CHARACTERIZATION OF LINEAR COMPLEMENTARITY PROBLEMS AS LINEAR PROGRAMS

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ABSTRACT

It is shown that the linear complementarity problem of finding an n-by-1 vector x such that $Mx + q \geqq 0$, $x \geqq 0$, and $x^T(Mx+q) = 0$, where M is a given n-by-n real matrix and q is a given n-by-1 vector, is solvable if and only if the linear program: minimize p^Tx subject to $Mx + q \geqq 0$, $x \geqq 0$, is solvable, where p is an n-by-1 vector which satisfies certain conditions. Furthermore each solution of the linear program, solves the linear complementarity problem. For a number of special cases including those when M has nonpositive off-diagonal elements, or when M is strictly or irreducibly diagonally dominant, or when M is a positive matrix with a dominant diagonal columnwise, p is very easily determined and the linear complementarity problem can be solved as an ordinary linear program. Examples of linear complementarity problems are given which can be solved as linear programs, but not by Lemke's method or the principal pivoting method.

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1. INTRODUCTION

We consider the linear complementarity problem of finding an x in R^n such that

(1) $$Mx + q \geq 0,\ x \geq 0,\ x^T(Mx+q) = 0 \qquad \text{(LCP)}$$

where M is a given n-by-n real matrix and q is a given vector in R^n. In earlier works [10,11,3] it was shown that for special cases of M a solution to the linear complementarity (1) can be obtained by solving the linear program

(2) $$\text{minimize } p^Tx \text{ subject to } w = Mx + q \geq 0,\ x \geq 0 \quad \text{(LP)}$$

for an easily determined p in R^n. The principal aim of this work is to show that for <u>any</u> n-by-n real matrix M, the linear complementarity problem (1) is solvable if and only there exists a p in R^n satisfying certain conditions (Theorem 1). Furthermore for such p, each solution of the linear program (1) solves the linear complementarity problem. Because of the presence of bilinear conditions (6c,d) among the otherwise linear conditions (6a,b,e,f) which p must satisfy, it is not easy in general to determine p for an arbitrary M. However for a number of special cases including those when M is a Z-matrix, that is a real square matrix with nonpositive off-diagonal elements, or when M is strictly or irreducibly diagonally dominant, p can be easily determined (Table 1) and the linear complementarity problem (1) can be solved as an ordinary linear program (2).

It will be convenient to state here for later reference the dual program to the linear program (2), which is

(3) $$\text{maximize } -q^Ty \text{ subject to } v = -M^Ty + p \geq 0,\ y \geq 0 \quad \text{(DP)}$$

In Section 2 of the paper we establish some prelimineary results including the fact the linear complementarity problem is equivalent to minimizing a piecewise linear <u>concave</u> function on a polyhedral set contained in the nonnegative orthant (Lemma 1), and consequently if it has a solution, it has one on a vertex of the polyhedral set (Lemma 2). In Section 3 we establish our principal results which include (Theorem 1) a characterization of the linear complementarity problem (1) as a linear program (2), and a number of special cases. In Section 4, the results of Section 3 are extended to other cases by employing an equivalent slack linear complementarity problem (8). Many of the specific cases derived in this paper are tabulated for convenience in Table 1. Finally in Section 5 we give two numerical examples of linear complementarity problems which are solvable as linear programs but not by Lemke's method [8,2] or the principal pivoting method [2].

We define here the notation employed in the paper and mention some of the well known results that will be used. All matrices and vectors considered are real ones. If A is an n-by-m matrix then this is denoted as $A \in R^{n\times m}$, and A_i and $A_{.j}$ denote respectively the i-th row and j-th column of A. Sometimes we shall use the notation Z_1, Z_2 to denote specific matrices. This will be made clear from the context. The letter I will denote a diagonal matrix of ones. If $x \in R^n$, then x_i denotes its i-th element, and $(x_i)_+ = \max\{0,x_i\}$. If $M \in R^{n\times n}$, then diag M is the matrix obtained from M by setting all its off-diagonal elements to zero. Similarly off-diag M is a matrix obtained from M by setting its diagonal equal to zero. If $c \in R^n$, then C = diag c is a matrix in $R^{n\times n}$ with zero off-diagonal elements and $C_{ii} = c_i, i=1,\ldots,n$. If $A \in R^{n\times m}$, then $|A|$ denotes the matrix obtained from A by replacing each A_{ij} by $|A_{ij}|$. The spectral radius of a square matrix M is the maximum of the absolute values of its eigenvalues and is denoted by $\rho(M)$. The letter e will denote a vector of ones in R^n or R^m. A square matrix is said to be irreducible if there exists no simultaneous permutation of rows and columns that will create a block of zeros in a southwest corner [12, pp. 102-105]. A square matrix with nonpositive off-diagonal elements is said to be a Z-matrix or it is said to belong to Z. Two subclasses, K and K_0, of Z will be used. These classes have been extensively investigated in [5,6,7]. The class K can be characterized in many ways of which we mention two. A matrix M is in K [5, Theorem 4,3] if it is in Z and either (a) M^{-1} exists and $M^{-1} \geq 0$, or (b) There exists a vector $r \geq 0$ such that $Mr > 0$ or $r^TM > 0$. A matrix M is in K_0 [5, Theorem 5,1] if it is in Z and if $M + \delta I \in K$ for each $\delta > 0$. It follows that $K \subset K_0 \subset Z$.

2. PRELIMINARIES

We begin by establishing, not in the most direct way, that every solvable linear complementary problem has a solution at a vertex of its feasible set

(4) $$S = \{x \mid Mx+q \geq 0,\ x \geq 0\}$$

We first establish the equivalence of the linear complementarity problem (1) to the minimization of a piecewise linear concave function on S.

