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Lorentz Invariant Computations

by

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Abstract

In this note, it is shown how to compute in special relativistic mechanics with numerical formulas which either are Lorentz invariant or are related directly to Lorentz invariant formulas. Applications are made to the study of a relativistic harmonic oscillator and to the motion of an electric particle in a magnetic field.

1. Introduction

In special relativity, only the simplest of problems formulated in terms of the Einstein dynamical equation

$$(1) \quad \frac{d}{dt}(m\vec{v}) = \vec{F}$$

can be solved analytically, where, in (1),

$$(2) \quad m = \frac{cm_0}{(c^2-v^2)^{1/2}}, \quad v<c,$$

m is the mass of a given particle P, m_0 is its rest mass, $\vec{v}$ is its velocity, v is its speed, and c is the velocity of light [1]. At the same time, the development of modern digital computers and their exceptional value in approximating solutions of nonlinear, Newtonian problems [2] suggests their possible value in approximating solutions of initial value problems for (1). And, indeed, since (1) is a second order, ordinary differential equation, the entire panoply of modern numerical methods [3], [4], [5] is immediately available. There is, however, an underlying problem of physics in the application of computers in this way, namely, computations in the lab frame and computations in the rocket frame would not, in general, be Lorentz invariant, thus violating the principle of relativity [6, p. 13]. It is, therefore, the purpose of this paper to show how to effect computations in a fashion which is Lorentz invariant. In clarity, we will do this by studying two prototype problems, each of which is typical of a larger class of problems to which the methodology can be applied.

2. The Relativistic Harmonic Oscillator Consider first a particle P whose motion is one dimensional, say, along an X-axis, and is governed by the particular form

$$(3) \quad \frac{d}{dt}(mv) = -x, \quad t > 0$$

of (1). In analogy with the Newtonian case, where m is constant, P is called a relativistic harmonic oscillator. We propose to study the initial value problem defined by (3) and

$$(4) \quad x(0) = x_0 = 0, \; v(0) = v_0.$$

To do this, let us first rewrite (3) in the equivalent form

$$(5) \quad \frac{c^2 m}{c^2 - v^2} \frac{dv}{dt} = -x, \quad t > 0.$$

Then, for $\Delta t > 0$ and $t_k = k\Delta t$, $k = 0,1,2,\ldots$, we approximate (5) by the difference equation

$$(6) \quad \frac{c^2 m_k}{[(c^2-v_k^2)(c^2-v_{k+1}^2)]^{1/2}} \frac{v_{k+1}-v_k}{t_{k+1}-t_k} = -x_k,$$

where $x(t_k) = x_k$, $v(t_k) = v_k$, $k = 0,1,2,\ldots$, and

$$(7) \quad v_k = \frac{x_{k+1}-x_k}{t_{k+1}-t_k}, \quad k = 0,1,\ldots.$$

$$(8) \quad m_k = \frac{cm_0}{(c^2-v_k^2)^{1/2}}, \quad k = 0,1,\ldots.$$

Then, numerical approximation (6) is Lorentz invariant [7], thus preserving the principle of relativity.

In order to proceed with the computer implementation, let us first simplify our formulas by adopting the absolute units $m_0 = c = 1$. We may then rewrite (6) and (7) in the equivalent forms

$$(9) \quad x_{k+1} = x_k + (\Delta t)v_k$$

$$(10) \quad v_{k+1} = \frac{v_k - (\Delta t)x_k(1-v_k^2)^{3/2}[1 + x_k^2 \Delta t^2(1-v_k^2)]^{1/2}}{1 + x_k^2 \Delta t^2 (1-v_k^2)^2}$$

and generate the motion recursively from initial conditions (4). This was done for 30,000 time steps with $\Delta t = 10^{-4}$ for each of the cases v_0 = 0.001, 0.01, 0.05, 0.1, 0.3, 0.5, 0.7, 0.9. The FORTRAN program is given in Appendix 1 of [8] and the total running time on the UNIVAC 1110 was under two minutes. Figure 1 shows the amplitude and period of the first complete oscillation for the case v_0 = 0.001. For such a relatively low velocity, the oscillator should behave like a Newtonian oscillator and, indeed, this is the case, with the amplitude being 0.001 and, to two decimal places, the period being 6.28($\sim 2\pi$). Subsequent motion of this oscillator continues to show almost no change in amplitude or period. At the other extreme, Figure 2 shows the motion for v_0 = 0.9, which is relatively close to the speed of light. To two decimal places, the amplitude of the first oscillation is 1.61 while the period is 8.88. These results are distinctly non-Newtonian, and to thirty thousand time steps, these results remain constant to two decimal places but do show small increments in the third decimal place. Finally, in Figure 3 is shown how the amplitude of a relativistic oscillator deviates from that of a Newtonian oscillator with increasing v_0.

