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The Arithmetic Basis of Special Relativity - II

by

Donald Greenspan

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University of Wisconsin
1210 West Dayton Street
Madison, Wisconsin 53706

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Abstract

In a previous paper, it was shown that, under the assumption that both particle and rocket frame motions were in the X-direction only, then all the basic concepts and results of special relativity were obtainable from arithmetic processes only. In this paper, the arithmetic approach is extended to more general particle and rocket frame motions. Particular attention is directed toward velocity, acceleration, proper time, momentum, energy and 4-vectors in both spacetime and Minkowski space, and to relativistic generalizations of Newton's second law.

1. Introduction

In a previous paper [1], it was shown that, under the assumption that both particle and rocket frame motions were in the X-direction only, then all the basic concepts and results of special relativity were obtainable from arithmetic processes only. In this paper, the arithmetic approach is extended to more general particle and rocket frame motions.

2. Basics

Consider two Euclidean coordinate systems XYZ and X'Y'Z' which at some initial time coincide. Let the X'Y'Z' system, called the rocket frame, be in constant uniform motion with respect to the XYZ frame, called the lab frame. Let this constant relative velocity be $\vec{u} = (u_1, u_2, u_3)$.

For $t_0 = 0$, let an observer in the lab frame make observations at the successive times t_k , $k = 0,1,2,\ldots$. Using an identical, synchronized clock, let an observer in the rocket frame make observations at the times t'_k , $k = 0,1,2,\ldots,$ where t'_k on the rocket clock corresponds to t_k on the lab clock.

If particle P is at (x_k, y_k, z_k) in the lab frame at time t_k , while it is at (x'_k, y'_k, z'_k) in the rocket frame at time t'_k , then we call x_k, y_k, z_k, t_k the spacetime coordinates of event (x_k, y_k, z_k, t_k) in the lab frame, and, correspondingly, call x'_k, y'_k, z'_k, t'_k the spacetime coordinates of event (x'_k, y'_k, z'_k, t'_k) in the rocket frame. The spacetime coordinates of the lab and the rocket frames are related by the Lorentz transformation, which is a linear algebraic relationship given as follows. Let

(1) $\vec{\beta} = (\beta_1, \beta_2, \beta_3) = \vec{u}/c$

(2) $u^2 = u_1^2 + u_2^2 + u_3^2 = c^2(\beta_1^2 + \beta_2^2 + \beta_3^2) = c^2\beta^2$

(3) $\gamma = (1 - \beta^2)^{-1/2}$,

where c is the speed of light. Let

$$\vec{r}_k = \begin{pmatrix} x_k \\ y_k \\ z_k \\ t_k \end{pmatrix}, \quad \vec{r}\,'_k = \begin{pmatrix} x'_k \\ y'_k \\ z'_k \\ t'_k \end{pmatrix} .$$

Then the Lorentz transformation $L = (L_{ij})$ is given [2,p.74] by

(5) $\vec{r}_k = L\vec{r}_k$,

where

$$(6) \quad (L_{ij}) = \begin{pmatrix} 1+\beta_1^2 \frac{\gamma^2}{\gamma+1} & \beta_1\beta_2 \frac{\gamma^2}{\gamma+1} & \beta_1\beta_3 \frac{\gamma^2}{\gamma+1} & -c\beta_1\gamma \\ \beta_1\beta_2 \frac{\gamma^2}{\gamma+1} & 1+\beta_2^2 \frac{\gamma^2}{\gamma+1} & \beta_2\beta_3 \frac{\gamma^2}{\gamma+1} & -c\beta_2\gamma \\ \beta_1\beta_3 \frac{\gamma^2}{\gamma+1} & \beta_2\beta_3 \frac{\gamma^2}{\gamma+1} & 1+\beta_3 \frac{\gamma^2}{\gamma+1} & -c\beta_3\gamma \\ -\frac{\beta_1}{c}\gamma & -\frac{\beta_2}{c}\gamma & -\frac{\beta_3}{c}\gamma & \gamma \end{pmatrix}$$