Lemma 1 Each solution $\bar{x}$ of the linear complementarity problem (1) satisfies

(5) $$0 = \sum_{i=1}^{n} \bar{x}_i - (\bar{x}_i - (M\bar{x}+q)_+ = \underset{x \in S}{\text{minimum}} \sum_{i=1}^{n} x_i - (x_i - (Mx+q)_i)_+$$

and conversely.

Proof First observe that for $x \in S$ the objective function of (5) is nonnegative, because

$$x_i - (x_i - Mx+q)_i)_+ = (Mx+q)_i \geq 0 \quad \text{if} \quad x_i - (Mx+q)_i \geq 0$$

$$x_i - (x_i - Mx+q)_i)_+ = x_i \geq 0 \quad \text{if} \quad x_i - (Mx+q)_i < 0$$

Suppose now that $\bar{x}$ solves (1). Then $\bar{x} \in S$ and $\bar{x}_i(M\bar{x}+q)_i = 0$ for $i = 1,\ldots,n$. Hence

$$\bar{x}_i - (\bar{x}_i - (M\bar{x}+q)_i)_+ = -(-(M\bar{x}+q)_i)_+ = 0 \quad \text{for} \quad \bar{x}_i = 0$$

$$\bar{x}_i - (\bar{x}_i - (M\bar{x}+q)_i)_+ = \bar{x}_i - \bar{x}_i = 0 \quad \text{for} \quad (M\bar{x}+q)_i = 0.$$

Thus $\bar{x}$ solves (5). Conversely if $\bar{x}$ solves (5) then $\bar{x}_i - (\bar{x}_i - (M\bar{x}+q)_i)_+ = 0$ for $i = 1,\ldots,n$, and

$$(M\bar{x}+q)_i = 0 \quad \text{if} \quad \bar{x}_i - (M\bar{x}+q)_i \geq 0$$

$$\bar{x}_i = 0 \quad \text{if} \quad \bar{x}_i - (M\bar{x}+q)_i < 0$$

Hence $\bar{x}$ solves (1). □

Remark 1 Because $(x_i-(Mx+q)_i)_+$ is the maximum of two linear functions: $x_i - (Mx+q)_i$ and 0, it is a piecewise linear convex function. Hence the objective function of (5) is a piecewise linear concave function, which is indeed a difficult function to minimize because it may have many local minima which are not global. Thus Lemma 1 shows that the linear complementarity problem is inherently a difficult nonconvex problem, but it also shows that

Lemma 2 If the linear complementarity problem (1) has a solution, it has a solution at a vertex of S.

Proof Because S is contained in the nonnegative orthant, it does not contain any straight lines (that go to infinity at both ends) and hence by Rockafellar's theorem [14, Corollary 32.3.4] the concave minimization problem (5) must have a solution at a vertex of S, which by Lemma 1 must also solve (1). □

Proposition 1 If the linear complementarity problem (1) has a solution, then there exists a p in R^n such that the linear program (2) has a (unique) solution $\bar{x}$ that also solves the linear complementarity problem (1).

Proof By Lemma 2 there exists a vertex $\bar{x}$ of S which solves (1). Because $\bar{x}$ is a vertex of S, there exist subsets (not necessarily disjoint) J and L of $\{1,\ldots,n\}$ such that $\bar{x}_{i\in J} = 0$, $(M\bar{x}+q)_{i\in L} = 0$, and $\{I_{i\in J}, M_{i\in L}\}$ are linearly independent. The desired p for the linear program (2) is determined by

$$p^T x = \sum_{i\in J} x_i + \sum_{i\in L} (Mx)_i$$

If $\bar{x}$ is not a unique solution of the linear program (2) with the above p, then there exists an $\hat{x} \neq \bar{x}$ such that

$$\hat{x}_i = 0 \text{ for } i \in J, \text{ and } (M\hat{x}+q)_i = 0 \text{ for } i \in L$$

Hence

$$\hat{x} - \bar{x} \neq 0, \quad I_{i\in J}(\hat{x}-\bar{x}) = 0, \quad M_{i\in L}(\hat{x}-\bar{x}) = 0$$

which contradicts the linear independence of $\{I_{i\in J}, M_{i\in L}\}$. Hence $\bar{x}$ is the unique solution of the linear program (2) with the p given above. □

3. PRINCIPAL RESULTS

We establish in this section the conditions that p of the linear program (2) must satisfy in order that each solution of (2) is also a solution of the linear complementarity problem (1).

Theorem 1 The linear complementarity problem (1) has a solution if and only if the linear program (2) is solvable for a p which together with vectors r, s, c, d, all in R^n, and matrices Z_1, Z_2, Y_1, Y_2 all in $R^{n\ n}$ satisfy the conditions

(6a) $p = r + M^Ts,\ (r,s) \geq 0$

(6b) $MZ_1 = Z_2 + qc^T$

(6c) $M^T(Y_1 - sc^T) + (Y_2 - rc^T) = 0$

(6d) $r^TZ_1 + s^TZ_2 - q^T(Y_1 - sc^T) = d^T$

(6e) $\operatorname{diag} d + \operatorname{diag}(Y_1 + Y_2) > 0$

(6f) $Z_1,\ Z_2 \in Z,\ Y_1, Y_2, c, d \geq 0$

Furthermore, each solution of the linear program (2), with a vector p satisfying (6) above, is also a solution of the linear complementarity problem (1).

