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LINEAR COMPLEMENTARITY PROBLEMS
SOLVABLE BY A SINGLE LINEAR PROGRAM

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Abstract

It is shown that the linear complementarity problem of finding a z in R^n such that $Mz + q \geq 0$, $z \geq 0$ and $z^T(Mz+q) = 0$ can be solved by a single linear program in some important special cases such as when M or its inverse is a Z-matrix, that is a real square matrix with nonpositive off-diagonal elements. As a consequence certain problems in mechanics, certain problems of finding the least element of a polyhedral set and certain quadratic programming problems, can each be solved by a single linear program.

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We consider the linear complementarity problem of finding a z in R^n such that

$$(1) \quad Mz + q \geq 0, \quad z \geq 0, \quad z^T(Mz+q) = 0$$

where M is a given real $n \times n$ matrix and q is a given vector in R^n . A number of authors [3,19,20,14,4,5,18,9] have recently considered an important special case of this problem under the restriction that M is a Z-matrix, that is a real square matrix with nonpositive off-diagonal elements, and have proposed a variety of methods for its solution. Originally Chandrasekaran [3] proposed solving a sequence of linear inequalities, Saigal [20] proposed Lemke's method, and Cottle, Golub and Sacher [4,5,18] proposed a modification of the principal pivoting method [6], a specialization of Chandrasekaran's method and a modification of the point successive overrelaxation technique [9]. Part of the significance of this problem arises from the fact that a number of free boundary problems of fluid mechanics can be solved by solving a linear complementarity problem (1) where M is a Z-matrix [8,5]. It is anticipated that many more physically significant free boundary value problems governed by elliptic partial differential equations will lead to complementarity problems (1) for which M is a Z-matrix [10].

The principal and somewhat surprising result of this paper is that each solution z of the linear program

$$(2) \quad \text{minimize } p^T z \text{ subject to } Mz + q \geq 0, \quad z \geq 0$$

for an easily determined p in R^n , solves the linear complementarity problem (1) for a number of special cases, including those when M or its inverse (if it exists) are Z-matrices (Theorems 1 and 2). In addition if M is a Z-matrix with a nonnegative inverse (or equivalently a Z-matrix with positive principal minors), we show that the

least element of the polyhedral set $\{z | Mz + q \geq 0, z \geq 0\}$ in the sense of Cottle-Veinott [7] can be obtained by a single linear program (Theorem 3). Finally, because the quadratic programming problem of minimizing $\frac{1}{2}z^T Mz + q^T z$ subject to $z \geq 0$ is equivalent [9, p. 386, 15, p. 111] to the linear complementarity problem (1) when M is symmetric and positive semidefinite, we will show (Theorem 4) that this quadratic programming problem can be solved by a single linear program whenever M or its inverse is a Z -matrix. We state now our principal result.

THEOREM 1: Let the set $\{z | Mz + q \geq 0, z \geq 0\}$ be nonempty, and let M satisfy

$$(3) \quad MZ_1 = Z_2$$

$$(4) \quad r^T Z_1 + s^T Z_2 > 0 \quad (r, s) \geq 0$$

where Z_1 and Z_2 are $n \times n$ Z -matrices, $r \in R^n$ and $s \in R^n$. Then the linear complementarity problem (1) has a solution which can be obtained by solving the linear program (2) with

$$(5) \quad p = r + M^T s$$

To prove this theorem it is convenient to first write the dual linear program to (2)

$$(6) \quad \text{maximize } -q^T y \quad \text{subject to } -M^T y + p \geq 0, y \geq 0$$

and to establish the following simple but key lemma.

LEMMA 1: If z solves the linear program (2) and if the corresponding optimal dual variable y satisfies

$$(I - M^T)y + p > 0$$

where I is the identity matrix, then z solves the linear complementarity problem (1).

Proof: $y^T(Mz+q) + z^T(-M^T y+p) = y^T q + z^T p = 0$

Since $y \geq 0$, $Mz + q \geq 0$, $z \geq 0$ and $-M^T y + p \geq 0$ it follows that

$$y_i (Mz+q)_i = 0, z_i (-M^T y+p)_i = 0 \quad i = 1, \dots, n$$

where subscripted quantities denote the i th element of a vector. But $y_i + (-M^T y+p)_i > 0$, $i = 1, \dots, n$, hence either $y_i > 0$ or $(-M^T y+p)_i > 0$, $i = 1, \dots, n$, and consequently $(Mz+q)_i = 0$ or $z_i = 0$, $i = 1, \dots, n$. \square