The transformation (6) is convenient from the physical point of view. From the geometric point of view, a more convenient form can be given as follows. Let new coordinates, called Minkowski coordinates [2,Chap. X], be defined by

$$(7) \quad \begin{cases} x_{1,k} = x_k,\ x_{2,k} = y_k,\ x_{3,k} = z_k,\ x_{4,k} = ict_k \\ x'_{1,k} = x'_k,\ x'_{2,k} = y'_k,\ x'_{3,k} = z'_k,\ x'_{4,k} = ict'_k . \end{cases}$$

If

$$(8) \qquad \vec{R}_k = \begin{pmatrix} x_{1,k} \\ x_{2,k} \\ x_{3,k} \\ x_{4,k} \end{pmatrix}, \qquad \vec{R}'_k = \begin{pmatrix} x'_{1,k} \\ x'_{2,k} \\ x'_{3,k} \\ x'_{4,k} \end{pmatrix},$$

Then the Lorentz transformation $L = (L_{ij})$ is given [2,p.74] by

$$(9) \qquad \vec{R}'_k = L\vec{R}_k ,$$

where

$$(10) \qquad (L_{ij}) = \begin{pmatrix} 1+\beta_1^2 \frac{\gamma^2}{\gamma+1} & \beta_1\beta_2 \frac{\gamma^2}{\gamma+1} & \beta_1\beta_3 \frac{\gamma^2}{\gamma+1} & i\beta_1\gamma \\ \beta_1\beta_2 \frac{\gamma^2}{\gamma+1} & 1+\beta_2^2 \frac{\gamma^2}{\gamma+1} & \beta_2\beta_3 \frac{\gamma^2}{\gamma+1} & i\beta_2\gamma \\ \beta_1\beta_3 \frac{\gamma^2}{\gamma+1} & \beta_2\beta_3 \frac{\gamma^2}{\gamma+1} & 1+\beta_3^2 \frac{\gamma^2}{\gamma+1} & i\beta_3\gamma \\ -i\beta_1\gamma & -i\beta_2\gamma & -i\beta_3\gamma & \gamma \end{pmatrix}$$

With regard to (10), note that

$$(11) \qquad \sum_{j=1}^{4} L_{ij} L_{kj} = \delta_{i,k}$$

$$(12) \qquad \sum_{j=1}^{4} L_{ji} L_{jk} = \delta_{i,k} ,$$

where $\delta_{i,k}$ is the Kronecker δ, and that

$$(13) \qquad L^T L = LL^T = I ,$$

where L^T is the transpose of L and I is the identity.

Note also that the classical relativistic implications of the Lorentz transformation, like time dilation and Lorentz contraction, are, of course, valid.

3. Velocity, Acceleration, and Proper Time

Let the forward difference operator Δ at time t_k be defined as usual by

$$\Delta F(k) = F(k+1) - F(k) .$$

Assume that particle P is in motion in the lab frame and at time t_k is at (x_k, y_k, z_k). Then P's velocity $\vec{v}_k$ and acceleration $\vec{a}_k$ at time t_k are defined by

$$(14) \qquad \vec{v}_k = \begin{pmatrix} v_{1,k} \\ v_{2,k} \\ v_{3,k} \end{pmatrix} = \begin{pmatrix} \frac{\Delta x_k}{\Delta t_k} \\ \frac{\Delta y_k}{\Delta t_k} \\ \frac{\Delta z_k}{\Delta t_k} \end{pmatrix}, \quad \vec{a}_k = \begin{pmatrix} a_{1,k} \\ a_{2,k} \\ a_{3,k} \end{pmatrix} = \begin{pmatrix} \frac{\Delta v_{1,k}}{\Delta t_k} \\ \frac{\Delta v_{2,k}}{\Delta t_k} \\ \frac{\Delta v_{3,k}}{\Delta t_k} \end{pmatrix} .$$

