,\varepsilon) \leqslant A, t_1 \leqslant t \leqslant t_2$;

(iv) *if* $A \leqslant -1$ *and* $-1 < B \leqslant 0$ $(A \leqslant 0$ *and* $0 < B \leqslant 1)$,
$A \leqslant y(t,\varepsilon) \leqslant \min\{B,t\}, \quad t_1 \leqslant t \leqslant t_2$.

Proof. We consider for simplicity only the problem (1.5), (1.6).

Define for $\varepsilon > 0$ and $-1 \le t \le 0$,

$\alpha(t,\varepsilon) = \min\{A,B\}$ and $\beta(t,\varepsilon) = \max\{A,B\}$.

Then $\alpha \le \beta$ and $\alpha(\beta)$ is a lower (upper) solution, i. e.,

$\alpha(-1,\varepsilon) \le A \le \beta(-1,\varepsilon), \quad \alpha(0,\varepsilon) \leqslant B \le \beta(0,\varepsilon)$, and

clearly

$$\varepsilon \alpha'' \ge (\alpha^2 - t^2)\alpha'$$

$$\varepsilon \beta'' \le (\beta^2 - t^2)\beta' .$$

Consequently, by Nagumo's theorem [18], [14;Sec.7] the problem (1.5), (1.6) has a solution $y = y(t,\varepsilon)$ with $\min\{A,B\} \le y(t,\varepsilon) \le \max\{A,B\}$, $-1 \le t \le 0$. In addition, if $-1 \le A < 0$ and $B \ge 0$ then $\alpha_1(t,\varepsilon) = t$ is clearly a lower solution and so (cf.[14]) $\max\{\alpha,\alpha_1\}$ is also a lower solution. Similarly, if $0 < A \le 1$ and $B \le 0$ then $\beta_1(t,\varepsilon) = -t$ is an upper solution and so $\min\{\beta,\beta_1\}$ is an upper solution too. The remaining estimates (iii) and (iv) are proved analogously.

It follows directly from the maximum principle (cf.[7]) that if $A = B$ then the problems (1.5), (1.6) and (1.7), (1.8) have the unique solution $y(t,\varepsilon) \equiv A$, and so $y'(t,\varepsilon) \equiv 0$. However, if $A \ne B$ then for any solution $y(t,\varepsilon)$ the derivative $y'(t,\varepsilon)$ is of one sign on $[t_1,t_2]$, i.e., $y'(t,\varepsilon) > 0$ if $A < B$ and $y'(t,\varepsilon) < 0$ if $A > B$. To see this note that $y'(t,\varepsilon) = y'(t_1,\varepsilon) \exp\left[\varepsilon^{-1}\int_{t_1}^{t} (y^2(s,\varepsilon)-s^2)ds\right]$, $t_1 \le t \le t_2$.

In addition, since (1.5) and (1.7) are quasilinear equations the boundary layers of any solution are of width $O(\varepsilon)$ (cf.[23]). Put differently, if $y(t,\varepsilon)$ has boundary layer behavior at $t = t_1$ $(t = t_2)$ then $y'(t_1,\varepsilon) = O(\varepsilon^{-1})$ $(y'(t_2,\varepsilon) = O(\varepsilon^{-1}))$. Using these facts we can now give the two lemmas which are central to our discussion of (1.5), (1.6).

<u>Lemma 2.2</u> <u>Suppose for</u> $A \ne B$ <u>that any solution</u> $y = y(t,\varepsilon)$ <u>of</u> (1.5), (1.6) <u>satisfies</u> $\lim_{\varepsilon\to 0^+} y(t,\varepsilon) = k$, <u>for</u> t <u>in</u> (-1,0) , (-1,0), (-1,0],

then

$$(*) \quad \lim_{\varepsilon\to 0^+} \{\varepsilon \ln|y'(0,\varepsilon)| - \varepsilon\ln|y'(-1,\varepsilon)|\} = k^2 - \frac{1}{3}.$$

<u>Proof.</u> Since $y'(t,\varepsilon) \ne 0$ we can rewrite (1.5) as

$$\varepsilon\frac{y''}{y'} = \varepsilon\frac{d}{dt}(\ln y') = y^2 - t^2.$$

Integrating this equation and using the boundary conditions (1.6) we have the stated result by the Dominated Convergence Theorem.