Proof

$$\langle \text{LCP(1) has solution} \rangle \Leftrightarrow \langle \exists p\text{: } x \text{ solves LP(2) and } x \text{ solves LCP(1).} \rangle$$

(By Proposition (1))

$$\Leftrightarrow \langle \exists p\text{: }(x,w) \text{ solve LP(2), } (y,v) \text{ solve DP(3), and for } i = 1,\dots,n \quad v_i + y_i = 0 \Longrightarrow x_i = 0 \text{ or } w_i = 0 \rangle$$

(When the optimal dual variable components v_i, y_i satisfy $v_i + y_i > 0$, it follows automatically from the complementarity condition $y_iw_i + x_iv_i = 0$, that $x_i = 0$ or $w_i = 0$.)

$$\Leftrightarrow \langle \exists p\text{: }(x,w) \text{ solve LP(2), } (y,v) \text{ solve DP(3), and for } i = 1,\dots,n \quad v_i + y_i \leq 0 \Longrightarrow x_i \leq 0 \text{ or } w_i \leq 0 \rangle$$

(Because $v, y, x, w \geq 0$)

$$\Leftrightarrow \langle \exists p\text{: } w = Mx + q \geq 0,\ x \geq 0,\ v = -M^Ty + p \geq 0,\ y \geq 0,\ p^Tx + q^Ty \leq 0, \text{ and for } i = 1,\dots,n \quad v_i + y_i \leq 0 \Longrightarrow x_i \leq 0 \text{ or } w_i \leq 0 \rangle$$

(The first set of conditions are the standard linear programming optimality conditions)

$$\Leftrightarrow \langle \exists p\text{: } p = r + M^Ts,\ (r,s) \geq 0,\ S \neq \phi, \text{ and for each } i \in \{1,\dots,n\} \ \langle w = Mx + q \geq 0,\ x \geq 0,\ v = -M^Ty + p \geq 0,\ y \geq 0,\ p^Tx + q^Ty \leq 0,\ v_i + y_i \leq 0,\ x_i > 0,\ w_i > 0 \rangle \text{ Has no solution } x, w, y, v \rangle$$

(The first condition is equivalent to dual feasibility which together with $S \neq \phi$ is equivalent to the solvability of the linear program (2).)

$$\Leftrightarrow \langle \exists p\text{: } p = r + M^Ts,\ (r,s) \geq 0,\ S \neq \phi \text{ and for each } i \in \{1,\dots,n\} \ \langle -M^Ty + p\zeta \geq 0,\ Mx + q\zeta \geq 0,\ y \geq 0,\ x \geq 0,\ (-I_i + M_i^T)y - p_i\zeta \geq 0,\ M_ix + q_i\zeta > 0,\ I_ix > 0,\ -q^Ty - p^Tx \geq 0,\ \zeta > 0,\ \zeta \in R \rangle \text{ Has no solution } x, y, \zeta \rangle$$

$$\Leftrightarrow \left\langle \begin{array}{l} S \neq \phi,\ \exists p: p = r + M^T s,\ (r,s) \geq 0,\ \exists R,S,U,X,D, \\ E,F \text{ all in } R^{n\times n}, \text{ with } D,E,F \text{ being diagonal, and} \\ \exists c,d \text{ both in } R^n: \\ \quad M(D-R) = qc^T + (D-U) \\ \quad M^T(E+S) = pc^T - (F+X) \\ \quad p^T(D-R) - q^T(E+S) = d^T \\ \quad \text{diag } d + E + F > 0 \\ \quad R,S,U,X,D,E,F,c,d \geq 0 \end{array} \right\rangle$$

(By Motzkin's theorem of the alternative [9])

(7)

$$\Leftrightarrow \left\langle \begin{array}{l} S \neq \phi,\ \exists p: p = r + M^T s,\ (r,s) \geq 0,\ \exists Z_1,Z_2,Y_1,Y_2 \text{ all} \\ \text{in } R^{n\times n}, \text{ and } \exists c,d \text{ both in } R^n: \\ \quad MZ_1 = Z_2 + qc^T \\ \quad M^T Y_1 = -Y_2 + pc^T \\ \quad p^T Z_1 - q^T Y_1 = d^T \\ \text{diag } d + \text{diag } (Y_1+Y_2) > 0 \\ Z_1,Z_2 \in Z,\ Y_1,Y_2,c,d \geq 0 \end{array} \right\rangle$$

(Follows by setting $Z_1 = D - R$, $Z_2 = D - U$, $Y_1 = E + S$, $Y_2 = F + X$.)

$$\Leftrightarrow \left\langle \begin{array}{l} S \neq \phi,\ \exists p,r,s,c,d \text{ all in } R^n, \text{ and } \exists Z_1,Z_2,Y_1,Y_2 \\ \text{all in } R^{n\times n}: \\ \quad p = r + M^T s,\ (r,s) \geq 0 \\ \quad MZ_1 = Z_2 + qc^T \\ \quad M^T(Y_1 - sc^T) = rc^T - Y_2 \\ \quad r^T Z_1 + s^T Z_2 - q^T(Y_1 - sc^T) = d^T \\ \text{diag } d + \text{diag } (Y_1+Y_2) > 0 \\ Z_1,Z_2 \in Z,\ Y_1,Y_2,c,d \geq 0 \end{array} \right\rangle$$

(Follows by substitution of $r + M^T s$ for p.)

□

Remark 2 Conditions (7) are an alternative equivalent statement of $S \neq \phi$ and conditions (6). Note that the presence of the bilinear conditions (6c,d) among the otherwise linear conditions (6a,b,e,f) precludes the possibility of easily solving (6) for the unknowns p,r,s,c,d,Z_1,Z_2,Y_1 and Y_2 for a general M. However for important special cases, this is possible, as is indicated in the subsequent results.