By the principle of relativity, P's velocity $\vec{v}'_k$ and acceleration $\vec{a}'_k$ in the rocket frame at time t'_k are defined by

$$(15) \qquad \vec{v}'_k = \begin{pmatrix} v'_{1,k} \\ v'_{2,k} \\ v'_{3,k} \end{pmatrix} = \begin{pmatrix} \frac{\Delta x'_k}{\Delta t'_k} \\ \frac{\Delta y'_k}{\Delta t'_k} \\ \frac{\Delta z'_k}{\Delta t'_k} \end{pmatrix}, \quad \vec{a}'_k = \begin{pmatrix} a'_{1,k} \\ a'_{2,k} \\ a'_{3,k} \end{pmatrix} = \begin{pmatrix} \frac{\Delta v'_{1,k}}{\Delta t'_k} \\ \frac{\Delta v'_{2,k}}{\Delta t'_k} \\ \frac{\Delta v'_{3,k}}{\Delta t'_k} \end{pmatrix} .$$

The respective magnitudes v_k, v'_k, a_k, a'_k of $\vec{v}_k, \vec{v}'_k, \vec{a}_k, \vec{a}'_k$ are defined in the customary way by

$$(16) \qquad v_k^2 = v_{1,k}^2 + v_{2,k}^2 + v_{3,k}^2 , \qquad v_k'^2 = v_{1,k}'^2 + v_{2,k}'^2 + v_{3,k}'^2$$

$$(17) \qquad a_k^2 = a_{1,k}^2 + a_{2,k}^2 + a_{3,k}^2 , \qquad a_k'^2 = a_{1,k}'^2 + a_{2,k}'^2 + a_{3,k}'^2 .$$

The quantity τ_k , defined in the lab frame by

(18) $\tau_k = (c^2t_k^2 - x_k^2 - y_k^2 - z_k^2)^{1/2}$

is invariant under L since

$$(c^2t_k'^2 - x_k'^2 - y_k'^2 - z_k'^2) = (c^2t_k^2 - x_k^2 - y_k^2 - z_k^2) .$$

When

(19) $c^2t_k^2 - x_k^2 - y_k^2 - z_k^2 > 0$,

τ_k is called the proper time of event (x_k,y_k,z_k,t_k) , and, throughout, we assume that (19) is valid for all k . The quantity $\delta\tau_k$, defined by

(20) $\delta\tau_k = [c^2(\Delta t_k)^2 - (\Delta x_k)^2 - (\Delta y_k)^2 - (\Delta z_k)^2]^{1/2}$

is, similarly, an invariant of L and is called the proper time between successive events (x_k,y_k,z_k,t_k) and $(x_{k+1},y_{k+1},z_{k+1},t_{k+1})$. Throughout, we assume that, in (20),

(21) $c^2(\Delta t_k)^2 - (\Delta x_k)^2 - (\Delta y_k)^2 - (\Delta z_k)^2 > 0$

or, equivalently, that $v_k < c$, since (21) implies

$$c^2 - (\frac{\Delta x_k}{\Delta t_k})^2 - (\frac{\Delta y_k}{\Delta t_k})^2 - (\frac{\Delta z_k}{\Delta t_k})^2 = c^2 - v_k^2 > 0 .$$

Note that $\delta\tau_k \neq \Delta\tau_k$ and $\delta\tau_k \neq d\tau_k$. For later convenience, observe also that

(22) $\delta\tau_k = \Delta t_k[c^2 - v_k^2]^{1/2} = \Delta t_k'[c^2 - v_k'^2]^{1/2}$.

Finally, note that

$$v_{j,k}' = \frac{L_{j1}v_{1,k} + L_{j2}v_{2,k} + L_{j3}v_{3,k} + L_{j4}}{L_{41}v_{1,k} + L_{42}v_{2,k} + L_{43}v_{3,k} + L_{44}} , \quad j = 1,2,3 ,$$

from which it follows that $\vec{v}_k$ does not transform into $\vec{v}_k'$ the way $\vec{r}_k$ transform into $\vec{r}_k'$. This is the basis of the usual statement that $\vec{v}_k$ is a vector in space, but not in spacetime.