3. <u>Motion of an Electric Charge in a Magnetic Field</u>

We turn next to motion in more than one dimension. In this case, it is known that, in general, (1) is <u>not</u> invariant under the Lorentz transformation [9, p. 63]. To resolve this failure of the principle of relativity, two approaches have been followed. First, one can proceed under the approximating assumption that if a rocket frame were attached to P, so that it can accelerate, and if at time t the velocity of P is $\vec{v}$, then one can treat the rocket frame at time t as being instantaneously in uniform relative motion with velocity $\vec{v}$ with respect to the lab frame [9, p. 63], [10, p. 258]. Second, one can formulate equations of motion in Minkowski space [1, pp. 165-167],

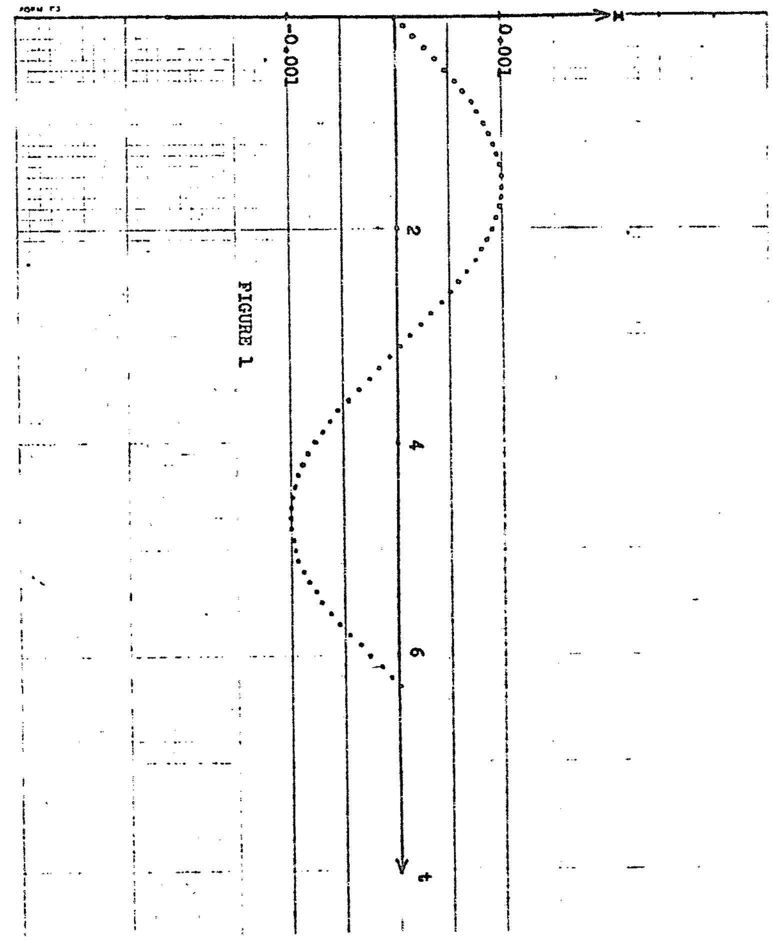

FIGURE 1

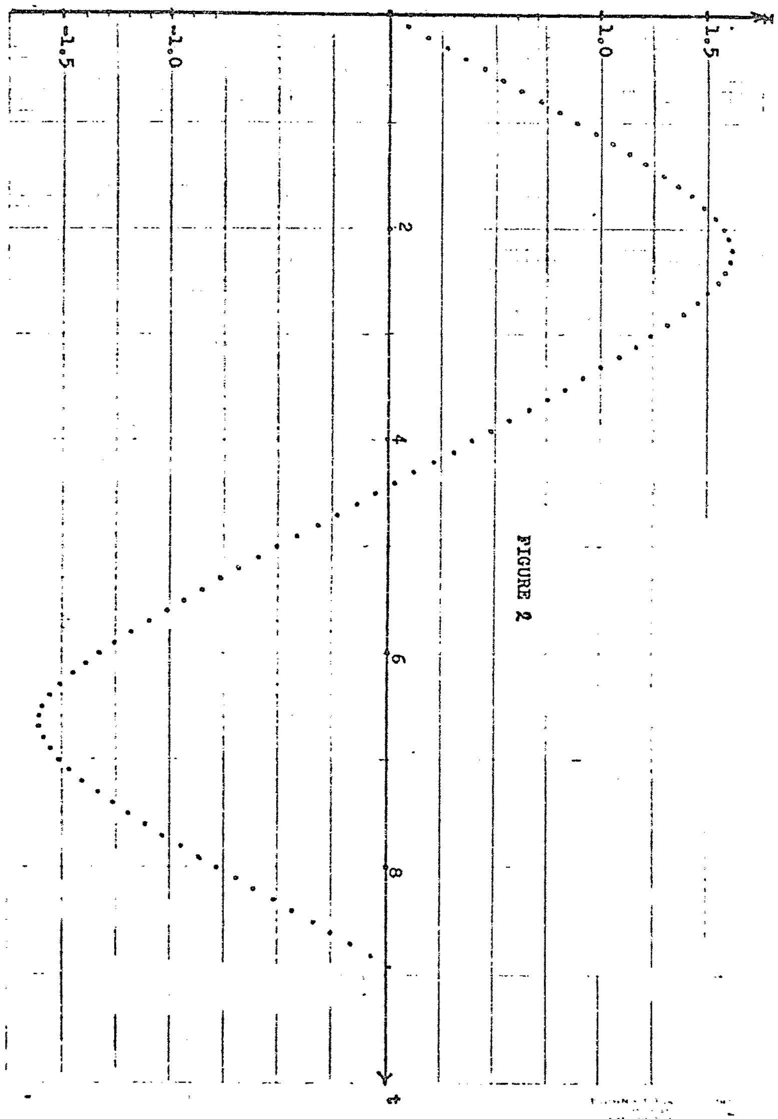

FIGURE 2

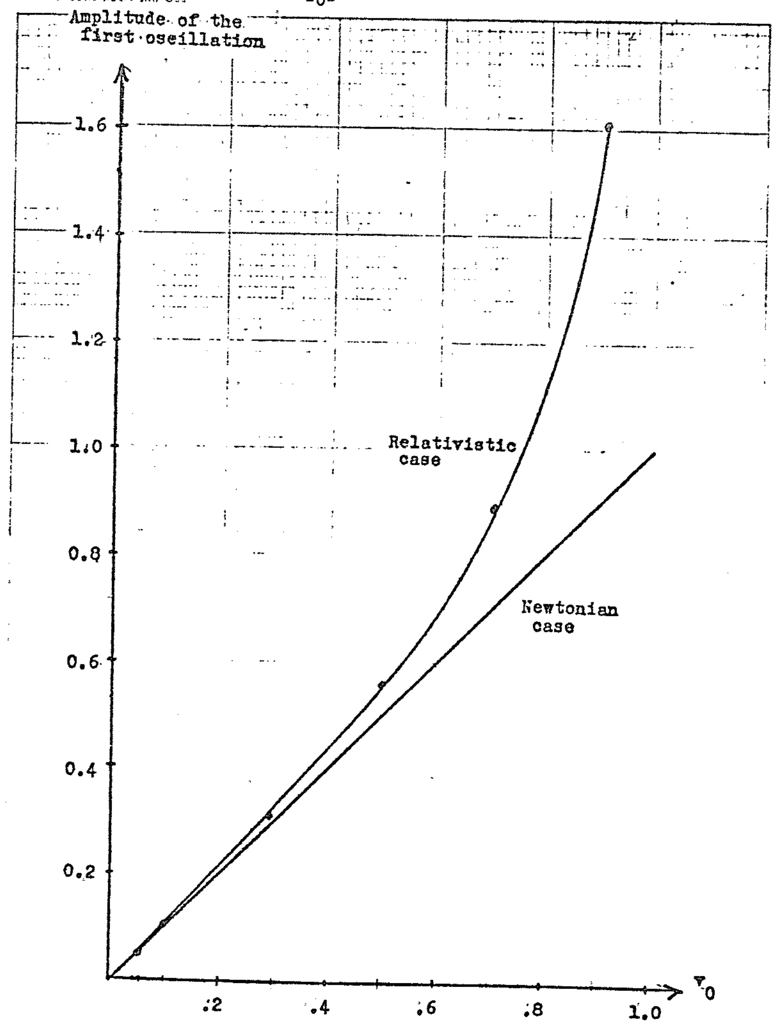

FIGURE 3

[11, p. 86], [12, pp. 103-104]. We shall proceed in the spirit of the latter approach.