All the results of [10,11] can be derived from the above theorem. For example Theorem 1 of [11] can be obtained from the above theorem by replacing in (7) the condition diag d + diag $(Y_1+Y_2) > 0$ by the more stringent condition $d > 0$. Theorem 2 of [11] follows by setting $Y_1 = Y_2 = 0$, and $c = 0$ in (7), or the equivalent Theorem 1 of [10] follows by setting $Y_1 = Y_2 = 0$ and $c = 0$ in (6).

Another interesting special case may be obtained from Theorem 1 above by setting $Y_1 = sc^T$, $Y_2 = rc^T$ and $C = \text{diag } c$. This gives the following theorem which again is a generalization of Theorem 1 of [10].

<u>Theorem 2</u> If $S \neq \phi$ and there exist r,s in R^n, Z_1,Z_2 in $R^{n\times n}$ such that

$$MZ_1 = Z_2 + qc^T$$
$$r^TZ_1 + s^TZ_2 \geqq 0$$
$$r^T(Z_1+C) + s^T(Z_2+C) > 0,\ C = \text{diag } c$$
$$Z_1,Z_2 \in Z,\ c,r,s \geqq 0$$

then the linear complementarity problem (1) has a solution which can be obtained by solving the linear program (2) with $p = r + M^Ts$.

Setting $c = \delta e$, $C = \delta I$ in the above where δ is some positive number and imposing the more stringent condition $(r+s)^TC > 0$ we obtain

<u>Corollary 1</u> If $S \neq \phi$ and there exist δ in R, r,s in R^n, Z_1,Z_2 in $R^{n\times n}$ such that

$$MZ_1 = Z_2 + \delta qe^T$$
$$r^TZ_1 + s^TZ_2 \geqq 0$$
$$r + s > 0,\ \delta > 0$$
$$Z_1,Z_2 \in Z,\ r,s \geqq 0$$

then the linear complementarity problem (1) has a solution which can be obtained by solving the linear program (2) with $p = r + M^Ts$.

By noting that if $Z_1 \in K_0$ and Z_1 is irreducible, then there exists an $r > 0$ such that $r^TZ_1 \geqq 0$ [5, Theorem 5,8], the following corollary is an immediate consequence of Corollary 1.

<u>Corollary 2</u> If $S \neq \phi$ and there exist $\delta \in R$, Z_1,Z_2 in $R^{n\times n}$ such that

$$MZ_1 = Z_2 + \delta qe^T$$
$$(Z_1,Z_2) \in Z,\ \delta > 0$$
$$Z_1 \text{ or } Z_2 \text{ irreducible and belongs to } K_0$$

then the linear complementarity problem (1) has a solution which can be obtained by solving the linear program (2) with

$p = r$, $r^TZ_1 \geqq 0$, $r > 0$ (if $Z_1 \in K_0$ and irreducible)

$p = M^Ts$, $s^TZ_2 \geqq 0$, $s > 0$ (if $Z_2 \in K_0$ and irreducible)

Another special case follows from Corollary 1 by noting that if $Z_1 \in Z$, $MZ_1 = \bar{Z}_2 \in Z$ and the off-diagonal elements of the rows $(\bar{Z}_2)_i$ are negative when $q_i < 0$, then by taking δ small enough we have that $\bar{Z}_2 - \delta qe^T \in Z$. By taking $s = 0$ and renaming $\bar{Z}_2$ (for simplicity) Z_2 we obtain the following

<u>Corollary 3</u> If $S \neq \phi$ and there exist r in R^n and Z_1,Z_2 in $R^{n\times n}$ such that

$$MZ_1 = Z_2$$
$$r^TZ_1 \geqq 0$$
$$r > 0,\ Z_1,Z_2 \in Z$$
$$(Z_2)_{ij} < 0 \text{ if } j \neq i \text{ and } q_i < 0$$

then the linear complementarity problem (1) has a solution which can be obtained by solving the linear program (2) with $p = r$.

Note that the conditions of Corollary 3 above [5, Theorem 5,4] imply that $Z_1 \in K_0$. Furthermore the conditions $Z_1 \in Z$, $r^T Z_1 \geq 0$, $r > 0$ are implied [5, Theorem 5,8] if we make the assumption that $Z_1 \in K_0$ and Z_1 is irreducible.

<u>Corollary 4</u> Let $n \geq 3$ and let M be a positive matrix which is diagonally dominant column-by-column, that is $M_{jj} \geq \sum_{i \neq j} M_{ij}$, $j = 1,\ldots,n$. Then the linear complementarity problem (1) has a solution which can be obtained by solving the linear program (2) with $p = e$.

<u>Proof</u> Without loss of genrality assume that M has been normalized so that it has ones along its diagonal. Let $M = I + F$ where $F \in R^{n \times n}$, $F_{ii} = 0$, $F_{ij} = M_{ij} > 0$ for $i \neq j$. Then $e^T(I-F) \geq 0$. Set $Z_1 = I - F$ in Corollary 3. Then $Z_2 = MZ_1 = I - F^2$. For $i \neq j$, the ijth element of F^2 is the scalar product of two nonnegative vectors in R^n, $n \geq 3$, F_i and $F_{.j}$, each of which containing exactly one zero element, and so $(F^2)_{ij} > 0$. Consequently $(Z_2)_{ij} = (-F^2)_{ij} < 0$ for $i \neq j$ and the assumptions of Corollary 3 are satisfied with $r = e$, $Z_1 = I - F$ and $Z_2 = I - F^2$. □

<u>Corollary 5</u> Let $S \neq \phi$ and let $f: R^{n \times n} \to R^{n \times n}$. If $f(M) \geq 0$ and $I + Mf(M) - M \geq 0$ and either

(a) $\rho(f(M)) < 1$ $\quad (p = r \geq 0,\ r^T(I - f(M)) > 0)$

or

(b) $\rho(I + Mf(M) - M) < 1$ $\quad (p = M^T s,\ s \geq 0,\ s^T M(I - f(M)) > 0)$

then the linear complementarity problem (1) has a solution which can be obtained by solving the linear program (2) with the p indicated above.