Proof of Theorem 1: Since $y = s$ is a dual feasible point, the dual linear programs (2) and (6) must have solutions, which we denote by z and y respectively. Let $Z_1 = D - V$ and $Z_2 = D - U$, where V and U are nonnegative matrices and D is a positive diagonal matrix. Then

$$\begin{aligned} 0 < r^T Z_1 + s^T Z_2 &= (r^T + s^T M) Z_1 = p^T (D-V) \\ &= p^T (D-V) + y^T (-MD + MV + D - U) \quad (\text{Since } M(D-V) = D-U) \\ &= (-y^T M + p^T) (D-V) + y^T (D-U) \\ &\leq (y^T (I-M) + p^T) D \quad (\text{Since } -y^T M + p^T \geq 0, V \geq 0, \\ &\quad y \geq 0, U \geq 0) \end{aligned}$$

Since D is a positive diagonal matrix, it follows that $y^T (I-M) + p^T > 0$ and by Lemma 1, z solves the linear complementarity problem (1). \square

Remark 1: The proof of Theorem 1 shows that conditions (3), (4) and (5) imply that there exists a dual feasible point for (6) and for each dual feasible point the condition $(I-M^T)y + p > 0$ of Lemma 1 is satisfied. The converse (which is not needed in the sequel of this paper) is also true and can be shown by using Motzkin's theorem of the alternative [15, p. 28]. Conditions (3), (4) and (5) are also equivalent to $MZ_3 \leq I$, $p^T Z_3 > 0$,

$p = r + M^T s$, $Z_3 \in Z$ and $(r, s) \geq 0$. The condition

$p = r + M^T s$, $(r, s) \geq 0$ is equivalent to dual feasibility.

Remark 2: The set Z of Z -matrices contains an important subset K which has been extensively characterized by Fiedler and Pták [13, p. 387]. This set K will play an important role in obtaining useful special cases of Theorem 1. In particular we shall employ the following equivalent characterizations of a K -matrix A : (a) A is a Z -matrix with a nonnegative inverse, (b) A is a Z -matrix with positive principal minors, (c) A is a Z -matrix and there exists an $r \in R^n$, $r \geq 0$, such that $r^T A > 0$, and (d) A is a Z -matrix and $z_i (Az)_i \leq 0$, $i = 1, \dots, n$, implies that $z = 0$. (A K -matrix is also called an M -matrix by some authors.)

The following immediate consequence of Theorem 1, is an existence result for the linear complementarity problem (1) which also provides a linear program (2) for solving (1) for important special cases such as when M or its inverse are Z -matrices.

THEOREM 2: Let $\{z | Mz + q \geq 0, z \geq 0\}$ be nonempty, and let e be any positive vector in R^n (in particular it may be taken as a vector of ones.) Then for each of the cases when

- (a) $M = Z_2 Z_1^{-1}$, $Z_1 \in K$, $Z_2 \in Z$ ($p=r \geq 0, r^T Z_1 > 0$)
- (b) $M = Z_2 Z_1^{-1}$, $Z_1 \in Z$, $Z_2 \in K$ ($p=M^T s, s \geq 0, s^T Z_2 > 0$)
- (c) $M \in Z$ ($p=e$)
- (d) $M^{-1} \in Z$ ($p=M^T e$)
- (e) $-M \in K$ ($p=-e$ or $p=M^T e$)
- (f) $-M^{-1} \in K$ ($p=-M^T e$ or $p=e$)

the linear complementarity problem (1) has a solution which can be obtained by solving the linear program (2) with the p indicated above.

Proof:

- (a) Follows from Theorem 1 by setting $s = 0$, and from Remark 2(c).
- (b) Follows from Theorem 1 by setting $r = 0$, and from Remark 2(c).
- (c) Follows from part (a) of this Theorem by setting $Z_1 = I, M = Z_2$ and $r = e$.
- (d) Follows from part (b) of this Theorem by setting $Z_2 = I, M = Z_1^{-1}$ and $s = e$.
- (e) The case $p = -e$ follows from part (b) of this Theorem by setting $Z_1 = -I, M = -Z_2$ and $s = -(M^T)^{-1}e$. The case $p = M^T e$ follows from part (d) of this Theorem by observing that, by Remark 2(a), $M^{-1} \leq 0$ and hence $M^{-1} \in Z$.
- (f) The case $p = -M^T e$ follows from part (a) of this Theorem by setting $Z_2 = -I, M = -Z_1^{-1}$ and $r = -M^T e$. The case $p = e$ follows from part (c) of this Theorem by observing that, by Remark 2(a), $M \leq 0$, and hence $M \in Z$. \square

Remark 3: Some special nonnegative matrices can be handled as special cases of Theorem 2 above. For example the case $M^{-1} \in K$ (and hence $M \geq 0$) is a special case of part (d) of Theorem 2. Also if in part (a) of Theorem 2 we impose the additional restriction that $Z_2 \geq Z_1$ then $M = (Z_2 - Z_1)Z_1^{-1} + I \geq I$. Similarly some special matrices with nonnegative inverses can be handled as special cases of Theorem 2. Thus the case $M \in K$ (and hence $M^{-1} \geq 0$) is a special case of part (c) of Theorem 2. Also if in part (b) of Theorem 2 we impose the additional restriction that $Z_1 \geq Z_2$ then $M^{-1} = (Z_1 - Z_2)Z_2^{-1} + I \geq I$.