<u>Lemma 2.3</u> <u>Suppose for</u> $A \ne B$ <u>that any solution</u> $y = y(t,\varepsilon)$ <u>of</u> (1.5), (1.6) satisfying $\lim_{\varepsilon\to 0^+} y(t,\varepsilon) = k$, $-1 < t < 0$, <u>has boundary layer behavior at each endpoint, then</u> $k = \pm\frac{1}{\sqrt{3}}$.

<u>Proof.</u> Since $y(t,\varepsilon)$ has boundary has boundary layer behavior at each endpoint, $y'(-1,\varepsilon) = O(\varepsilon^{-1})$ and $y'(0,\varepsilon) = O(\varepsilon^{-1})$. Hence $\lim_{\varepsilon\to 0^+} \varepsilon\ln|y'(0,\varepsilon)| = \lim_{\varepsilon\to 0^+} \varepsilon\ln|y'(-1,\varepsilon)| = 0$ since $\lim_{\varepsilon\to 0^+} \varepsilon|\ln\varepsilon| = 0$, and the result follows by the relation (*) of Lemma 2.2.

Lemmas 2.2 and 2.3 allow us to discuss the limiting behavior of $y(t,\varepsilon)$. This is the content of the next section.

3. <u>Boundary Layer Behavior.</u>

Consider now

(1.5) $\quad \varepsilon y'' = (y^2 - t^2)y' = f(t,y,y')$, $-1 < t < 0$,

(1.6) $\quad y(-1,\varepsilon) = A$, $y(0,\varepsilon) = B$, and the corresponding reduced problems

$$(R)_\ell \quad \begin{cases} (u^2-t^2)u' = 0, & -1 < t < 0, \\ u_\ell(-1) = A, \end{cases}$$

$$(R)_r \quad \begin{cases} (u^2-t^2)u' = 0, & -1 < t < 0, \\ u_r(0) = B, \end{cases}$$

$$(R) \qquad (u^2-t^2)u' = 0, \quad -1 < t < 0.$$

Clearly $u_\ell \equiv A$ and $u_r \equiv B$ are solutions of $(R)_\ell$ and $(R)_r$, respectively, while $u \equiv$ constant and $u(t) = \pm t$ are solutions of (R). Since the maximum principle shows that any solution $y = y(t,\varepsilon)$ of (1.5), (1.6) satisfies $|y(t,\varepsilon)| \le K$, $-1 \le t \le 0$, for all $\varepsilon > 0$ and $K = \max\{|A|, |B|\}$, it follows that for small values of ε, $y(t,\varepsilon)$ must be close to a solution of $(R)_\ell$, $(R)_r$ or (R) in $(-1,0)$ except possibly at a countable set of values (cf.[7]). Let us now examine the stability of the solutions of these reduced problems. Classical singular perturbation theory (cf.[3], [20], [24]) shows that the solution $u_\ell \equiv A$ of $(R)_\ell$ is stable on $[-1,0]$ if $f_{y'}[A] = A^2 - t^2 > 0$, $-1 \le t \le 0$, i.e., if $|A| > 1$, while $u_r \equiv B$ is unstable on $[-1,0]$ if $|B| > 1$ since $f_{y'}[B] = B^2 - t^2 > 0$. If $|A| \le 1$ or $|B| \le 1$ then u_ℓ and u_r change stability when these functions cross the lines $\pm t$ since $f_{y'}[\pm t] \equiv 0$. For instance, if $0 < A < 1$ then

$$f_{y'}[A] = A^2 - t^2 \quad \begin{cases} < 0, & -1 \le t < -A, \\ = 0, & t = -A \\ > 0, & -A < t \le 0. \end{cases}$$