<u>Proof:</u> Set in Theorem 2, $c = 0$, $Z_1 = I - f(M)$, and $Z_2 = M(I - f(M))$. For case (a) take $s = 0$ and by [6, Theorem 2,1], $Z_1 \in K$ and hence there exists $r \geq 0$ such that $r^T Z_1 = r^T(I - f(M)) > 0$. Similarly for case (b) take $r = 0$ and again $Z_2 \in K$ and hence there exists $s \geq 0$ such that $s^T Z_2 = s^T M(I - f(M)) > 0$. □

<u>Corollary 6</u> Let $S \neq \phi$, let $f: R^{n \times n} \to R^{n \times n}$, let $X \in R^{n \times n}$, $X \geq 0$, and let $M = I + f(X)$. If $f(X) \leq (I + f(X))X$ and either

(a) $\rho(X) < 1$ $\quad (p = r \geq 0,\ r^T(I - X) > 0)$

(b) $\rho(X + f(X)X - f(X)) < 1$ $\quad (p = M^T s,\ s \geq 0,\ s^T(I - X + f(X) - f(X)X) > 0)$

then the linear complementarity problem (1) has a solution which can be obtained by solving the linear program (2) with the p indicated above.

<u>Proof:</u> Set in Theorem 2, $c = 0$, $Z_1 = I - X$, and $Z_2 = I - (X - f(X) + f(X)X)$. For case (a) take $s = 0$ and by [6, Theorem 2,1], $Z_1 \in K$ and hence there exists $r \geq 0$ such that $r^T Z_1 = r^T(I - X) > 0$. Similarly for case (b) take $r = 0$ and again $Z_2 \in K$ and hence there exists $s \geq 0$ such that $s^T Z_2 = s^T(I - X + f(X) - f(X)X) > 0$. □

4. EXTENSIONS AND SPECIAL CASES

In order to enlarge the class of matrices for which the linear complementarity problem can be easily set up as a linear program we consider the following linear complementarity problem with a slack variable x_0 in R^m which is slightly more general than considered earlier [11]

$$(8) \qquad \begin{bmatrix} w \\ w_0 \end{bmatrix} = \begin{bmatrix} M & A \\ E & B \end{bmatrix} \begin{bmatrix} x \\ x_0 \end{bmatrix} + \begin{bmatrix} q \\ 0 \end{bmatrix} \geq 0, \quad \begin{bmatrix} x \\ x_0 \end{bmatrix} \geq 0, \quad x^T w + x_0^T w_0 = 0$$

where $A \in R^{n\times m}$, $E \in R^{m\times n}$ and $B \in R^{m\times m}$.

<u>Lemma 3</u> Let $E \geq 0$ and let B be strictly copositive, that is $x^T Bx > 0$ whenever $0 \leq x \neq 0$. If x solves the linear complementarity problem (1) then $(x, x_0=0)$ solves the slack linear complementarity problem (8). If (x, x_0) solves (8) then $x_0 = 0$, and moreover x solves (1).

<u>Proof</u> If x solves (1) then obviously $(x, x_0=0)$ solves (8). If (x, x_0) solves (8) then

$$0 = x_0^T w_0 = x_0^T Ex + x_0^T Bx_0 \geq x_0^T Bx_0$$

Since $x_0 \geq 0$ and B is strictly copositive, it follows that $x_0 = 0$ and x solves (1). □

By combining this lemma with the previous results we obtain a more general result than [11, Theorem 3].

<u>Theorem 3</u> If $S \neq \phi$ and there exist matrices Z_1, Z_2 in $R^{n\times n}$, B, Z_3, Z_4 in $R^{m\times m}$, A, H, N in $R^{n\times m}$ and E, G, L in $R^{m\times n}$ such that $E, G, H, L, N \geq 0$, B is strictly copositive, $Z_1, Z_2, Z_3, Z_4 \in Z$ and such that

$$(9) \quad \begin{cases} MZ_1 - AG = Z_2, & MH - AZ_3 = N \\ -EZ_1 + BG = L & -EH + BZ_3 = Z_4 \end{cases}$$

and

$$(10) \quad (r^T r_0^T s^T s_0^T) \begin{pmatrix} Z_1 & -H \\ -G & Z_3 \\ Z_2 & -N \\ -L & Z_4 \end{pmatrix} > 0, \text{ for some } r, r_0, s, s_0 \geq 0 \quad (r, r_0, s, s_0) \in R^{n+m+n+m}$$

then the linear complementarity problem (1) has a solution which can be obtained by solving the linear program (2) with

$$(11) \qquad p = r + M^T s + E^T s_0$$

<u>Proof</u> By applying Theorem 2 with $c = 0$ to the slack linear complementarity problem (8) we have that (8) has a solution which can be obtained by solving the linear program

minimize $p^T x + p_0^T x_0$

subject to $\begin{bmatrix} M & A \\ E & B \end{bmatrix} \begin{bmatrix} x \\ x_0 \end{bmatrix} + \begin{bmatrix} q \\ 0 \end{bmatrix} \geq 0, \quad \begin{bmatrix} x \\ x_0 \end{bmatrix} \geq 0$

where

$$\begin{bmatrix} p \\ p_0 \end{bmatrix} = \begin{bmatrix} r \\ r_0 \end{bmatrix} + \begin{bmatrix} M^T & E^T \\ A^T & B^T \end{bmatrix} \begin{bmatrix} s \\ s_0 \end{bmatrix}$$

