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ELEMENTARY PROOFS OF ALGEBRAIC RELATIONSHIPS FOR THE EXPONENTIAL AND LOGARITHM FUNCTIONS*

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ABSTRACT

This paper uses elementary algebraic methods to obtain new proofs for theorems on algebraic relationships between the logarithmic and exponential functions. The main result is multivariate version of a special case of the Structure Theorem due to Risch that gives in a very explicit fashion the possible algebraic relationships between the exponential and logarithm functions. In addition there are some more results that give new information about the forms of elementary integrals of elementary functions as well as a new treatment of some algebraic dependence theorems previously discussed by Ostrowski, Kolchin and Ax.

1. Introduction

In this paper the structure of fields of elementary functions that are obtained from the rational functions by the use of rational operations and the use of the logarithm and exponential functions is studied. The main result is the Structure Theorem of section 5 which explicitly gives the form of any possible algebraic relationship among various exponential and logarithm functions.

The Structure Theorem is a multivariate version of a special case of a result due to Risch [26]. Risch's result has been generalized by Ax [1]. Both the proofs of Risch and Ax involve elements of algebraic function theory or algebraic geometry. By eliminating consideration of algebraic cases an important version of the Structure Theorem is obtained herein by using only very elementary algebraic arguments.

The study of the structure of fields of functions has recently been revived by the interest in exact mathematical computation (or symbolic mathematical computation) but the interest in such topics has a long history. Joseph Liouville was the first to write extensively on this subject as he did in the period 1833-1841 primarily in connection with his work on integration in finite terms. See references [14]-[20] for his results.

The study of the structure of the elementary functions has been a fundamental part of the work on integration in finite terms. Hardy [9], Ritt [29], Ostrowski [23], Risch [27,28] and Rosenlicht [31,32] have written about elementary functions as a part of their work on integration. The results of Ostrowski have been reformulated and generalized by Kolchin [12]. Kolchin's generalizations of Ostrowski's work is also treated by Ax [1] and in section 6 of this paper another treatment of Ostrowski's theorem and its generalizations is given.

With the recent development of computing systems such as Altran [8], Macsyma [21], Reduce [10], SAC-1 [5] and Scratchpad [7] for exact mathematical computation, the problem of simplification of mathematical expressions has been recognized as fundamental. Brown [2], Caviness [3], Fateman [34], Fitch [35], Johnson [36], Moses [37] and Richardson [24,25] have written explicitly about the simplification problem and have explored both unsolvability aspects of the problem as well as having a suggested partial methods for carrying out simplification. The results of this paper are similar in spirit to those of Brown.

However algorithms for elementary transcendental function arithmetic based on the results of this paper and the generalizations found in the previously mentioned work of Risch and Ax provide the most powerful methods currently known for dealing with the simplification problem for classes of expressions involving the elementary functions, i.e., algebraic, trigonometric, inverse trigonometric, hyperbolic, inverse hyperbolic, logarithmic and exponentials functions.

Epstein [6] has developed algorithms in the SAC-1 system for performing arithmetic in certain subfields (To be precise, in regular Liouville extension fields of $Q(i,z_1,\ldots,z_n)$ where Q denotes the field of rational numbers. See section 4 for a definition of Liouville extension fields.) of the field of elementary transcendental functions. The Structure Theorem is used to find canonical representations for the functions and hence to solve the simplification problem. The work of Epstein is much like that suggested and anticipated by Moses [22]. Some of the algorithms of Fateman [34] use similar ideas but the full power of the Structure Theorem is apparently not employed.

The results of this paper are obtained within the framework of differential algebra. In section 2 standard definitions and concepts from differential algebra are presented. Although this paper is mostly self-contained, readers desiring more information about differential algebra may consult Kaplansky [11], Kolchin [13] or Ritt [30]. Some of the definitions of section 2 are from Risch [27]. Readers need only be familiar with elementary algebraic material on polynomials, fields, finite extensions of fields, and homomorphisms to read this paper.

Presented in section 3 are the basic results upon which the proofs of the remaining sections are based. Most of these results for the univariate case occur at least implicitly in Risch [27] and Rosenlicht [32]. An exception is theorem 3.7 which, although suggested by the work of Risch, is new and interesting in its own right. This theorem gives rather complete information about the form of integrals of certain special integrands composed of exponentials and logarithms and may have useful applications in algorithms for integration in finite terms. Results for differential rings that are analogous to some of the results of section 3 are given by Caviness and Rothstein [4].