Thus u_ℓ is stable near $t = 0$ in the classical sense, becoming unstable at $t = -A$ in $(-1, 0)$. We note also that the solutions $u(t) = \pm t$ of (R) annihilates $f_{y'}$ on $[-1, 0]$ and so (cf. [2], [10], [11]) their stability is determined by the sign of $f_y = 2yy'$ evaluated along $u(t)$. Since $f_y[\pm t] = 2t < 0$, $-1 < t < 0$, these solutions are unstable. Finally, the constant solutions $u = \pm \frac{1}{\sqrt{3}}$ of (R) are stable with respect to boundary layer behavior (cf. [11]) in the sense that

$$f_{y'}[\pm \tfrac{1}{\sqrt{3}}] = \frac{1}{3} - t^2 \quad \begin{cases} < 0, & -1 \le t < -\frac{1}{\sqrt{3}}, \\ > 0, & -\frac{1}{\sqrt{3}} < t \le 0. \end{cases}$$

Solutions of (1.5), (1.6) for small $\varepsilon > 0$ will follow the stable solution of $(R)_\ell$, $(R)_r$ or (R). Thus no branch of the function $u(t) = \pm t$ can be the limit of a solution of (1.5), (1.6) for small $\varepsilon > 0$ (cf.[11]).

With this as background we are now ready to discuss the limiting behavior of solutions of (1.5), (1.6).

Suppose first that $\frac{1}{\sqrt{3}} < A \le 1$ and $B < -\frac{1}{\sqrt{3}}$.

It follows from Lemma 2.1 that (1.5), (1.6) has a solution $y = y(t,\varepsilon)$ such that

$$B \le y(t,\varepsilon) \le \min\{A,-t\}, \quad -1 < t < 0 .$$

We claim that $\lim_{\varepsilon \to 0^+} y(t,\varepsilon = -\frac{1}{\sqrt{3}}$, $-1 < t < 0$.

To see this note that by our previous remarks and by Lemma 2.3 the limit of $y(t,\varepsilon)$ as $\varepsilon \to 0^+$ is either $u_\ell \equiv A$, $u_r \equiv B$ or $u_\pm \equiv \pm \frac{1}{\sqrt{3}}$.

Clearly u_ℓ and u_+ cannot be the limit of $y(t,\varepsilon)$ since $y(t,\varepsilon) < A$, $-A < t \le 0$, and $y(t,\varepsilon) < \frac{1}{\sqrt{3}}$, $-\frac{1}{\sqrt{3}} < t \leqslant 0$, which contradict the fact that $y(t,\varepsilon) < \min\{A-t\}$, $-1 \leqslant t \leqslant 0$.

Suppose however that

$$\lim_{\varepsilon \to 0^+} y(t,\varepsilon) = B, \; -1 < t \le 0 .$$

Since $y(-1,\varepsilon) = A \neq B$ there is a boundary layer at $t = -1$ and so $y'(-1,\varepsilon) = O(\varepsilon^{-1})$, i.e. $\lim_{\varepsilon \to 0^+} \varepsilon \,\ell n|y'(-1,\varepsilon)| = 0$. In addition, since $y(0,\varepsilon) = B$, $|y'(0,\varepsilon)|$ is very close to zero for small ε (cf.[24 ; Chap. 10]) and so $\lim_{\varepsilon \to 0^+} \varepsilon \,\ell n\, |y'(0,\varepsilon)| \le 0$. Thus

$$\lim_{\varepsilon \to 0^+} \{\varepsilon \,\ell n\, |y'(0,\varepsilon)| - \varepsilon \,\ell n\, |y'(-1,\varepsilon)|\} < 0,$$ and since

