SOLUTION OF THE LINEAR INVERSE VECTOR OPTIMIZATION PROBLEM BY A SINGLE LINEAR PROGRAM

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ABSTRACT

It is shown that finding a solution to a linear vector optimization problem which is efficient with respect to the constraints as well as to the objectives is equivalent to solving a single linear program.

Key words: Vector Optimization, Linear Programming, Efficient Points

Running-head title: Inverse Vector Optimization

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We are concerned here with the pair of inverse vector optimization problems that were introduced in [3]:

<u>Definition</u> (IVP) Given the matrices $C \in R^{pxn}$ and $D \in R^{qxn}$, find vectors $a \in R^p$, $b \in R^q$ (if they exist) such that

$$\text{(IVP)} \quad \begin{aligned} a &= \text{vector max } \{Cx \mid Dx \leqq b,\ x \geqq 0\} && (1) \\ b &= \text{vector min } \{Dx \mid Cx \geqq a,\ x \geqq 0\} && (2) \end{aligned}$$

or equivalently find vectors $x^1, x^2 \in R^n$, $a \in R^p$, $b \in R^q$ such that

$$\text{(IVP)} \quad \begin{aligned} a &= Cx^1,\ Dx^1 \leqq b,\ x^1 \geqq 0,\ \nexists x \in R^n:\ Dx \leqq b,\ x \geqq 0,\ Cx \geq Cx^1 && (1') \\ b &= Dx^2,\ Cx^2 \geqq a,\ x^2 \geqq 0,\ \nexists x \in R^n:\ Cx \geqq a,\ x \geqq 0,\ Dx \leq Dx^2 && (2') \end{aligned}$$

where the vector ordering $\geq$ means that at least one strict inequality must hold between the vector components.

Taken separately each of problems (1) and (2) is the well known linear vector optimization problem and for which a number of algorithms have been proposed [6,2]. Taken together problems (1) and (2) constitute the inverse vector optimization problem IVP in which the vector maximum a (minimum b) of one problem is simultaneously the right hand side of the constraint of the other problem. Hence IVP having a solution (a,b) simply means that the vector a is maximal for problem (1) given b, while b is a minimal vector among those which yield this level of the objective in (1). Results on existence, duality, geometry and computation for this problem are given in [3]. The main purpose of this work is to show that a solution to the inverse vector optimization problem IVP can be obtained if and only if the following linear program is solvable for some arbitrary nonnegative vector x^0 in R^n

$$\text{Max } (e^T C - e^T D)x$$

$$\text{(LP)} \qquad \text{subject to} \quad \begin{aligned} Dx &\leqq Dx^0 \\ Cx &\geqq Cx^0 \\ x &\geqq 0 \end{aligned} \qquad (3)$$

where e is a vector of ones in the appropriate Euclidean space. This linear program is similar to that proposed by Ecker and Kouada [2] for solving the single linear vector maximization problem (1). We state and prove our principal result now. (Parts (ii) to (iv) of the Theorem have been given in [3] and are included here for completeness.)

<u>Theorem</u> The following statements are equivalent

(i) IVP is solvable

(ii) There exists $(\bar{x},a,b)\in R^{n+p+q}$ such that

$$a = C\bar{x} = \text{vector max } \{Cx \mid Dx \leqq b,\ x \geqq 0\}$$

$$b = D\bar{x} = \text{vector min } \{Dx \mid Cx \geqq a,\ x \geqq 0\}$$

(iii) There exists $(\bar{x},u,v,a,b)\in R^{n+p+q+p+q}$ such that

$$a = C\bar{x},\ b = D\bar{x},\ \bar{x}\geqq 0,\ C^T u \leqq D^T v,\ b^T v = a^T u,\ u>0,\ v>0$$

(iv) There exists $(u,v)\in R^{p+q}$ such that $C^T u \leqq D^T v$, $u>0$, $v>0$.

(v) There exists $X\in R^{(p+q)\times(p+q)}$ such that

$$(I+X)\begin{pmatrix}-C\\ D\end{pmatrix} \geqq 0, \quad X \geqq 0$$

(vi) For some arbitrary nonnegative $x^0\in R^n$, $\bar{x}$ solves LP.

Proof

(ii) $\Rightarrow$ (i): By (1) and (2).

(ii) $\Leftarrow$ (i): Let $\bar{x}$ solve (1). Hence $a = C\bar{x}$ and $\bar{x}$ is feasible for problem (2). If $D\bar{x}\leq b$, then we have a contradiction to the assumption that b is the vector minimum value of (2). Hence $D\bar{x} = b$.

(ii) $\Longleftrightarrow$ (iii):

$$\text{(ii)} \Longleftrightarrow \begin{cases} \exists \bar{x} \in R^n: a = C\bar{x},\ b = D\bar{x},\ \bar{x} \geqq 0 \\ \nexists x \in R^n: Dx \leqq b,\ x \geqq 0,\ Cx \geq a \\ \nexists x \in R^n: Cx \geqq a,\ x \geqq 0,\ Dx \leq b \end{cases}$$

$$\Longleftrightarrow \left\langle \begin{array}{l} \exists \bar{x} \in R^n: a = C\bar{x},\ b = D\bar{x},\ \bar{x} \geqq 0 \\ \nexists (x,\zeta) \in R^{n+1}: Cx - a\zeta \geq 0,\ -Dx + b\zeta \geqq 0,\ x \geqq 0,\ \zeta > 0 \\ \nexists (x,\zeta) \in R^{n+1}: Cx - a\zeta \geqq 0,\ -Dx + b\zeta \geq 0,\ x \geqq 0,\ \zeta > 0 \end{array} \right.$$

$$\Longleftrightarrow \left\langle \begin{array}{l} \exists \bar{x} \in R^n: a = C\bar{x},\ b = D\bar{x},\ \bar{x} \geqq 0 \\ \exists (u,v) \in R^{p+q}: C^Tu - D^Tv \leqq 0,\ -a^Tu + b^Tv \leqq 0,\ u > 0,\ v \geqq 0 \\ \exists (u,v) \in R^{p+q}: C^Tu - D^Tv \leqq 0,\ -a^Tu + b^Tv \leqq 0,\ u \geqq 0,\ v > 0 \end{array} \right.$$