Since each solution (x, x_0) of this linear program solves (8), we have by Lemma 3 that $x_0 = 0$ and x solves (1). Since $x_0 = 0$ at each solution of the above linear program, we can set $x_0 = 0$ at the outset which reduces the above linear program to (2). □

Corollary 7 Let $m = n$. Theorem 3 holds with condition (10) replaced by any one of the following four more special conditions

(12a) $s = s_0 = 0,\ Z_1 - G = Z_5,\ Z_3 - H = Z_6,\ Z_5,\ Z_6 \in K.$

$r = r_0 \geq 0,\ r^T Z_5 > 0,\ r^T Z_6 > 0$

(12b) $r_0 = s = 0,\ (E+I)Z_1 - BG = Z_5,\ BZ_3 - (E+I)H = Z_6$

$Z_5,\ Z_6 \in K,\ r = s_0 \geq 0,\ r^T Z_5 > 0,\ r^T Z_6 > 0$

(12c) $r = s_0 = 0,\ MZ_1 - (A+I)G = Z_5,\ (A+I)Z_3 - MH = Z_6$

$Z_5,\ Z_6 \in K,\ r_0 = s \geq 0,\ s^T Z_5 > 0,\ s^T Z_6 > 0$

(12d) $r = r_0 = 0,\ (M+E)Z_1 - (A+B)G = Z_5,\ (A+B)Z_3 - (M+E)H = Z_6$

$Z_5,\ Z_6 \in K,\ s = s_0 \geq 0,\ s^T Z_5 > 0,\ s^T Z_6 > 0$

We give now special cases, most of which are generalizations of cases given earlier [10, 11].

Theorem 4 Let $S \neq \phi$, and let M satisfy any of the conditions below. Then the linear complementarity problem (1) has a solution which can be obtained by solving the linear program (2) with the p indicated

(a) $M = Z_2 + ab^T,\ Z_2 \in K,\ a,\ b \in R^n,\ 0 \neq a \geq 0,\ 0 \neq b \geq 0,$

$p_i = (1+\delta)b_i$ for $b_i > 0$, $p_i = \delta$ for $b_i = 0$, for sufficiently small $\delta > 0$. (See (13) below.)

(b) $M = 2Z_2 - Z_5,\ Z_2 \in Z,\ Z_5 \in K,\ Z_2 \geq Z_5,\ p = M^T s,$

where $s \geq 0$ and $s^T Z_5 > 0$.

(c) $D - |F| \in K$ where $D = \text{diag } M$, $F = \text{off-diag } M$, $p = M^T s$

$s \geq 0$ and $s^T(D-|F|) > 0$.

(d) $M_{jj} > \sum_{i \neq j} |M_{ij}|,\ j = 1,\dots,n,\ p = M^T e$

(e) $M_{ii} > \sum_{j \neq i} |M_{ij}|,\ i = 1,\dots,n$

or

$M_{ii} \geq \sum_{j \neq i} |M_{ij}|,\ i = 1,\dots,n$, M irreducible, strict inequality holding for at least one i

or

$M_{jj} \geq \sum_{i \neq j} |M_{ij}|,\ j = 1,\dots,n$, M irreducible, strict inequality holding for at least one j

Here $p = M^T s$, where $s \geq 0$, $s^T(D-|F|) > 0$, $D = \text{diag } M$ and $F = \text{off-diag } M$.

(f) $M = I + \sum_{i=1}^{k} \alpha_i X^i,\ 1 \geq \alpha_i \geq \alpha_{i+1} \geq 0,\ i = 1,\dots,k-1,$

$k \geq 1,\ X \in R^{n \times n},\ X \geq 0,\ \rho(X) < 1,\ p = r \geq 0,\ r^T(I-X) > 0.$
If $k = \infty$ then it is required in addition that $\rho(X) < \bar{\rho}$ where $\bar{\rho}$ is the radius of convergence of the scalar power series $\sum_{i=1}^{\infty} \alpha_i x^i$.

(g) $M = e^X$, or $M = I + \sinh X$, or $M = \cosh X$, $X \in R^{n \times n},\ X \geq 0,$

$\rho(X) < 1,\ p = r \geq 0,\ r^T(I-X) > 0.$

(h) $M \geq 0,\ \rho(M) < \frac{1}{2},\ 2M \leq (I-M)^{-1},\ p = r \geq 0,\ r^T(2I-(I-M)^{-1}) > 0.$

Proof (a) Set in Theorem 3, $s = 0$, $s_0 = 0$, $Z_1 = I$, $E = 0$, $B = I$, $A = a$, $G = b^T$, and $H = h \geq 0$ where h satisfies $Z_2 h > 0$ which is possible because $Z_2 \in K$. Define

$$Z_5 = Z_3 - b^T h$$

and note that Z_5 and Z_3 are real numbers here. Conditions (9) are satisfied provided that

$$N = Z_2 h - Z_5 a \geq 0$$

This is achieved by taking $Z_5 = \min_i \frac{(Z_2 h)_i}{a_i > 0} > 0$. To satisfy (10) we need

$$r^T - r_0 b^T > 0 \quad \text{and} \quad -r^T h + r_0(Z_5 + b^T h) > 0$$

These inequalities are satisfied if we take $r_0 = 1$, $r_i = (1+\delta)b_i$ for $b_i > 0$ and $r_i = \delta$ for $b_i = 0$ where

$$(13) \qquad 0 < \delta < Z_5/(b^T h + \sum_{b_j=0} h_j) = \min_i \frac{(Z_2 h)_i}{(b^T h + \sum_{b_j=0} h_j) a_i > 0}$$

(b) Set in Corollary 7 and condition (12c) thereof $A = B = Z_1 = H = I$, $Z_2 = Z_3 = Z_4$, $Z_5 = Z_6$, $L = N = G = Z_2 - Z_5$ and $E = 0$.