$|B| > \frac{1}{\sqrt{3}}$, $B^2 > \frac{1}{\sqrt{3}}$. This however contradicts the relation (*) of Lemma 2.2 which states that

$$\lim_{\varepsilon \to 0^+} \{\varepsilon \,\ell n\, |y'(0,\varepsilon)| - \varepsilon \,\ell n|y'(-1,\varepsilon)|\} = B^2 - \frac{1}{3} > 0.$$

We conclude that $u_r \equiv B$ cannot be the limit of $y(t,\varepsilon)$ and therefore

$$\lim_{\varepsilon \to 0^+} y(t,\varepsilon) = -\frac{1}{\sqrt{3}} , \quad -1 < t < 0.$$

See figure 1.

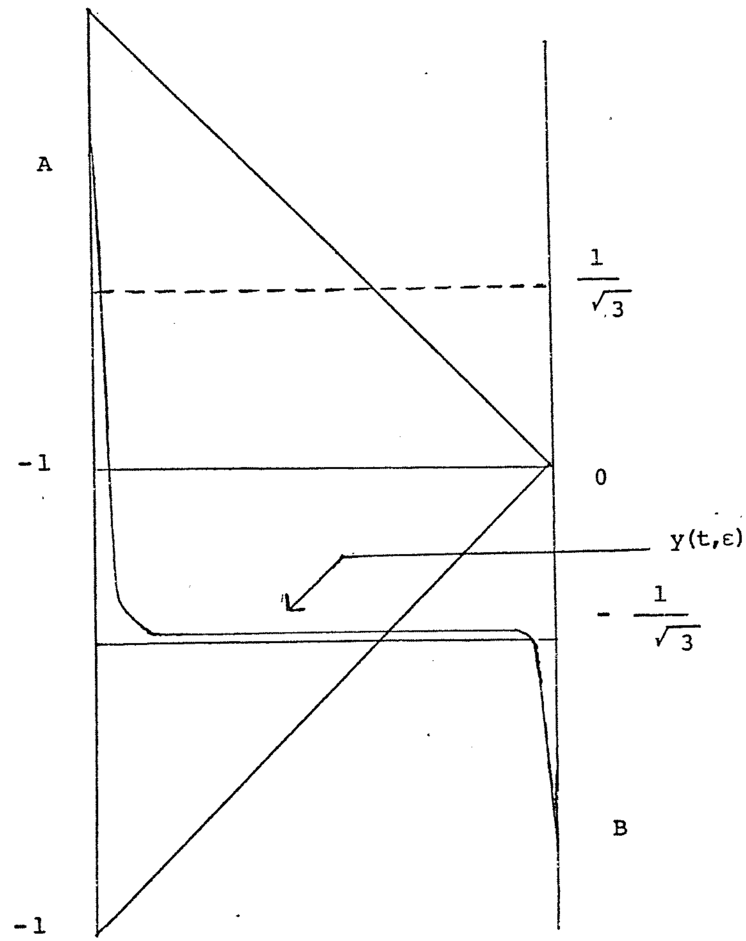

Fig. 1

SOLUTION $y(t,\varepsilon)$ OF (1.5), (1.6) FOR $\frac{1}{\sqrt{3}} < A \le 1$, $-\infty < B < -\frac{1}{\sqrt{3}}$

Suppose next that $0 \leq A < \frac{1}{\sqrt{3}}$ and $B > \frac{1}{\sqrt{3}}$.

For this range of A and B Lemma 2.1 shows that (1.5), (1.6) has a solution $y = y(t,\varepsilon)$ for all $\varepsilon > 0$ such that

$$A \leq y(t,\varepsilon) \leq B, \quad -1 \leq t \leq 0.$$

We argue as above.

