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Strong Duality for a Class of Integer Programs

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It is well-known [1,2] that primal-dual formulations for integer and mixed-integer programming problems generally exhibit a so-called duality gap: i.e., the optimal value of the primal and dual problems need not be equal. The purpose of this note is to exhibit a class of non-trivial integer programs that have the property that for each "primal" problem in the class there exists a corresponding "dual" problem whose optimal value always coincides with the optimal value of the primal problem. Furthermore, the integrality constraints are crucial to the duality results in the sense that deletion of the integrality constraints leads to an infinite duality gap for the resulting problems.

Consider the following two problems:

(P) Maximize x

subject to $c_j/x = y_j (j = 1,2,\ldots,n)$

y_j integer $(j = 1,2,\ldots,n)$

(D) Minimize $\sum_{j=1}^{n} c_j z_j$

subject to $\sum_{j=1}^{n} c_j z_j > 0$

z_j integer $(j = 1,2,\ldots,n)$.

When the c_j's are integers not all zero, it is easily seen that the optimal objective value of (P) is the greatest common divisor of the c_j's . Thus, (P) may be considered as a generalization of the concept of greatest common divisor to non-integer data sets. When the c_j's are all integers and $c \neq 0$, it is noted by Greenberg [3] that the greatest common divisor of the c_j's is the minimum of cz subject to $cz \geqq 1$ and z integer. However, the generalizations to <u>non-integer</u> data presented here and their characterizations as <u>duality</u> theorems do not appear to have been previously described.

Before establishing the main result (a "strong duality" theorem), we will first prove that "weak duality" holds for the pair (P) - (D). Note that (D) will have a feasible solution if and only if $c \neq 0$, and (P) will have a feasible solution of c is a <u>rational</u> vector, but (P) may or may not have a feasible solution otherwise.

<u>Lemma</u>: (Weak Duality)

If $(\bar{x},\bar{y})$ is feasible for (P) and $\bar{z}$ is feasible for (D) then $K\bar{x} = c\bar{z}$, where K is a non-zero integer, so that $\bar{x} \leqq c\bar{z}$.

<u>Proof</u>:

Using the feasibility of $(\bar{x},\bar{y})$ we have $0 < c\bar{z} = (\bar{x}\bar{y})\bar{z} = \bar{x}(\bar{y}\bar{z}) = K\bar{x}$, where $K = \bar{y}\bar{z}$. Since $c\bar{z} > 0$, it follows that $K \neq 0$ and thus, by the integrality of K, $\bar{x} \leqq c\bar{z}$. □

<u>Theorem 1</u>: (Strong Duality)

If (P) and (D) both have feasible solutions, then (P) and (D) both have optimal solutions and the optimal values of (P) and (D) are equal.

<u>Proof</u>:

Suppose that (P) has a feasible solution pair $(\bar{x},\bar{y})$. Since $(-\bar{x},-\bar{y})$ is also feasible, we can assume without loss of generality that $\bar{x} > 0$. Since (D) is feasible, note that $c \neq 0$. It is easily seen that the optimal value of the problem

(P') Maximize x

subject to $c_j/x = y_j \quad (j = 1,2,\ldots,n)$

y_j integer $(j = 1,2,\ldots,n)$

$$\bar{x} \leq x \leq \min_{c_j \neq 0} \{|c_1|,|c_2|,\ldots,|c_n|\}$$

must exist (since the feasible region of (P') is compact) and is equal to the optimal value of (P) . Moreover, if (P) has (x^*,y^*) as an optimal solution, then the integers $y_1^*,y_2^*,\ldots,y_n^*$ must be relatively prime (otherwise they would have a common factor $\mu \geqq 2$ and $(\mu x^*,\mu^{-1}y^*)$ would be feasible for (P) , contradicting the fact that the optimal value of (P) is x^*). Thus, there exists an integer vector z^* such that $y^*z^* = 1$ (this may be established constructively via the Euclidean algorithm, see [6]). Now note that z^* is feasible for (D) , since $cz^* = (x^*y^*)z^* = x^*(y^*z^*) = x^* > 0$. Since the objective function value for z^* in (D) coincides with the objective function value for (x^*,y^*) in (P) , it follows from the preceding lemma that z^* is an optimal solution of (D) and that the optimal values of the two problems coincide. □

Theorem 2:

If c is a rational vector and $c \neq 0$, then (P) and (D) have optimal solutions with equal optimal values.

Proof:

When $c \neq 0$ and rational, (P) and (D) both have feasible solutions, so the previous theorem applies. □

If the hypothesis of the preceding theorem does not hold, then either (P) is infeasible or (D) is infeasible. Both cannot be infeasible because (D) is infeasible if and only if $c = 0$ in which case (P) is feasible. The following theorem describes the properties of the pair (P) - (D) in these cases.

Theorem 3: (Infeasible Cases)

If (P) is infeasible, then there exists a sequence $\{z^{(i)}\}$ such that each $z^{(i)}$ is feasible for (D) and $\lim_{i\to\infty} cz^{(i)} = 0$, hence (D) has no optimal solution. If (D) is infeasible, then $c = 0$ and (P) is an unbounded problem.

Proof:

If (P) is infeasible, we will show that there exist indices r and s such that c_r/c_s is irrational. Suppose this is not the case. Since (P) is infeasible, $c \neq 0$, and there exists an s such that $c_s \neq 0$. If c_r/c_s is rational for all $r = 1,2,\ldots,n$, there would be a rational number $\bar{x}$ such that $c_r/(c_s\bar{x})$ is integer for $r = 1,2,\ldots,n$, contradicting the infeasibility of (P). As noted in Meyer [5], it follows from the irrationality of c_r/c_s and an approximation result from number theory [6] that there exists a sequence of integer pairs $(\hat{z}_r^{(i)}, \hat{z}_s^{(i)})$ such that $\lim_{i\to\infty} (c_r/c_s)\hat{z}_r^{(i)} = z_s^{(i)} = 0$. From this sequence, we may construct a corresponding sequence of $z^{(i)}$ feasible for (D) such that $\lim_{i\to\infty} cz^{(i)} = 0$ (namely, $z_j^{(i)} = \hat{z}_j^{(i)} \operatorname{sgn}\{c_r\hat{z}_r^{(i)} + c_s\hat{z}_s^{(i)}\}$ if $j = r,s$, and 0 if $j \neq r,s$).

The proof of the second part of the theorem is an obvious consequence of the fact that $(\bar{x},0)$ is feasible for (P) for all $\bar{x} \neq 0$.

The following table, where m denotes the <u>optimal</u> value of (D) (if (D) is infeasible, $m = +\infty$ by convention) and M denotes the <u>optimal</u> value of (P) (if (P) is infeasible, $M = -\infty$ by convention), summarizes Theorems 1 and 3:

(P)/(D)	Feasible	Infeasible (c=0)
Feasible	$m = M \varepsilon (0,\infty)$	$m = M = +\infty$
Infeasible	$M = -\infty$ m does not exist	Cannot occur

From this table, the following Theorem may be deduced.

Theorem 4:

(P) has an optimal solution if an only (D) has an optimal solution, in which case the optimal values are equal.

Finally, it is interesting to note that this approach suggests that the Euclidean algorithm, which has been called "the grandaddy of all algorithms" by Knuth [4], should be considered a "dual" method, since it computes the greatest common divisor by generating feasible solutions for the "dual" problem (D) rather than for the "natural" formulation (P) of the greatest common divisor problem. The Euclidean algorithm is thus not only the "oldest non-trivial algorithm" [4], but also the oldest dual algorithm.

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