The concept of a well-structured field is introduced in section 4. It is particularly easy to prove a structure theorem for well-structured fields as is done in section 4. The key to the proof of such a theorem is very explicit knowledge about the form of integrals in well-structured fields. The necessary information about such integrals is given in the Structured Liouville Theorem, i.e., theorem 4.1, and corollary 4.2.

The primary purpose of the results of section 4 is to facilitate the proof of the Structure Theorem for regular Liouville extension fields in section 5. In section 5 it is shown that every regular Liouville extension field is differentially isomorphic to a well-structured field. Through this isomorphism and the structure theorem for well-structured fields, a structure theorem is obtained for the more general regular Liouville extension fields.

As was mentioned earlier section 6 contains new proofs of Ostrowski's theorem [23] and its generalizations due to Kolchin [12]. Kolchin's proofs utilize a theorem on algebraic groups whereas our proofs are simple induction proofs based on the lemmas of section 3. Theorem 6.4 is also proven by Ax [1, section 4] by algebraic geometry methods. Section 6 is not an integral part of the paper but is presented to show how the lemmas of section 3 can lead to elementary proofs of other well-known results.

2. Differential Fields

Let R be a commutative ring with unity and let $D_1, \ldots, D_n$ be mappings from R into R such that for any a,b in R and $1 \leq j, k \leq n$

$$D_j(a+b) = D_j(a) + D_j(b) \tag{2.1}$$

$$D_j(ab) = aD_j(b) + bD_j(a) \tag{2.2}$$

and

$$D_j(D_k(a)) = D_k(D_j(a)) . \tag{2.3}$$

R is called a partial (ordinary) differential ring if $n > 1$ $(n=1)$ with derivation operators $D_1, \ldots, D_n$. If F is a field satisfying (2.1), (2.2) and (2.3), F is called a differential field.

Let F_1 and F_2 be fields such that every element of F_1 is an element of F_2 . Then F_2 is called an extension field of F_1 and F_1 is called a subfield of F_2 . Let F_1 be a subfield of the differential field F_2 . If for each derivation operator D of F_2 and for each a in F_1 , Da is in F_1 than the restriction of D to F_1 is a derivation operator for F_1 and F_1 is a differential subfield of F_2 . The derivation operators for F_1 are the derivation operators of F_2 restricted to F_1 . Also, in this case F_2 is called a differential extension field of F_1 .

Let F_1 be a subfield of F_2 and S a collection of elements in F_2 . There are fields, such as F_2 , which contain both F_1 and S . The intersection of all such fields is called the *adjunction* of S to F_1 and is denoted by $F_1(S)$. Clearly, $F_1(S)$ is the smallest field containing both F_1 and S. When $S = \{\theta_1, \ldots, \theta_n\}$ is a finite set, $F_1(\theta_1, \ldots, \theta_n)$ may be written for $F_1(S)$. If θ is an element of F_2 such that $\sum_{j=0}^{n} a_j \theta^j = 0$ for some choice of a_j in F_1, θ is said to be *algebraic* over F_1.

In [33] it is proven that if θ is algebraic over a field F_1, there is a unique monic polynomial $P(x)$ in $F_1[x]$ having minimum degree such that $P(\theta) = 0$. This polynomial is called the *minimum polynomial* of θ. Van der Waerden also proves that if θ_1 is algebraic over F_1 and θ_2 is algebraic over $F_1(\theta_1)$, θ_2 is algebraic over F_1.

If there is a polynomial $P(x_1, \ldots, x_n)$ in $F_1[x_1, \ldots, x_n]$ such that $P(\theta_1, \ldots, \theta_n) = 0$, $\theta_1, \ldots, \theta_n$ are said to be *algebraically dependent* over F_1. Otherwise, $\theta_1, \ldots, \theta_n$ are said to be *algebraically independent*. If there is no polynomial $P(x_1)$ in $F_1[x_1]$ such that $P(\theta_1) = 0$, θ_1 is said to be *transcendental* over F_1. If every element in F_2 is algebraic over F_1, F_2 is said to be an *algebraic extension* of F_1. If $F_2 = F_1(\theta_1, \ldots, \theta_m)$ for some choice of $\theta_1, \ldots, \theta_m$, then F_2 is said to be *finitely generated* over F_1.

Let F_1 and F_2 be two fields and let σ be a mapping from F_1 into F_2. If for every a, b, c in F_1

$$\sigma(a(b+c)) = \sigma(a)(\sigma(b) + \sigma(c)) \tag{2.4}$$

σ is said to be a homomorphism from F_1 to F_2 . If the only element, a of F_1 such that $\sigma(a) = 0$ is 0 and each element of F_2 is the image of an element of F_1 under σ, then σ is called an isomorphism.