(c) Set in part (b) of this theorem $Z_2 = D - (-F)_+$, $Z_5 = D - |F| = D - (F)_+ - (-F)_+$, where $((F)_+)_{ij} = \max\{0, F_{ij}\}$.

(d) Set $s = e$ in part (c) of this theorem.

(e) From [12, Theorem 6.2.17] we have that $D - |F| \in K$ and hence we can apply part (c) of this theorem.

(f) Set in Corollary 6 part (a), $f(X) = \sum_{i=1}^{k} \alpha_i X^i$. If $k = \infty$, the requirement that $\rho(x) < \bar{\rho}$ is sufficient for the power series $\sum_{i=1}^{\infty} \alpha_i X^i$ to converge [1, Theorem 4-1-1].

(g) These are special cases of part (f) of this theorem and for each case of which $\bar{\rho} = \infty$.

(h) Set in Corollary 5, part (a), $f(M) = \sum_{i=1}^{\infty} M^i$. Then $I + Mf(M) - M = (I-M)^{-1} - 2M \geq 0$, and $\rho(f(M)) = \sum_{i=1}^{\infty} (\rho(M))^i = \frac{\rho(M)}{1-\rho(M)} < 1$, since $\rho(M) < \frac{1}{2}$. We also have that $I - f(M) = 2I - (I-M)^{-1} \in K$, because $\rho(f(M)) < 1$. □

Remark 3 Cases (a), (d) and the first case of (e), of Theorem 4 above, are generalizations of corresponding cases of Theorem 6 of [11]. These generalizations have also been obtained independently in [3] by a different approach. Note that in case (a), if $b > 0$, then $p = (1+\delta)b$ and hence the constant factor $(1+\delta)$ can be ignored in solving the linear program (2). This reduces to Theorem 6(a) of [11].

Remark 4 In solving the linear program (2) associated with case (a) of Theorem 4, instead of calculating the upper bound on δ given by (13), which may involve a linear program for determining h, we can start with an arbitrary positive value of δ and solve (2) for successive decreasing values of δ using the techniques of parametric linear programming [4] until we arrive at a solution of the linear complementarity problem (1).

We give in Table 1 a convenient summary of some of the cases for which the linear complementarity problem (1) is solvable as a linear program together with the conditions that M and p must satisfy.

TABLE 1

Linear Complementarity Problems Solvable as Linear Programs

Matrix M of (1)	Conditions on M	Vector p of (2)
$M = Z_2 Z_1^{-1}$	$Z_1, Z_2 \in Z$ $r^T Z_1 + s^T Z_2 > 0$ $r, s \geq 0$	$p = r + M^T s$
M	$M \in Z$	$p > 0$
M	$M^{-1} \in Z$	$p = M^T s,\ s > 0$
$M = Z_2 + ab^T$	$Z_2 \in K$ $0 \neq a \geq 0$ $0 \neq b \geq 0$	$p_i = (1+\delta) b_i$ for $b_i > 0$ $p_i = \delta$ for $b_i = 0$ $\delta > 0$, sufficiently small (See (13))
$M = 2Z_2 - Z_5$	$Z_2 \in Z,\ Z_5 \in K$ $Z_2 \geq Z_5$	$p = M^T s,\ s \geq 0$ $s^T Z_5 > 0$
M	$D - \lvert F \rvert \in K$ D = diag M F = off-diag M	$p = M^T s,\ s \geq 0$ $s^T (D - \lvert F \rvert) > 0$
M	$M > 0,\ n \geq 3$ $M_{jj} \geq \sum_{i \neq j} M_{ij},\ j = 1,\ldots,n$	$p = e$
M	$M_{jj} > \sum_{i \neq j} \lvert M_{ij} \rvert, j = 1,\ldots,n$	$p = M^T e$

TABLE 1 (Continued)

Matrix M of (1)	Conditions on M	Vector p of (2)
M	$M_{ii} > \sum_{j \neq i} \lvert M_{ij} \rvert, i = 1,\ldots,n$	$p = M^T s,\ s \geq 0$ $s^T (D - \lvert F \rvert) > 0$ D = diag M F = off-diag M (applies to all three rows)
M	$M_{ii} \geq \sum_{j \neq i} \lvert M_{ij} \rvert, i = 1,\ldots,n$ strict inequality holds for at least one i, M irreducible	
M	$M_{jj} \geq \sum_{i \neq j} \lvert M_{ij} \rvert, j = 1,\ldots,n$ strict inequality holds for at least one j, M irreducible	
$MZ_1 = Z_2 + \delta q e^T$	$Z_1, Z_2 \in Z,\ \delta > 0$ $r^T Z_1 + s^T Z_2 \geq 0$ $r, s \geq 0,\ r + s > 0$	$p = r + M^T s$
$MZ_1 = Z_2 + \delta q e^T$	$Z_1, Z_2 \in Z,\ \delta > 0$ $Z_1 \in K_0$, Z_1 irreducible	$p = r > 0$ $r^T Z_1 \geq 0$
$MZ_1 = Z_2 + \delta q e^T$	$Z_1, Z_2 \in Z,\ \delta > 0$ $Z_2 \in K_0$, Z_2 irreducible	$p = M^T s,\ s > 0$ $s^T Z_2 > 0$
$MZ_1 = Z_2$	$Z_1, Z_2 \in Z,\ r^T Z_1 \geq 0,\ r > 0$ $(Z_2)_{ij} < 0$ for $q_i < 0$ and $j \neq i$	$p = r$
$M = I + \sum_{i=1}^{k} \alpha_i X^i$	$1 \geq \alpha_i \geq \alpha_{i+1} \geq 0,\ i = 1,\ldots,k-1$ $X \geq 0,\ \rho(X) < 1$ (For $k = \infty$, $\rho(X) < \bar{\rho}$ is also required, where $\bar{\rho}$ is the radius of convergence of $\sum_{i=1}^{\infty} \alpha_i x^i$)	$p = r \geq 0$ $r^T (I - X) > 0$