Consider, in particular, the motion of an electric charge e, moving in the XY-plane under the influence of a magnetic field which acts in the direction of the Z-axis. As in [1, p. 170], assume that in the XY-plane, the force acting on the charge is

$$(11) \qquad \vec{F} = (eHv_y, -eHv_x) \quad ,$$

where v is the speed of the charge and H is the intensity of the field. The relativistic differential equations of motion are

$$(12) \qquad \frac{d}{dt}(mv_x) = e\,Hv_y$$

$$(13) \qquad \frac{d}{dt}(mv_y) = -e\,Hv_x \quad .$$

If H is uniform, then (12) and (13) can be solved analytically [1, p. 171] to yield circular motion. If H is not uniform, then, in general, (12) and (13) cannot be solved analytically.

Using absolute units $m_0 = c = e = 1$, let us begin with the general numerical approximations [13, p. 14]:

$$F_{k,x} = \frac{m_k}{[(1-v_k^2)(1-v_{k+1}^2)]^{1/2}} \frac{v_{k+1,x}-v_{k,x}}{\Delta t_k} ,$$

$$F_{k,y} = \frac{m_k}{[(1-v_k^2)(1-v_{k+1}^2)]^{1/2}} \frac{v_{k+1,y}-v_{k,y}}{\Delta t_k} ,$$

or, equivalently,

$$(14) \quad F_{k,x} = \frac{v_{k+1,x}-v_{k,x}}{\Delta t_k(1-v_k^2)(1-v_{k+1}^2)^{1/2}} , \quad F_{k,y} = \frac{v_{k+1,y}-v_{k,y}}{\Delta t_k(1-v_k^2)(1-v_{k+1}^2)^{1/2}} \, ,$$

where

$$(15) \qquad x_{k+1} = x_k + v_{k,x}\,\Delta t_k \quad ; \quad y_{k+1} = y_k + v_{k,y}\,\Delta t_k \quad .$$

Then (11) and (14) yield the following approximations of (12) and (13):

$$(16) \qquad v_{k+1,x}-v_{k,x}-Hv_{k,y}(1-v_{k,x}^2-v_{k,y}^2)(1-v_{k+1,x}^2-v_{k+1,y}^2)^{1/2}\Delta t_k = 0$$

$$(17) \qquad v_{k+1,y}-v_{k,y}+Hv_{k,x}(1-v_{k,x}^2-v_{k,y}^2)(1-v_{k+1,x}^2-v_{k+1,y}^2)^{1/2}\Delta t_k = 0.$$

From (14), (16) and (17), one can construct readily a related 4-force and a related set of Lorentz invariant dynamical difference equations in Minkowski 4-space [13]. The numerical computations, however, are done more simply in cartesian space using (15)-(17), so we continue to concentrate on these.

Let us consider the particular initial conditions

$$(18) \qquad x_0 = y_0 = v_{0,x} = 0 \quad , \quad v_{0,y} = 0.01 \quad .$$

For the parameter choices $\Delta t = 0.0001$ and $H = 100$, Figure 4 shows the resulting circular trajectory T_1 with center at (0.0001,0), radius $r = 0.0001$, and period $\tau = 0.063$, in complete agreement with the analytical solution [1, p. 171]. Equations (16) and (17) are solved at each step by Newton's method with the velocity components at the previous time step being used to initiate the iteration. A comprehensive FORTRAN program for this example and for the one which follows is given in Appendix 2 of [7].

Consider next the initial value problem defined by (15)-(18), but in a nonuniform magnetic field with a "$\frac{1}{r^2}$" intensity given by

$$(19) \quad H = \frac{100}{1 + \alpha(x^2+y^2)} \quad , \alpha \geq 0.$$

Of course, for $\alpha = 0$, (19) reduces to the uniform case above, where $H = 100$. For the parameter choices $\Delta t = 0.0001$ and $\alpha = 10^7$, the resulting particle trajectory T_2 is shown also in Figure 4. The particle motion is initially similar to the circular motion of the first example, but as (x^2+y^2) increases and decreases, the varying effect of H results in the spiral type motion shown up to $t = 0.2$ in the figure.