Suppose F_1 and F_2 are differential fields with n derivation operators $D_1, \ldots, D_n$ and $D_1^*, \ldots, D_n^*$ respectively. Let π be a permutation of $1, 2, \ldots, n$ and let σ be a homomorphism (isomorphism) from F_1 to F_2 such that $\sigma(D_j(a)) = D^*_{\pi(j)}(\sigma(a))$ for each D_j . Then σ is called a differential homomorphism (isomorphism).

Let F be a field and suppose there is a positive integer p such that $pa = 0$ for every a in F . The smallest such integer is called the characteristic of F . If there is no such integer, F is said to have characteristic 0 . Henceforth it is assumed that all fields have characteristic 0 .

Let F be a differential field and D a derivation operator on F . Then $D(1) = D(1 \cdot 1) = D(1) + D(1)$. Therefore $D(1) = 0$. So for any $a \neq 0$ in F , $0 = D(1) = D(a/a) = aD(1/a) + 1/a\, D(a)$. Thus $D(1/a) = -1/a^2\, D(a)$. Using induction on m it follows easily that for any $a \neq 0$ in F and any integer m

$$D(a^m) = m\, a^{m-1} D(a) . \tag{2.5}$$

Combining (2.2) and (2.5) yields

$$D(a/b) = (bD(a) - aD(b))/b^2 . \qquad (2.6)$$

Suppose F is a differential field with derivation operators $D_1, \ldots, D_n$. If $D_j(a) = 0$ and $a \neq 0$, it follows from (2.5) that $D_j(a^{-1}) = D_j(1/a) = 0$. Together with (2.1) and (2.2) this implies that $C_j^{(F)} = \{c \text{ in } F\colon D_j(c) = 0\}$ is a differential subfield of F called the field of constants with respect to the j^{th} variable. By taking the intersection $C^{(F)} = \bigcap_{j=1}^{n} C_j^{(F)}$ we obtain the constant field of F. $C_j^{(F)}$ is a differential extension field of $C^{(F)}$. When no confusion arises in doing so, C will often be written in place of $C^{(F)}$ and C_j will be written for $C_j^{(F)}$. Since C contains 1, and F has characteristic 0, C contains a field isomorphic to the field of rational numbers.

For the remainder of this chapter suppose that F is a differential field with n derivation operators $D_1, \ldots, D_n$. The gradient operator, ∇, is defined by

$$\nabla a = (D_1 a, \ldots, D_n a) \qquad (2.7)$$

for a in F. $\nabla a = (0,0,\ldots,0)$ will be abbreviated $\nabla a = 0$. It is clear that $C = \{a \text{ in } F : \nabla a = 0\}$. From equations (2.1), and (2.2) it follows that for any a,b in F and c in C

$$\nabla(c(a+b)) = c\nabla a + c\nabla b . \qquad (2.8)$$

Thus $c\nabla a = \nabla b$ if and only if $ca = b+k$ where k is in C since $\nabla(ca-b) = 0$.

Let U be a differential extension field of F with the property that any finitely generated differential extension of F is differentially isomorphic to a subfield of U . Then U is said to be a universal extension of F . In [13, page 92], Kolchin proves that every differential field of characteristic zero has a universal extension field. Note that what we call a universal extension is called a semi-universal extension by Kolchin. A field F will be identified with its isomorphic image in U .

Let θ be an element of U such that each $D_j\theta$ is in F . Then θ is said to be primitive over F . If there is an non-zero f in F such that

$$f\nabla\theta = \nabla f \tag{2.9}$$

then θ is called a logarithm over F , written $\theta = \log f$. If there is an f in F such that

$$\nabla\theta = \theta\nabla f \tag{2.10}$$

θ is said to be exponential over F , written $\theta = \exp f$.

Observe that θ in U cannot be primitive, exponential and transcendental over F unless θ is a constant. For if θ is exponential, there is an f in F such that (2.9) holds. If $\nabla f \neq 0$, there is a $D_j f \neq 0$. Therefore $\theta = D_j\theta / D_j f$ which is in F since θ is primitive. Thus, either $\nabla f = 0$ or θ is in F . If $\nabla f = 0$, $\nabla\theta = 0$ and thus θ is a constant.

A <u>monomial</u> over F is an element of U which satisfies the following properties.

(1) θ is transcendental over F ;

(2) θ is primitive or exponential over F .