TABLE 1 (Continued)

Matrix M of (1)	Conditions on M	Vector p of (2)
$M = e^X$ $M = I + \sinh X$ $M = \cosh X$	$X \geq 0,\ \rho(X) < 1$	$p = r \geq 0$ $r^T(I-X) > 0$
M	$M \geq 0,\ \rho(M) < \frac{1}{2}$ $2M \leq (I-M)^{-1}$	$p = r \geq 0$ $r^T(2I-(I-M)^{-1}) > 0$

5. NUMERICAL EXAMPLES

We give now some examples for which the linear complementarity problem can be solved by a linear program but not by Lemke's method [8,2] or the principal pivoting procedure [2].

Example 1

$$M = \begin{pmatrix} 0 & 3 & 4 \\ 1 & -1 & 0 \\ 0 & -1 & -3 \end{pmatrix} \qquad q = \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$$

This example satisfies the conditions of Theorem 2 with $c = s = e,\ r = 0,\ Z_1 = -I$ and

$$Z_2 = \begin{pmatrix} 2 & -1 & -2 \\ -1 & 1 & 0 \\ -1 & 0 & 2 \end{pmatrix}$$

By using $p = M^T e = e$ and solving the linear program (2) we obtain

	$-x_1$	$-x_2$	$-x_3$	1
$w_1 =$	0	-3	-4	-2
$w_2 =$	-1	1	0	0
$w_3 =$	0	1	3	1
	1	1	1	

Three pivots of dual simplex algorithm $\longrightarrow$

	$-w_2$	$-w_3$	$-w_1$	1
$x_3 =$				1/5
$x_1 =$				2/5
$x_2 =$				2/5
	1	1	1	

Hence the solution of the linear program which is $x_1 = 2/5,\ x_2 = 2/5,\ x_3 = 1/5,\ w_1 = w_2 = w_3 = 0$, solves the linear complementarity problem. Application of Lemke's algorithm by adding one artificial variable x_0

	$-x_1$	$-x_2$	$-x_3$	$-x_0$	
$w_1 =$	0	-3	-4	(-1)	-2
$w_2 =$	-1	1	0	-1	0
$w_3 =$	0	1	3	-1	1
$w_0 =$	0	0	0	-1	0

to make the problem feasible, and pivoting on the circled -1 above gives

	$-x_1$	$-x_2$	$-x_3$	$-w_1$	1
x_0 =	0	3	4	-1	2
w_2 =	-1	4	4	-1	2
w_3 =	0	4	7	-1	3
w_0 =	0	3	4	-1	2

The complement of w_1 which just became nonbasic is x_1 and can be made arbitrarily large and hence we have encountered a ray.

If we apply the principal pivoting method, then in the tableau

	$-x_1$	$-x_2$	$-x_3$	1
w_1 =	0	-3	-4	-2
w_2 =	-1	1	0	0
w_3 =	0	1	3	1

x_1 is the driving variable and w_1 is the distinguished variable. Again by increasing x_1 an unbounded (infeasible) ray is generated.

<u>Example 2</u>

$$M = \begin{pmatrix} -1 & 1 \\ 2 & -1 \end{pmatrix} \qquad q = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

This example which is due to Ponstein [13] arose from different considerations regarding the linear complementarity problem. This example satisfies the conditions of Theorem 2 with $r^T = (0\ 1)$, $s^T = (1\ 0)$, $c^T = (0\ 2)$,

$$Z_1 = \begin{pmatrix} -\frac{1}{2} & 0 \\ 0 & 2 \end{pmatrix}, \quad Z_2 = \begin{pmatrix} \frac{1}{2} & 0 \\ -1 & 2 \end{pmatrix},$$

and $p^T = (-1\ \ 2)$. Hence the linear program (2) with $p^T = (-1\ \ 2)$ has the (unique) solution $x_1 = 1$, $x_2 = 0$, $w_1 = 0$, $w_2 = 1$ which also solves the linear complementarity problem. Using Lemke's procedure we have the tableaus

	$-x_1$	$-x_2$	$-x_0$	1
w_1 =	1	-1	-1	1
w_2 =	-2	1	(-1)	-1
w_0 =	0	0	-1	0

	$-x_1$	$-x_2$	$-w_2$	1
w_1 =	3	-2	-1	2
x_0 =	2	-1	-1	1
w_0 =	2	-1	-1	1

and an unbounded ray has been obtained by letting $x_2 \to \infty$. Using the principal pivoting algorithm we have

	$-x_1$	$-x_2$	1
w_1 =	1	-1	1
w_2 =	-2	(1)	-1

	$-x_1$	$-w_2$	
w_1 =	-1	1	0
x_2 =	-2	(1)	-1

and the next tableau would be identical to the first one and hence the method cycles.

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