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EQUIVALENCE OF THE COMPLEMENTARITY PROBLEM TO A SYSTEM OF NONLINEAR EQUATIONS

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EQUIVALENCE OF THE COMPLEMENTARITY PROBLEM TO A SYSTEM OF NONLINEAR EQUATIONS[1)]

by

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Abstract

It is shown that the complementarity problem of finding a z in R^n satisfying $zF(z) = 0$, $F(z) \geqq 0$, $z \geqq 0$, where $F: R^n \to R^n$, is completely equivalent to solving the system of n nonlinear equations in n unknowns

$$\theta(|F_i(z) - z_i|) - \theta(F_i(z)) - \theta(z_i) = 0 , \quad i = 1, \ldots, n$$

where $F_i(z)$ and z_i denote the components of $F(z)$ and z respectively and θ is any strictly increasing function from R into R that passes through the origin. If in addition, F is differentiable on R^n , θ is differentiable on R and $\theta'(0) = 0$, then the above equations are globally differentiable, and at any solution z which satisfies the strict complementarity condition $F(z) + z > 0$, the system of equations has a nonsingular Jacobian if F has a nonsingular Jacobian with nonsingular principal minors.

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Let $F:R^n \to R^n$, let $z \in R^n$ and let F_i and z_i , $i=1,\dots,n$, denote the components of F and z respectively. The celebrated complementarity problem of finding a $z \in R^n$ such that

$$(1) \qquad zF(z) = 0 \qquad F(z) \geqq 0 \qquad z \geqq 0$$

has received wide attention in the mathematical programming literature [2,3,4,5,8]. The main method for its solution has been that of simplicial approximation which is a constructive method for finding fixed points of continuous mappings. We give here a completely equivalent formulation of the complementarity problem as a system of n nonlinear equations in n unknowns and thereby make possible the use of the powerful tools of nonlinear equations theory [9] in solving the complementarity problem. Our principal result is the following theorem which can be obtained by symmetrizing the key Lemma 2.7 of [6] or from Lemma 3 of [7].

<u>THEOREM</u> Let θ be any strictly increasing function, from R into R , that is $a > b \Longleftrightarrow \theta(a) > \theta(b)$, and let $\theta(0) = 0$. Then, z solves the complementarity problem (1) if and only if

$$(2) \qquad \theta(|F_i(z) - z_i|) - \theta(F_i(z)) - \theta(z_i) = 0 \ , \ i = 1,\dots,n$$

<u>Proof:</u> (only if) For each $i = 1,\dots,n$, either $z_i = 0$ or $F_i(z) = 0$. If $z_i = 0$, then $\theta(|F_i(z) - z_i|) - \theta(F_i(z)) - \theta(z_i) = \theta(F_i(z)) - \theta(F_i(z)) - 0 = 0$. If $F_i(z) = 0$, $\theta(|F_i(z) - z_i|) - \theta(F_i(z)) - \theta(z_i) = \theta(z_i) - 0 - \theta(z_i) = 0$.

(if) (a) To show that $F(z) \geqq 0$, assume the contrary, that is $F_i(z) < 0$ for some $i = 1,\ldots,n$. Then $0 \leqq \theta(|z_i - F_i(z)|) = \theta(F_i(z)) + \theta(z_i) < \theta(z_i)$, from which it follows by the strict increasing property of θ that $z_i > 0$ and $z_i > |z_i - F_i(z)| = z_i - F_i(z)$. This contradicts $F_i(z) < 0$.

(b) To show that $z \geqq 0$, interchange the roles of z_i and $F_i(z)$ in (a) above.

(c) From (a) and (b) we have that $z \geqq 0$ and $F(z) \geqq 0$. To show that $zF(z) = 0$, assume the contrary, that is, $z_i > 0$ and $F_i(z) > 0$ for some $i = 1,\ldots,n$. If $F_i(z) \geqq z_i$ then

$$\theta(|F_i(z) - z_i|) = \theta(F_i(z) - z_i) < \theta(F_i(z)) < \theta(F_i(z)) + \theta(z_i)$$

This however contradicts $\theta(|F_i(z) - z_i|) - \theta(F_i(z)) - \theta(z_i) = 0$. Similarly, to show that the case $z_i \geqq F_i(z)$ also leads to a contradiction, interchange the roles of z_i and $F_i(z)$ in the last two sentences. QED

In many computational algorithms for solving nonlinear equations (e.g. Newton and quasi-Newton methods) it is often required that the Jacobian be nonsingular at the solution to which the algorithm is supposed to converge. The following corollary gives <u>sufficient</u> conditions for the Jacobian of (θ) to be nonsingular.