It is clear that there are only three possibilities:

$y \to A$, $y \to B$ or $y \to \frac{1}{\sqrt{3}}$, as $\varepsilon \to 0^+$. We claim that

$\lim_{\varepsilon\to 0^+} y(t,\varepsilon) = \frac{1}{\sqrt{3}}$, $-1 < t < 0$. Otherwise $y \to A$ or $y \to B$. If

$\lim_{\varepsilon\to 0^+} y(t,\varepsilon) = A$, $-1 \leq t < 0$ then $y'(-1,\varepsilon) > 0$ is close to zero

and $y'(0,\varepsilon) = O(\varepsilon^{-1})$ for ε small, and so

$$\lim_{\varepsilon\to 0^+} \{\varepsilon \ln y'(0,\varepsilon) - \varepsilon \ln y'(-1,\varepsilon)\} \geqslant 0$$

since $\lim_{\varepsilon\to 0^+} \varepsilon \ln y'(0,\varepsilon) = 0$. This however contradicts the relation

(*) of Lemma 2.2 which in the present case states that

$$\lim_{\varepsilon\to 0^+} \{\varepsilon \ln y'(0,\varepsilon) - \varepsilon \ln y'(-1,\varepsilon)\} = A^2 - \frac{1}{\sqrt{3}} < 0.$$

Thus $y \not\longrightarrow A$, as $\varepsilon \to 0^+$. Similarly if $\lim_{\varepsilon\to 0^+} y(t,\varepsilon) = B$, $-1 < t \leq 0$,

then $\lim_{\varepsilon\to 0^+} \{\varepsilon \ln y'(0,\varepsilon) - \varepsilon \ln y'(-1,\varepsilon)\} \leqslant 0$; however, by (*)

$$\lim_{\varepsilon\to 0^+} \{\varepsilon \ln y'(0,\varepsilon) - \varepsilon \ln y'(-1,\varepsilon)\} = B^2 - \frac{1}{3} > 0.$$

Thus $y \not\longrightarrow B$, as $\varepsilon \to 0^+$, and therefore

$$\lim_{\varepsilon\to 0^+} y(t,\varepsilon) = \frac{1}{\sqrt{3}} , \quad -1 < t < 0 .$$

See figure 2.

Suppose finally that $0 \leq A < \frac{1}{\sqrt{3}}$ and $B < -\frac{1}{\sqrt{3}}$. Then as before there are only three possible limits of a solution $y = y(t,\varepsilon)$ of (1.5), (1.6), namely, A, B or $-\frac{1}{\sqrt{3}}$. The relation (*) of Lemma 2.1 clearly rules out the first two and we conclude that

$$\lim_{\varepsilon\to 0^+} y(t,\varepsilon) = -\frac{1}{\sqrt{3}} , \quad -1 < t < 0 .$$

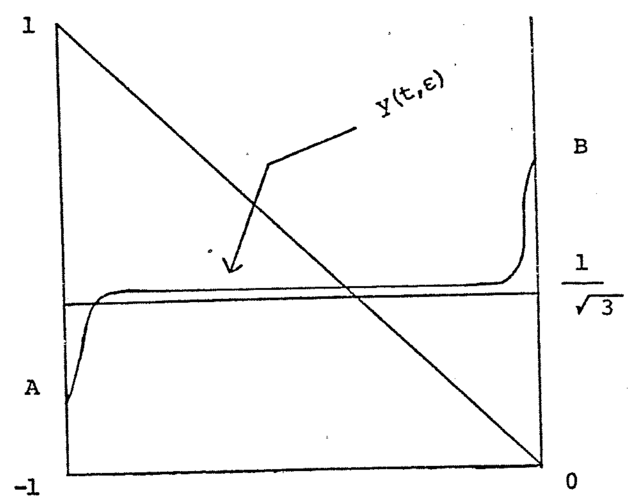

Figure 2

SOLUTION $y(t,\varepsilon)$ OF (1.5), (1.6) FOR $0 \leq A < \frac{1}{\sqrt{3}}$, $\frac{1}{\sqrt{3}} < B < \infty$