In Minkowski coordinates, (18) can be rewritten as

(24) $\tau_k = \{\sum_{i=1}^{4} [-(x_{i,k})^2]\}^{1/2}$

while (20) becomes

(25) $\delta\tau_k = \{\sum_{i=1}^{4} [-(\Delta x_{i,k})^2]\}^{1/2}$,

and we define 4-velocities, or world velocities, and 4-accelerations, or world accelerations, in the following way. In Minkowski space, any quantity which has four components and is given in the lab frame by, say,

$$W = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{pmatrix}$$

and in the rocket frame by, say,

$$W' = \begin{pmatrix} w_1' \\ w_2' \\ w_3' \\ w_4' \end{pmatrix}$$

is called a 4-vector if

(26) $W' = LW$.

The prototype 4-vector in Minkowski space is, of course, $\vec{R}_k$, given by (8).

Now, suppose particle P is in motion and in the lab frame it is at (x_k, y_k, z_k) at time t_k while in the rocket frame it is at (x_k', y_k', z_k') at the corresponding time t_k'. At time t_k in the lab frame, we define P's Minkowski 4-velocity $\vec{V}_k$ and Minkowski 4-acceleration $\vec{A}_k$ by

$$(27)\qquad \vec{V}_k = \begin{pmatrix} V_{1,k} \\ V_{2,k} \\ V_{3,k} \\ V_{4,k} \end{pmatrix} = \begin{pmatrix} \frac{\Delta x_{1,k}}{\delta\tau_k} \\ \frac{\Delta x_{2,k}}{\delta\tau_k} \\ \frac{\Delta x_{3,k}}{\delta\tau_k} \\ \frac{\Delta x_{4,k}}{\delta\tau_k} \end{pmatrix} ; \quad \vec{A}_k = \begin{pmatrix} A_{1,k} \\ A_{2,k} \\ A_{3,k} \\ A_{4,k} \end{pmatrix} = \begin{pmatrix} \frac{\Delta V_{1,k}}{\delta\tau_k} \\ \frac{\Delta V_{2,k}}{\delta\tau_k} \\ \frac{\Delta V_{3,k}}{\delta\tau_k} \\ \frac{\Delta V_{4,k}}{\delta\tau_k} \end{pmatrix}$$

By the principle of relativity, and recalling that $\delta\tau_k$ is invariant, $\vec{V}_k'$ and $\vec{A}_k'$ are defined by

$$(28)\qquad \vec{V}_k' = \begin{pmatrix} V_{1,k}' \\ V_{2,k}' \\ V_{3,k}' \\ V_{4,k}' \end{pmatrix} = \begin{pmatrix} \frac{\Delta x_{1,k}'}{\delta\tau_k} \\ \frac{\Delta x_{2,k}'}{\delta\tau_k} \\ \frac{\Delta x_{3,k}'}{\delta\tau_k} \\ \frac{\Delta x_{4,k}'}{\delta\tau_k} \end{pmatrix} ; \quad \vec{A}_k' = \begin{pmatrix} A_{1,k}' \\ A_{2,k}' \\ A_{3,k}' \\ A_{4,k}' \end{pmatrix} = \begin{pmatrix} \frac{\Delta V_{1,k}'}{\delta\tau_k} \\ \frac{\Delta V_{2,k}'}{\delta\tau_k} \\ \frac{\Delta V_{3,k}'}{\delta\tau_k} \\ \frac{\Delta V_{4,k}'}{\delta\tau_k} \end{pmatrix}$$

Direct computation with (26) reveals easily that both $\vec{V}_k$ and $\vec{A}_k$ are 4-vectors. The relationship between components of $\vec{v}_k$ and the first three components of $\vec{V}_k$ can be established readily from (14), (22), and (27). Similar connections can be established between $\vec{a}_k$ and the first three components of $\vec{A}_k$.

The magnitude V_k of $\vec{V}_k$ is defined by

$$(29)\qquad V_k^2 = \sum_{j=1}^{4} V_{j,k}^2 .$$

An analogous definition holds also for $(V_k')^2$. Note that (25) and (29) imply

$$(30)\qquad V_k^2 = -1 .$$

Thus, since (30) is valid for all k, the concept of 4-velocity, though geometrically convenient, is more restrictive physically than the three dimensional velocity concept given by (14).