If also $C^{(F)} = C^{(F(\theta))}$, θ is said to be a <u>regular monomial.</u> If θ in U is primitive or exponential over F , then θ is not a regular monomial over F precisely when there is a c in $C^{(F(\theta))}$ such that θ is algebraic over F(c) . Certainly, if such a c exists θ is not a regular monomial. Conversely, if θ is not a regular monomial either θ is algebraic, in which case $c = 1$, or θ is transcendental over F but there is a c in $C^{(F(\theta))}$ not in $C^{(F)}$. However, c in $F(\theta)$ may be written $P(\theta)/Q(\theta)$ where $P(\theta)$, $Q(\theta)$ are relatively prime polynomials in $F[\theta]$. Then $P(\theta) - cQ(\theta) = 0$ so that θ is algebraic over F(c) .

This section is concluded with the following lemma.

<u>Lemma 2.1</u> Suppose f,g are non-zero elements of U which are exponential over the differential field F . Let m be a non-zero integer. Then, f/g^m is a constant in $C^{(U)}$ if and only if

$$\frac{1}{mf}\nabla f = \frac{1}{g}\nabla g \qquad (2.11)$$

Furthermore, when (2.11) holds, f is not a regular monomial over $F(g)$.

Proof. f/g^m is a constant if and only if $\nabla(f/g^m) = 0$. But $\nabla(f/g^m) = - mfg^{-m-1}\nabla g + g^{-m}\nabla f = mfg^{-m}(\frac{\nabla g}{-g} + \frac{1}{mf}\nabla f)$. Thus, $\nabla(f/g^m) = 0$ precisely when (2.11) holds since $m \neq 0$ and f and g are non-zero. It is clear that f is not a regular monomial over $F(g)$ when (2.11) holds, since f/g^m is a constant. ∎

3. Basic Lemmas

Throughout this section θ shall be a regular monomial over the differential field F . Δ will represent the set of derivation operators on F . Elements of $F(\theta)$ will be considered quotients of elements in $F[\theta]$, the ring of polynomials in θ with coefficients in F . Thus f in $F(\theta)$ may be written in the form P/Q where P,Q are relatively prime elements of $F[\theta]$ with Q monic. For P in $F[\theta]$, the <u>degree of ∇P</u> in θ , written $\deg_\theta(\nabla P)$, is maximum $\{\deg_\theta D(P)\colon\ D \text{ in } \Delta\}$. P in $F[\theta]$ is called <u>square-free</u> if there does not exist Q in $F[\theta]$ with $\deg_\theta Q > 0$ such that $Q^2 | P$.

The next two lemmas are multivariate versions of results appearing in [27] .

<u>Lemma 3.1</u> Suppose $P(\theta) = \sum_{j=0}^{k} p_j\, \theta^j$ is a polynomial of degree k in $F[\theta]$. If θ is primitive over F , $k-1 \leq \deg_\theta(\nabla P) \leq k$. Furthermore, $\deg_\theta \nabla P = k-1$ if and only if p_k is in C , the constant field of F .

Proof. $\nabla P = \sum_{j=0}^{k} [\nabla p_j\, \theta^j + j p_j\, \theta^{j-1} \nabla\theta]$. If $\deg_\theta(\nabla P) < k$, ∇p_k must be zero and by definition p_k is in C . If $\nabla p_k = 0$, then $\nabla p_{k-1} + k\, p_k \nabla\theta$ cannot be zero. For if it were, then $\nabla p_{k-1} + k\, p_k \nabla\theta = \nabla(p_{k-1} + k\, p_k\, \theta) = 0$ which implies that θ is not a regular monomial by the discussion following the definition of regular monomial. ∎

<u>Lemma 3.2</u> If $P(\theta) = \sum_{j=0}^{k} p_j \theta^j$ is a polynomial of degree k in $F[\theta]$ where θ is exponential over F, $\deg_\theta \nabla P = k$.

Proof. Since θ is exponential over F, there is an f in F such the $\nabla\theta = \theta\nabla f$. Then $\nabla P = \sum_{j=0}^{k} (\nabla p_j + j\, p_j \nabla f)\,\theta^j$. If $\nabla p_k + k p_k \nabla f = 0$, lemma 2.1 implies that θ is not a regular monomial over F which is contrary to the hypotheses. Thus $\deg_\theta(\nabla P) = k$. $\blacksquare$

<u>Lemma 3.3</u> Suppose $P(\theta) = \sum_{j=0}^{k} p_j \theta^j$ is a polynomial in $F[\theta]$ of degree $k > 0$. If, for each D in Δ, $P \mid DP$ (as polynomials in $F[\theta]$), then θ is an exponential monomial and $P = p\theta^k$ where p is in F.