Increasing the input parameter $v_{0,y}$ in both the above examples reveals quickly the price being paid for computational Lorentz invariance, for the numerical formulas being used are of relatively low order and suffer from the usual shortcomings of such formulas. Thus, increasing $v_{0,y}$ to 0.1 in (18) results in having to reduce Δt to 10^{-7} to obtain reasonable accuracy on the UNIVAC 1110. Also, even for initial data (18), extended calculations with $\Delta t = 0.0001$ yield the inevitable, relatively large error accumulation associated with low order methods. A most interesting and relevant question, then, which remains unanswered as yet, is whether or not there exist higher order, Lorentz invariant numerical formulas.

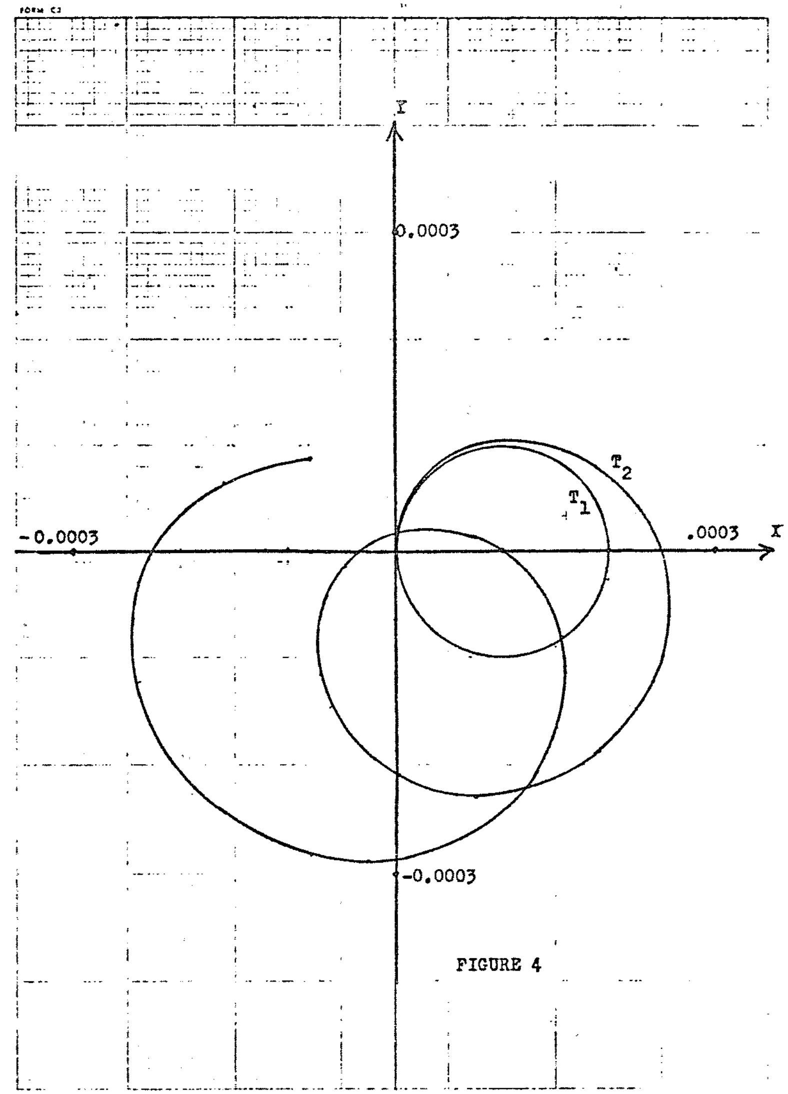

FIGURE 4

REFERENCES

1. J. L. Synge, Relativity: The Special Theory, North-Holland, Amsterdam, 1965.
2. D. Greenspan, "Computer power and it impact on applied mathematics", in Studies in Mathematics, Vol. 7, edited by A. H. Taub, published by the Mathematical Association of America, distributed by Prentice-Hall Englewood Cliffs, N. J., 1971, pp. 65-89.
3. D. Greenspan, Discrete Numerical Methods in Physics and Engineering, Academic, N.Y., 1974.
4. G. Dahlquist and A. Bjork, Numerical Methods, Prentice-Hall, Englewood Cliffs, N.J., 1974.
5. C. W. Gear, Numerical Initial Value Problems in Ordinary Differential Equations, Prentice-Hall, Englewood Cliffs, N.J., 1971.
6. E. F. Taylor and J. A. Wheeler, Spacetime Physics, Freeman, San Francisco, 1966.
7. D. Greenspan, "The arithmetic basis of special relativity-I", TR 232, Dept. Comp. Sci., Univ. Wis., Madison, Dec., 1974.
8. D. Greenspan, "Lorentz invariant computations", TR 251, Dept. Comp. Sci., Univ. Wis., Madison, 1975.
9. H. M. Schwartz, Introduction to Special Relativity, McGraw-Hill, N.Y. 1968.
10. H. Arzelies, Relativistic Kinematics, Pergamon, N.Y., 1966.
11. M. Muirhead, The Special Theory of Relativity, Wiley, N.Y., 1973.
12. P. Bergmann, Introduction to the Theory of Relativity, Prentice-Hall, Englewood Cliffs, N.J., 1942.
13. D. Greenspan, "The arithmetic basis of special relativity-II," TR 240, Dept. Comp. Sci., Univ. Wisconsin, Madison, March, 1975.