For completeness, let us show finally that

$$(V_k')^2 = (V_k)^2 ,$$

the validity of which follows since

$$\sum_{j=1}^{4} (V_{j,k}')^2 = \sum_{j=1}^{4} (V_{j,k}')(V_{j,k}')$$
$$= \sum_{j=1}^{4} \left(\sum_{m=1}^{4} L_{jm} V_{m,k}\right)\left(\sum_{n=1}^{4} L_{jn} V_{n,k}\right)$$
$$= \sum_{m=1}^{4} \sum_{n=1}^{4} \delta_{mn} V_{m,n} V_{n,k}$$
$$= \sum_{j=1}^{4} V_{j,k}^2 .$$

Note that 4-vectors with respect to spacetime coordinates x_k, y_k, z_k, t_k can also be defined easily merely by replacing L with $\mathcal{L}$ in (26).

4. Momentum and Energy

We proceed now under the assumption that, without the presence of an external force, the interaction of two particles conserves linear momentum. To be precise, let particle P of mass m be in motion in the lab frame. At time t_k, the linear momentum $\vec{p}_k$ of P is defined by

$$(31) \qquad \vec{p}_k = m\,\vec{v}_k \,.$$

Similarly, in the rocket frame, let

$$(32) \qquad \vec{p}'_k = m'\,\vec{v}'_k \,.$$

The validity of momentum conservation follows [3, pp. 101-110] if we require that in the lab frame

$$(33) \qquad m = \frac{c\, m_0}{(c^2 - v_k^2)^{1/2}}$$

and, at the corresponding time in the rocket frame,

$$(34) \qquad m' = \frac{c\, m_0}{(c^2 - v_k'^2)^{1/2}} \,,$$

where m_0 is a constant called the rest mass of P.

We continue then by assuming the validity of (33) and (34).

The total energy E of particle P of mass m is defined by

$$(35) \qquad E = mc^2$$

Extensive experimental evidence now exists [4, p. 15-11] to support the validity of (35), and the usual formula for rest energy $E_0 = m_0 c^2$ follows readily.

To establish a relationship between momentum, energy, and rest mass, note that (31) and (33) imply

$$E_0^2 = p_k^2 c^2 + m_0^2 c^4 \,,$$

where p_k is the magnitude of $\vec{p}_k$.

5. The Energy Momentum 4-Vector

Thus far we have not placed particular emphasis on any special units of measurement. In this connection, we will now be relatively more specific in the following way. Let

$$(36) \qquad E^* = E/c^2$$

be a normalized energy in the sense that the units of E^* are units of mass. Then, from (35) and (36),

$$(37) \qquad E^* = m \,.$$

Our present purpose is to show that the quantity

$$\begin{pmatrix} m\, v_{1,k} \\ m\, v_{2,k} \\ m\, v_{3,k} \\ E^* \end{pmatrix}$$

is a 4-vector, called the energy momentum vector, with respect to L. To do this observe that, with the help of (22), (33) and (34), one has

$$L \begin{pmatrix} m\, v_{1,k} \\ m\, v_{2,k} \\ m\, v_{3,k} \\ E^* \end{pmatrix} = L \begin{pmatrix} m\, v_{1,k} \\ m\, v_{2,k} \\ m\, v_{3,k} \\ m \end{pmatrix} = L \begin{pmatrix} \dfrac{c\, m_0\, \Delta x_k}{(c^2-v_k^2)^{1/2}\, \Delta t_k} \\ \dfrac{c\, m_0\, \Delta y_k}{(c^2-v_k^2)^{1/2}\, \Delta t_k} \\ \dfrac{c\, m_0\, \Delta z_k}{(c^2-v_k^2)^{1/2}\, \Delta t_k} \\ \dfrac{c\, m_0}{(c^2-v_k^2)^{1/2}} \end{pmatrix}$$