Since the equation $\varepsilon y'' = (y^2 - t^2)y'$ is invariant under the change of dependent variable $y \to -y$, we have also described the limiting behavior in the reflected cases $-1 \le A < -\frac{1}{\sqrt{3}}$, $B > \frac{1}{\sqrt{3}}$, $-\frac{1}{\sqrt{3}} < A \le 0$, $B < \frac{1}{\sqrt{3}}$ and $-\frac{1}{\sqrt{3}} < A \le 0$, $B > \frac{1}{\sqrt{3}}$. Namely, if $-1 \le A < -\frac{1}{\sqrt{3}}$ and $B > \frac{1}{\sqrt{3}}$ then

$$\lim_{\varepsilon \to 0^+} y(t,\varepsilon) = \frac{1}{\sqrt{3}}, \quad -1 < t < 0;$$

if $-\frac{1}{\sqrt{3}} < A \le 0$ and $B < -\frac{1}{\sqrt{3}}$ then

$$\lim_{\varepsilon \to 0^+} y(t,\varepsilon) = -\frac{1}{\sqrt{3}}, \quad -1 < t < 0,$$ and if $-\frac{1}{\sqrt{3}} < A \le 0$ and $B > \frac{1}{\sqrt{3}}$

then $\lim_{\varepsilon \to 0^+} y(t,\varepsilon) = \frac{1}{\sqrt{3}}, -1 < t < 0.$

It is not difficult to see that such behavior - characterized by boundary layers at both boundaries - can only occur in these six cases. One simply uses the relation (*) of Lemma 2.2 together with the estimate: if

$$\lim_{\varepsilon \to 0^+} y(t,\varepsilon) = k \quad (**) \quad \lim_{\varepsilon \to 0^+} \{\varepsilon y'(0,\varepsilon) - \varepsilon y'(-1,\varepsilon)\} = \frac{1}{3}(B^3 - A^3) + A - k$$

which is obtained by integrating (1.5) and then integrating again by parts.

We conclude our discussion of (1.5), (1.6) by summarizing the asymptotic behavior of solutions for $A > 0$ and raising some questions about the possible existence of more than one solution for certain values of A and B. Our remarks apply equally to the case $A < 0$ via the change of dependent variable $y \to -y$.

If $A > 1$ then $u_\ell \equiv A$ is globally stable and by the classic result of Coddington and Levinson [3] the problem (1.5), (1.6) has a unique solution $y = y(t,\varepsilon)$ satisfying

$$\lim_{\varepsilon \to 0^+} y(t,\varepsilon) = A, \quad -1 \le t < 0, \qquad \text{for all B.}$$

Suppose now that $\frac{1}{\sqrt{3}} \le A \le 1$. If $B \underset{(-)}{\geq} A$ or $\frac{1}{\sqrt{3}} \le B < A$ then it is possible to show by using Lemma 2.2 that (1.5), (1.6) has a unique solution $y(t,\varepsilon)$ such that

$$\lim_{\varepsilon \to 0^+} y(t,\varepsilon) = A, \quad -1 \le t \underset{(-)}{<} 0.$$

However, if $-\frac{1}{\sqrt{3}} \le B < \frac{1}{\sqrt{3}}$ then it may happen that (1.5), (1.6) has two solutions $y_1(t,\varepsilon)$ and $y_2(t,\varepsilon)$ such that

$$\lim_{\varepsilon \to 0^+} y_1(t,\varepsilon) = A, \quad -1 \le t < 0, \quad \text{and}$$

$$\lim_{\varepsilon \to 0^+} y_2(t,\varepsilon) = B, \quad -1 < t \le 0, \quad \text{if } B = -\frac{1}{2}A,$$

since both limits are compatible with relations (*) of Lemma 2.2 and (**) above. Finally if $B < -\frac{1}{\sqrt{3}}$ then we know (cf. figure 1) that (1.5), (1.6) has a solution $y(t,\varepsilon)$ satisfying

$$\lim_{\varepsilon \to 0^+} y(t,\varepsilon) = -\frac{1}{\sqrt{3}}, \quad -1 < t < 0.$$