<u>Corollary:</u> Let z solve the complementarity problem (1) and satisfy the strict complementarity condition $z + F(z) > 0$. Let $\nabla F(z)$, the Jacobian of F at z , have nonsingular principal minors, let θ be

a differentiable strictly increasing function from R into R such that $\theta'(0) + \theta'(\zeta) > 0$ for all $\zeta > 0$. Then z solves (2) and the Jacobian of (2) at z is nonsingular.

Proof: Let $G_i(z) = 0$, denote the i^{th} equation of (2) and let

$$\operatorname{sgn}\zeta = \begin{cases} 1 & \text{if } \zeta > 0 \\ -1 & \text{if } \zeta < 0 \end{cases}, \qquad \delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

Then

$$\frac{\partial G_i(z)}{\partial z_j} = \theta'(|F_i(z) - z_i|)\operatorname{sgn}(F_i(z) - z_i)\left(\frac{\partial F_i(z)}{\partial z_j} - \delta_{ij}\right) - \theta'(F_i(z))\frac{\partial F_i(z)}{\partial z_j} - \theta'(z_i)\,\delta_{ij}$$

Assume for the moment that $F_i(z) = 0$ for $i = 1,\ldots,\bar{n} \leq n$ and $F_i(z) > 0$ for $i = \bar{n}+1,\ldots,n$. Hence by strict complementarity $z_i > 0$ for $i = 1,\ldots,\bar{n}$, $z_i = 0$ for $i = \bar{n}+1,\ldots,n$, and

$$\nabla G(z) = \begin{bmatrix} -\theta'(z_1) - \theta'(0) & & & & & & 0 \\ & \ddots & & & & & \\ & & -\theta'(z_{\bar{n}}) - \theta'(0) & & & & \\ & & & 0 & & & \\ 0 & & & & \ddots & & \\ & & & & & & 0 \end{bmatrix} \nabla F(z) + \begin{bmatrix} 0 & & & & & 0 \\ & \ddots & & & & \\ & & 0 & & & \\ & & & -\theta'(F_{\bar{n}+1}(z)) - \theta'(0) & & \\ 0 & & & & \ddots & \\ & & & & & -\theta'(F_n(z)) - \theta'(0) \end{bmatrix}$$

The nonsingularity of $\nabla G(z)$ follows from $\theta'(0) + \theta'(\zeta) > 0$ for $\zeta > 0$ and the fact that the principal minor $\frac{\partial F_i(z)}{\partial z_j}$, $i = 1,\ldots,\bar{n}$, $j = 1,\ldots,\bar{n}$, is nonsingular. The argument is similar for the case when $F_i(z) = 0$ for $i \epsilon I \subset \{1,\ldots,n\}$ and $I \neq \{1,\ldots,\bar{n}\}$. QED

The simplest possible realization of (2) is obtained by taking $\theta(z) = z$. This gives

(3) $$|F_i(z) - z_i| - F_i(z) - z_i = 0 \qquad i = 1,\ldots,n$$

Note that the Jacobian of (3) is nonsingular under the assumptions of the Corollary. However equations (3) are only <u>locally</u> differentiable near a solution satisfying strict complementarity. This is because the absolute value function is not differentiable at zero and hence (3) is not differentiable when $F_i(z) - z_i = 0$. Note however that at a solution satisfying the strict complementarity condition, $F_i(z) - z_i$ is equal to either $F_i(z) > 0$ or $-z_i < 0$.

In order for equations (2) to possess <u>global differentiability</u> we require that $\theta'(0) = 0$. The simplest function having this property and which is strictly increasing is $\theta(z) = z|z|$. Equations (2) become

(4) $$(F_i(z) - z_i)^2 - F_i(z)|F_i(z)| - z_i|z_i| = 0 \qquad i = 1,\ldots,n .$$

Note that by the Theorem above, <u>solving either of the systems (3) or (4) is equivalent to solving the complementarity problem (1)</u>. <u>This is true without any assumptions on F</u> . Under the additional assumptions of the Corollary on F, both (3) and (4) have nonsingular Jacobians at certain solution points.

As a consequence of the above Corollary, locally superlinearly convergent algorithms for solving the complementarity problem take the form

$$z^{j+1} - z^{j} = -H^{j} G(z^{j})$$

where H^j may be taken as $\nabla G(z^j)^{-1}$ (Newton method) or an approximation thereof (quasi-Newton methods) [1]. It is hoped that the use of such methods independently or in conjunction with simplicial approximation methods would lead to an improvement in computational efficiency.

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