$$= L \begin{pmatrix} \dfrac{c\, m_0}{\delta\tau_k} \Delta x_k \\ \dfrac{c\, m_0}{\delta\tau_k} \Delta y_k \\ \dfrac{c\, m_0}{\delta\tau_k} \Delta z_k \\ \dfrac{c\, m_0}{\delta\tau_k} \Delta t_k \end{pmatrix} = \begin{pmatrix} \dfrac{c\, m_0}{\delta\tau_k} \Delta x_k' \\ \dfrac{c\, m_0}{\delta\tau_k} \Delta y_k' \\ \dfrac{c\, m_0}{\delta\tau_k} \Delta z_k' \\ \dfrac{c\, m_0}{\delta\tau_k} \Delta t_k' \end{pmatrix} = \begin{pmatrix} m' v_{1,k}' \\ m' v_{2,k}' \\ m' v_{3,k}' \\ m' \end{pmatrix},$$

and the assertion is proved.

6. Dynamics

Finally, we examine possible relativistic extensions of Newton's second law and the invariance (called symmetry by some authors, as in [4], and covariance by others, as in [5]) of such extensions under the Lorentz transformation.

It is an unfortunate mathematical consequence of continuous special relativistic theory [6, pp. 103-104] that the simple Einstein generalization

$$(38) \qquad \vec{F} = \frac{d}{dt}(m\vec{v})$$

does not, in general, transform under L into

$$(39) \qquad \vec{F}' = \frac{d}{dt'}(m'\vec{v}'),$$

although, interestingly enough, if both the particle and the rocket frames move in the same direction, then, indeed, does (38) transform into (39). To resolve this failure of the principle of relativity with respect to (38), two approaches have been followed. First ([2, p. 268], [5, p. 63]), one can proceed under the approximating assumption that if a rocket frame were attached to P, so that it can have accelerated motion, and if at time t the velocity of P is $\vec{v}$, then one can treat the rocket frame at time t as being instantaneously in uniform relative motion with velocity $\vec{v}$ with respect to the lab frame. Indeed, such an assumption is tacitly made in the "clock paradox" [2, p. 63]. Second ([6, pp. 103-104], [7, p. 86], [8, pp. 165-167]), one can formulate equations of motion directly in Minkowski space.

To develop the arithmetic analogues of the concepts and results described above, we will assume in Minkowski space the dynamical difference equation

$$(40) \qquad \vec{F}_k = \alpha_k m_k \vec{A}_k - \frac{\Delta(\alpha_k m_k)}{\delta\tau_k}\left(\frac{\vec{V}_{k+1}+\vec{V}_k}{2}\right), \quad \alpha_k m_k = m_0,$$

and in spacetime a relative projection [8, p. 167] of the form

$$(41) \qquad \vec{F}_k^P = c^2\left[m_k \vec{A}_k^P - \frac{\Delta m_k}{\delta\tau_k}\vec{V}_k^P\right],$$

where c^2 has replaced α_k in (40), and where the superscript p denotes the dropping of the fourth component of the given quantity. Note that (40) is analogous to the expanded form

$$\vec{F} = m\vec{a} + \vec{v}\,\frac{dm}{dt}$$

of (38) except for the sign between the terms. However, one can redefine $\vec{V}_k$ readily to yield agreement of signs also.

Let us show first that (41) is invariant under L provided that P and the rocket frame have velocities in the same direction. To do this, let us choose the lab frame and rocket frame coordinates so that motions are in the X-direction only. Our problem then is to show that

$$(42) \qquad F_{1,k} = c^2[m_k A_{1,k} - \frac{\Delta m_k}{\delta\tau_k} V_{1,k}]$$

and

$$(43) \qquad F'_{1,k} = c^2[m'_k A'_{1,k} - \frac{\Delta m'_k}{\delta\tau_k} V'_{1,k}]$$