Nevertheless, we cannot rule out the possibility that there is another solution $y(t,\varepsilon)$ such that

$$\lim_{\varepsilon \to 0^+} y_1(t,\varepsilon) = A, \quad -1 \le t < 0.$$

This is the only range of $A > 0$ for which our results are incomplete. If $0 < A < \frac{1}{\sqrt{3}}$ then application of Lemmas 2.2 and 2.3 shows that (1.5), (1.6) has a unique solution $y = y(t,\varepsilon)$ such that

$$\lim_{\varepsilon \to 0^+} y(t,\varepsilon) = \begin{cases} \frac{1}{\sqrt{3}}, & -1 < t \underset{(-)}{<} 0, \text{ if } B \underset{(-)}{>} \frac{1}{\sqrt{3}}, \\ B, & -1 < t \le 0, \text{ if } -\frac{1}{\sqrt{3}} \le B < \frac{1}{\sqrt{3}}, \\ \frac{1}{\sqrt{3}}, & -1 < t < 0, \text{ if } B < -\frac{1}{\sqrt{3}}. \end{cases}$$

The last limiting relation was proved above (cf. figure 2).

4. Interior Layer Behavior

We observe now that solutions of the problem

(1.7) $\quad \varepsilon y'' = (y^2 - t^2)y' = f(t,y,y'), \quad 0 < t < 1,$

(1.8) $\quad y(0,\varepsilon) = A, \quad y(1,\varepsilon) = B,$ cannot have boundary layer behavior at both endpoints. This follows from the fact that for any constant k in (-1,1)

$$f_{y'}[k] = k^2 - t^2 \quad \begin{cases} > 0, & 0 \le t < |k|, \\ < 0, & |k| < t \le 1, \end{cases}$$

i. e., the function $u \equiv k$ is unstable with respect to boundary layer behavior at each endpoint (cf. [11]). For the relevant choices of A and B solutions $y = y(t,\varepsilon)$ of (1.7), (1.8) exhibit interior crossing phenomena and shock layer phenomena in place of twin boundary layer behavior since the solutions $u(t) = \pm t$ of the reduced equation are stable on [0,1],

i. e., $f_y[+t] = 2t \ge 0$. As an example, if $0 \le A \le B \le 1$ then it is easy to prove that

$$\lim_{\varepsilon \to 0^+} y(t,\varepsilon) = \begin{cases} A, & 0 \le t \le A, \\ t, & A \le t \le B, \\ B, & B \le t \le 1. \end{cases}$$

See figure 3.

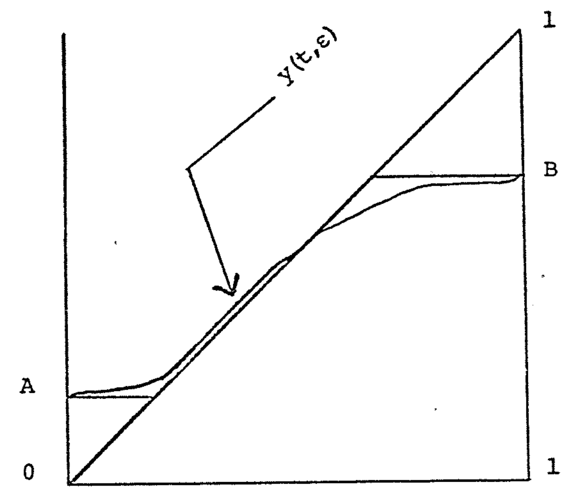

Figure 3.

SOLUTION $y(t,\varepsilon)$ OF (1.7), (1.8) FOR $0 \le A \le B \le 1$.