APPENDIX 1


```
'RUN GREENSPAN,9724
'FOR,SIZ
C DOUBLE PRECISION MODE
      IMPLICIT DOUBLE PRECISION (A-H,O-Z)
C INITIAL POSITIONS AND VELOCITIES
5     READ (5,10,END=500)X,V
10     FORMAT (2F10.5)
C PRINT INITIAL POSITION AND VELOCITY
      WRITE (6,15)X,V
15    FORMAT(1X,2F20.10)
C INITIALIZE PARAMETERS
      I=0
      IPRINT=100
      DT=0.0001
C CALCULATE
20     Y=X+DT*V
      A=DT*X*(1.0-V*V)
      B=1.0+A*A
      C=B*(1.0-V*V)
      W=(V-A*SQRT(C))/B
C COUNTER
      I=I+1
C PRINT EVERY HUNDRETH CALCULATION
      IF(MOD(I,IPRINT).EQ.0) WRITE(6,30)I,Y,W
30      FORMAT(1X,I10,2F20.10)
      X=Y
      V=W
C CALCULATE 25,000 TIME STEPS
      IF (I.LT.25000) GO TO 20
40    GO TO 5
500    STOP
      END
```

APPENDIX 2

```
°RUN,/R GREENSPAN, 9724
°FOR,SIZ
C ELECTRIC PARTICLE IN MAGNETIC FIELD - CASE OF 1/R*R DECAY
C DOUBLE PRECISION MODE
      IMPLICIT DOUBLE PRECISION (A-H,O-Z)
C INIT POSITION, VELOC, AND ALPHA
5       READ (5,10,END=500) X,Y,VX,VY,ALPHA
10      FORMAT (4F10.5,F20.5)
C PRINT INITIAL DATA
      WRITE (6,15) X,Y,VX,VY,ALPHA
15      FORMAT (1X,4F10.5,F20.10)
C INITIALIZE PARAMETERS
      I=0
      IPRINT=10
      DT=0.0001
      VX1=0.0
      VY1=0.0
      EPS=1.0E-15
      ZERO=0.0
      W=1.0
C CALCULATE NEW POSITIONS
20      X2=X+VX*DT
      Y2=Y+VY*DT
C CALCULATE H
17      H=100.0/(1.0+ALPHA*(X*X+Y*Y))
C CALCULATE NEW VELOCITIES
      F=1.0-VX*VX-VY*VY
      A=-H*VY*DT*F
      B=H*VX*DT*F
      VX1=VX
      VY1=VY
24      C1=1.0-VX1*VX1-VY1*VY1
      IF (C1.LT.ZERO) GO TO 70
      C=SQRT(C1)
25      VX2=(VX1-W*(VX1-VX+A*C)*C)/(C-A*VX1)
      D1=1.0-VX2*VX2-VY1*VY1
      IF (D1.LT.ZERO) GO TO 70
      D=SQRT(D1)
26      VY2=(VY1-W*(VY1-VY+B*D)*D)/(D-B*VY1)
      S=ABS(VX2-VX1)
      T=ABS(VY2-VY1)
27      IF(S.LE.EPS) GO TO 28
      VX1=VX2
      VY1=VY2
      GO TO 24
28      IF(T.LE.EPS) GO TO 29
      VX1=VX2
      VY1=VY2
      GO TO 24
C COUNTER
29      I=I+1
C PRINT EVERY 10 STEPS
      IF (MOD(I,IPRINT).EQ.0) WRITE (6,50) I,X2,Y2,VX2,VY2
50    FORMAT (1X,I5,4F11.8)
      X=X2
      Y=Y2

      VX=VX2
      VY=VY2
C CALCULATE 5000 STEPS
      IF (I.LT.5000) GO TO 20
70      GO TO 5
500     STOP
      END
```