imply that

$$F_{1,k} = F'_{1,k} \; .$$

From (42), then,

$$F_{1,k} = \frac{c^2}{\delta\tau_k}\,[m_k V_{1,k+1} - m_{k+1} V_{1,k}] \; .$$

Now, under the present assumptions, (22) simplifies to

$$\delta\tau_k = \Delta t_k[c^2 - v_k^2]^{1/2} \; ,$$

so that

$$m_k = \frac{c m_0 \Delta t_k}{\delta\tau_k} \; .$$

Thus,

$$F_{1,k} = \frac{c m_k}{m_0 \Delta t_k}\,(m_k V_{1,k+1} - m_{k+1} V_{1,k})$$

$$= \frac{c^2 m_k}{\frac{\delta\tau_k}{\Delta t_k} , \frac{\delta\tau_{k+1}}{\Delta t_{k+1}}} \left(\frac{\frac{\Delta x_{k+1}}{\delta\tau_{k+1}} \frac{\delta\tau_{k+1}}{\Delta t_{k+1}} - \frac{\Delta x_k}{\delta\tau_k} \frac{\delta\tau_k}{\Delta t_k}}{\Delta t_k} \right) ,$$

so that

$$(44) \qquad F_{1,k} = \frac{c^2 m_k}{[(c^2 - v_k^2)(c^2 - v_{k+1}^2)]^{1/2}} \; \frac{\Delta v_{1,k}}{\Delta t_k} \; ,$$

the invariance of which follows readily, as in [1].

Note that as $\Delta t_k \to 0$, (44) reduces to the special form

$$F = \frac{c^2 m}{c^2 - v^2}\,\frac{dv}{dt}$$

of

$$F = \frac{d}{dt}\,(mv) \; .$$

In Minkowski space there is a basic problem in the study of (40), which will be written now as

$$(45) \qquad \vec{F}_k = m_0 \vec{A}_k - \frac{\Delta m_0}{\delta\tau_k}\,\left(\frac{\vec{V}_{k+1} + \vec{V}_k}{2}\right) \; .$$

Indeed, equations (20) and (45) constitute nine equations for the eight quantities $x_{j,k+1}$, $V_{j,k+1}$, $j = 1,2,3,4$, a type of complexity which did not exist when considering (41) in cartesian three-space [8, p.166]. In Minkowski space, then, one is forced to generate another unknown quantity, and the only candidate is the rest mass m_0 . So, for the present, we must continue under the assumption that m_0 depends on time through $\vec{F}_k$. Under this assumption, by taking inner products of both sides of (45) with $(\vec{V}_{k+1}+\vec{V}_k)/2$ and by using (27) and (30), one finds

(46) $$\vec{F}_k \cdot \left(\frac{\vec{V}_{k+1}+\vec{V}_k}{2}\right) = -\frac{\Delta m_0}{\delta\tau_k}\left(\frac{\vec{V}_{k+1}+\vec{V}_k}{2}\right) \cdot \left(\frac{\vec{V}_{k+1}+\vec{V}_k}{2}\right) .$$

If one then chooses $\vec{F}_k$ in such a manner that

(47) $$\vec{F}_k \cdot \left(\frac{\vec{V}_{k+1}+\vec{V}_k}{2}\right) = 0 ,$$

then, from (46), one can always choose $\Delta m_0 = 0$. Thus, restricting attention to forces which satisfy (47) yields

(48) $$\vec{F}_k = m_0\vec{A}_k ,$$

which is covariant under the Lorentz transformation and is completely analogous in structure to Newton's equation of motion.

Condition (47) restricts attention to forces which are orthogonal to the average velocity of a particle in motion. In the limit, it requires the force to be orthogonal to the particle's instantaneous velocity. This is, of course, the case in the most important application of special relativistic mechanics, that is, to the study of the motion of a charged particle in an electromagnetic field.

Note, finally, with regard to (48), that for the special case $\vec{F}_k \equiv \vec{0}$, one has

(49) $$m_0\frac{\vec{V}_{k+1}-\vec{V}_k}{\delta\tau_k} \equiv \vec{0} ,$$

so that

(50) $$\vec{V}_k \equiv \vec{V}_0 , \quad k = 0,1,2,\ldots .$$

But, (50) implies

$$\frac{\vec{R}_{k+1}-\vec{R}_k}{\delta\tau_k} = \vec{V}_0 , \quad k = 0,1,2,\ldots ,$$

so that

$$\vec{R}_k = \vec{R}_0 + \vec{V}_0\sum_{j=0}^{k-1}\delta\tau_j ; \quad k = 0,1,2,\ldots ,$$

which is lineal in Minkowski space.

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