This result follows by observing that the functions

$$\alpha(t,\varepsilon) = \begin{cases} A - \varepsilon^{1/2}\sigma_1^{-1}, & 0 \le t \le A, \\ t - \varepsilon^{1/2}\sigma_1^{-1}, & A \le t \le B, \\ B - \varepsilon^{1/2}\sigma_1^{-1}\,(1+\sigma_1(t-B)\,\varepsilon^{-1/2})^{-1}, & B \le t \le 1, \end{cases}$$

and

$$\beta(t,\varepsilon) = \begin{cases} A + \varepsilon^{1/2}\sigma_2^{-1}\,(1+\sigma_2(A-t)\,\varepsilon^{-1/2})^{-1}, & 0 \le t \le A, \\ t + \varepsilon^{1/2}\sigma_2^{-1}, & A \le t \le B, \\ B + \varepsilon^{1/2}\sigma_2^{-1}, & B \le t \le 1, \end{cases}$$

with $0 < \sigma_1 < \sqrt{B}$ and $0 < \sigma_2 < \sqrt{A}$, are lower and upper solutions, respectively, of the problem (1.7), (1.8), and then applying an extension of Nagumo's theorem (see [12; Thm. 2.1]). Note that α (β) is not differentiable at $t = A (t=B)$; however, $\alpha'(A^-,\varepsilon) = 0 < \alpha'(A^+,\varepsilon) = 1$ ($\beta(B^-,\varepsilon) = 1 > \beta'(B^+,\varepsilon)=0$ and so Theorem 2.1 of [12] is applicable.

On the other hand, if $0 \le B < A \le 1$ then $y(t,\varepsilon)$ exhibits shock layer behavior at $t_0 = \frac{1}{\sqrt{3}}(A^2 + AB + B^2)$ in $(0,1)$, i. e.,

$$\lim_{\varepsilon \to 0^+} y(t,\varepsilon) = \begin{cases} A, & 0 \le t < t_0, \\ B, & t_0 < t \le 1; \end{cases}$$

See figure 4.

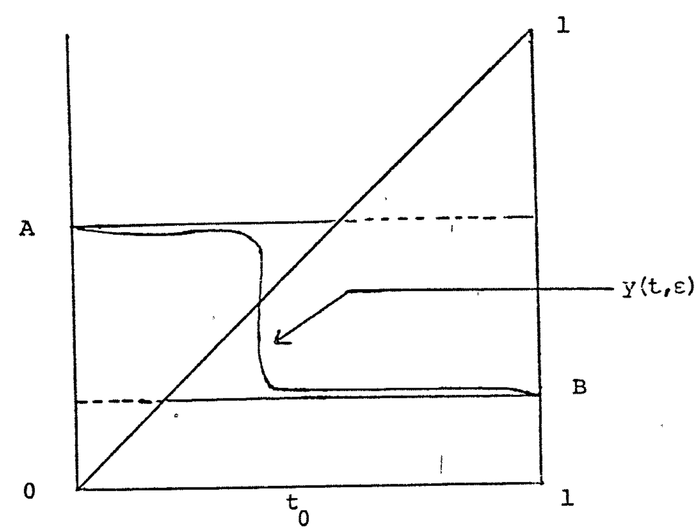

Figure 4

SOLUTION $y(t,\varepsilon)$ OF (1.7), (1.8) FOR $0 \le B < A \le 1$.

The point t_0 is the value of t which annihilates the functional

$$J[t] = \int_B^A (s^2 - t^2)ds \quad \text{(cf.[8] and [13]).}$$

We remark finally that it is possible to give similar discussions of the behavior of solutions of the more general problem

$$\varepsilon y'' = h(t,y)\, y', \quad a < t < b,$$

$$y(a,\varepsilon), \quad y(b,\varepsilon) \text{ prescribed,}$$

in which $h(t,y) = 0$ has one or more continuous solution branches (cf.[ ]).

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