Remark 4: Note that whenever $p \geq 0$, as is the case in parts (a), (c) and (f) of Theorem 2 above, $y = 0$ is a dual feasible point, and hence the dual simplex algorithm [11] should be used.

Remark 5: Note that when $M^{-1} \geq 0$, as is the case for example when $M \in K$, the set $\{z | Mz+q \geq 0, z \geq 0\}$ is nonempty for any q . For, choose $a \in R^n$ such that $a \geq 0$ and $a \geq q$, and define $z = M^{-1}(a-q) \geq 0$. Then $Mz + q = a \geq 0$.

Our next result shows how to find by a single linear program the least element, in the sense of Cottle and Veinott [7], of a polyhedral set defined by a K -matrix.

THEOREM 3: If $M \in K$, then for each q the polyhedral set $\{z | Mz+q \geq 0, z \geq 0\}$ contains a unique least element \bar{z} , that is $\bar{z} \leq z$ for all $z \in \{z | Mz+q \geq 0, z \geq 0\}$, which is also the unique solution of the linear complementarity problem (1), and which can be obtained by solving the linear program (2) with any $p > 0$.

Proof: Because the condition $\bar{z} \leq z$ is equivalent to $p^T(z-\bar{z}) \geq 0$ for all $p > 0$, it follows that \bar{z} is the desired least element of $\{z | Mz+q \geq 0, z \geq 0\}$ if and only if it solves the linear program (2) for all $p > 0$. By Remark 5 and Theorem 2(c), for any q the linear complementarity problem has a solution which can be obtained by solving the linear program (2) with any $p > 0$. Suppose \bar{z} and \hat{z} are solutions to the linear program (2) with $p = \bar{p} > 0$ and $p = \hat{p} > 0$ respectively. (We do not exclude the possibility that $\bar{p} = \hat{p}$.) Then by Theorem 2(c) both \bar{z} and \hat{z} solve the linear complementarity problem (1). Hence, similarly to Gale-Murty [17, p. 76], we have that for $i = 1, \dots, n$

$$\begin{aligned} (\bar{z}-\hat{z})_i (M(\bar{z}-\hat{z}))_i &= (\bar{z}-\hat{z})_i (M\bar{z}+q - (M\hat{z}+q))_i \\ &= -\bar{z}_i (M\hat{z}+q)_i - \hat{z}_i (M\bar{z}+q)_i \leq 0 \end{aligned}$$

Thus by Remark 2(d), $\bar{z} - \hat{z} = 0$ and \bar{z} is a unique solution of the linear program (2) no matter what $p > 0$ is used. Hence \bar{z} is the desired least element. \square

Remark 6: For very large linear complementarity problems such as those arising from discretization of numerical analysis problems [9,10,5] and in which M is a Z -matrix, we propose the use of large scale linear programming codes for solving (2) or alternatively the use of relaxation methods or projection methods [16,1,2,12] to solve the system of linear inequalities and equalities $Mz + q \geq 0, z \geq 0, -M^T y + p \geq 0, y \geq 0, p^T z + q^T y = 0$, which constitutes the Kuhn-Tucker conditions of the linear program (2). Since these methods do not disturb the sparsity, if any, of the matrix M , it would be interesting to compare them with the methods proposed in [5] for large sparse matrices.

We conclude by showing that under suitable conditions the quadratic program

$$(7) \quad \text{minimize} \quad \frac{1}{2}z^T Mz + q^T z \quad \text{subject to} \quad z \geq 0$$

can be solved by solving the linear program (2).

THEOREM 4: (a) Let M be a symmetric positive semidefinite matrix, let $\{z | Mz + q \geq 0, z \geq 0\}$ be nonempty and let either $M \in Z$ or $M^{-1} \in Z$. Then the quadratic program (7) has a solution which can be obtained by solving the linear program (2) with $p = e$, any positive vector in R^n , when $M \in Z$; and $p = M^T e$ when $M^{-1} \in Z$. (b) Let M be symmetric. If $M \in K$, or in particular if $M \in Z$ and M has a positive strictly dominant diagonal, then the quadratic program (7) has a unique solution which can be obtained by solving the linear program (2) with $p = e$, any positive vector in R^n .

Proof: (a) The necessary and sufficient Kuhn-Tucker optimality conditions for (7) are precisely conditions (1) [9, p. 386, 15, p. 111]. This part of the theorem follows then from Theorem 2, parts (c) and (d).

(b) By Remark 2(c), M is a K -matrix, and by Theorem 3, the conditions (1) have a unique solution for each q which is also the unique solution of the quadratic program (7) and the linear program (2) for any $p > 0$. \square

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