(By Slater's theorem of the alternatives [5, Theorem 2.4.1] and the fact that $a = C\bar{x}$, $b = D\bar{x}$, $\bar{x} \geqq 0$ and $C^Tu - D^Tv \leqq 0$, $-a^Tu + b^Tv < 0$, $u \geqq 0$, $v \geqq 0$ are mutually exclusive.)

$$\Longleftrightarrow \left\langle \begin{array}{l} \exists \bar{x} \in R^n: a = C\bar{x},\ b = D\bar{x},\ \bar{x} \geqq 0 \\ \exists (u,v) \in R^{p+q}: C^Tu \leqq D^Tv,\ b^Tv = a^Tu,\ u > 0,\ v > 0 \end{array} \right.$$

(The equality $b^Tv = a^Tu$ follows from $b^Tv \leqq a^Tu$ and $b^Tv - a^Tu = \bar{x}^T(D^Tv - C^Tu \geqq 0.)$

(vi) $\Rightarrow$ (iii): Let x^0 be some arbitrary nonnegative vector in R^n, let $\hat{x}$ solve LP and let $(\hat{u},\hat{v}) \in R^{p+q}$ be the corresponding optimal dual variables. Then $\hat{x}, \hat{u}, \hat{v}$ satisfy the optimality conditions [1]

$$\begin{array}{l} D\hat{x} \leqq Dx^0,\ C\hat{x} \geqq Cx^0,\ \hat{x} \geqq 0,\ D^T\hat{v} - C^T\hat{u} \geqq C^Te - D^Te,\ \hat{u} \geqq 0,\ \hat{v} \geqq 0, \\ \hat{v}^TD(\hat{x}-x^0) = 0,\ \hat{u}^TC(\hat{x}-x^0) = 0,\ \hat{x}^TD^T(\hat{v}+e) = \hat{x}^TC^T(\hat{u}+e) \end{array} \qquad (4)$$

By setting $\bar{x} = \hat{x}$, $u = \hat{u} + e$, $v = \hat{v} + e$, $a = C\hat{x}$ and $b = D\hat{x}$ (iii) is satisfied.

(vi) $\Leftarrow$ (iii): For any arbitrary nonnegative x^0 in R^n, LP is feasible (take $x=x^0$) and (iii) implies that the dual to LP [1]

$$\begin{aligned} &\text{Min} \quad (v^T D - u^T C)x^0 \\ &\text{subject to} \quad D^T v - C^T u \geqq C^T e - D^T e \qquad (5) \\ &\qquad v, u \geqq 0 \end{aligned}$$

is also feasible. Hence LP is solvable.

(iv) $\Longleftrightarrow$ (vi): Because LP is feasible for any nonnegative x^0 in R^n, and (iv) is equivalent to dual feasibility, it follows that (iv) and (vi) are equivalent.

(v) $\Longleftrightarrow$ (iv):

$$(\text{iv}) \Longleftrightarrow \left\langle \begin{array}{l} \exists (u,v,\zeta) \in R^{p+q+1} : \\ -C^T u + D^T v \geqq 0,\ u - e\zeta \geqq 0,\ v - e\zeta \geqq 0,\ \zeta > 0 \end{array} \right.$$

$$\Longleftrightarrow \left\langle \nexists x \in R^n : \begin{pmatrix} Cx \\ -Dx \end{pmatrix} \geq 0,\ x \geqq 0 \right.$$

(By Motzkin's theorem of the alternatives [5, Theorem 2.4.2])

$$\Longleftrightarrow \quad \left\langle Cx \geqq 0,\ -Dx \geqq 0,\ x \geqq 0 \ \Rightarrow\ -Cx \geqq 0,\ Dx \geqq 0 \right\rangle$$

$$\Longleftrightarrow \quad -C = X_1 C - X_2 D + X_3,\ D = X_4 C - X_5 D + X_6$$

for some $X_1, X_2, X_3, X_4, X_5, X_6 \geqq 0$

(By (2") of [4]))

$$\Longleftrightarrow \quad (I+X_1)\ C \leqq X_2 D,\ X_4 C \leqq (I+X_5) D$$

for some $X_1, X_2, X_4, X_5 \geqq 0$

$$\Longleftrightarrow \quad (I+X) \begin{pmatrix} -C \\ D \end{pmatrix} \geqq 0,\ X \geqq 0$$

for some $X \in R^{(p+q)X(p+q)}$ □

REFERENCES

1. G. B. Dantzig, "Linear Programming and Extensions" (Princeton University Press, Princeton, New Jersey, 1963).

2. J. G. Ecker and I. A. Kouada, "Finding efficient points for linear multiple objective programs", Mathematical Programming 8 (1975) 375-377.

3. D. F. Gray and W. R. S. Sutherland, "Inverse programming and the linear vector maximization problem", Mathematics Department, Dalhousie University, Halifax, Nova Scotia (September 1976).

4. O. L. Mangasarian, "Characterizations of real matrices of monotone kind", SIAM Review 10 (1968) 439-441.

5. O. L. Mangasarian, "Nonlinear Programming" (McGraw-Hill, New York, New York, 1969).

6. J. Philip, "Algorithms for the vector maximization problem", Mathematical Programming 2 (1972) 207-229.