Proof. If for each D in Δ, $P \mid DP$, then for each D there is an f in F such that $DP = fP$. Suppose θ is primitive. Since $k = \deg_\theta P = \deg_\theta(fP) = \deg_\theta(DP)$, $Dp_k \neq 0$. Equating the leading coefficient of DP with the leading coefficient of fP yields $Dp_k = fp_k$. Thus $p_k DP - PDp_k = 0$ and $\nabla(P/p_k) = 0$. Therefore, $P = c\,p_k$ for some c in $C^{(F(\theta))}$ contradicting the fact that θ is a regular monomial over F. Thus θ cannot be primitive.

Assume θ is exponential and, therefore, there is a g in F such that $\nabla\theta = \theta\nabla g$. The coefficient of the term of degree j in DP then becomes $Dp_j + jp_j Dg = fp_j$. Thus, for each D in Δ, $p_k(Dp_j + jp_j Dg) = p_k(f\,p_j) = (f\,p_k)\,p_j = p_j(Dp_k + kp_k Dg)$. Then, if $p_j \neq 0$, $\nabla(p_k/p_j) = (j-k)(p_k/p_j)\nabla g$ which implies that $\theta = (j-k)(p_k/p_j)$ for any $j < k$. Thus, if there is a $j < k$ with $p_j \neq 0$, lemma 2.1 implies that θ is not a regular monomial. ∎

Lemma 3.4 Let P, Q be relatively prime elements of $F[\theta]$ with $\deg_\theta Q > 0$. Suppose for each derivation operator, D, $D(P/Q) = A/B$ where A, B are relatively prime elements of $F[\theta]$. Then each B is square-free if and only if θ is exponential over F and $Q = q\theta$ where q is in F.

Proof. It is easily seen that if θ is exponential and $Q = q\theta$, $D(P/Q)$ can be written in the form A/B with A, B relatively prime elements of $F[\theta]$ and B square-free.

Conversely, suppose $D(P/Q) = (Q\,D\,P - P\,D\,Q)/Q^2 = A/B$. Then $B(Q\,D\,P - P\,D\,Q) = Q^2 A$. If B is square-free, $Q \mid (Q\,D\,P - P\,D\,Q)$. Since $Q \mid Q\,D\,P$ this implies $Q \mid P\,D\,Q$. Since P and Q are relatively prime (polynomials in $F[\theta]$), $Q \mid D\,Q$. By lemma 3.3 $Q = q\theta^k$ where q is in F and θ is exponential.

It remains only to show that $k = 1$. Suppose $k > 1$ and $P = \sum_{j=0}^{m} p_j \theta^j$ where p_j is in F. Then for some f in F, $\nabla\theta = \theta\nabla f$ so $A/B = D(P/q\theta^k)$ $= (D(P/q) - k\,P/q\,D\,f)/\theta^k$ for each D in Δ. Thus $\theta \mid (D(P/q) - k\,P/q\,D\,f)$ since $k > 1$ and θ^2 does not divide B. So the trailing coefficient of $D(P/q) - k\,P/q\,D\,f$ is zero. Therefore, $\nabla(p_o/q) - k(p_o/q)\nabla f = 0$. So by lemma 2.1 θ is not a regular monomial. This contradiction shows that $k = 1$. $\blacksquare$

Lemma 3.5 Let $R(\theta) \neq 0$ be in $F(\theta)$. If $D(R)/R = A/B$ where A and B are relatively prime elements of $F[\theta]$, B is square-free.

Proof. If R is in F , the lemma is clearly true. Otherwise, write $R = f \prod_{j=1}^{m} R_j^{e_j}$ where f is in F , the e_j are non-zero integers and the R_j are distinct, monic, irreducible members of $F[\theta]$. Then for each D in Δ

$$DR/R = [Df \prod_{j=1}^{m} R_j + \sum_{k=1}^{m} e_k f D R_k \prod_{j \neq k} R_j]/(f \prod_{j=1}^{m} R_j) \ . \qquad (3.1)$$

from which the lemma follows. $\blacksquare$

Lemma 3.6 Suppose S is in U and for each D in Δ, $DS = P/Q$ where P and Q are relatively prime elements of $F[\theta]$ with Q square-free. Suppose further that

$$DS = \sum_{j=1}^{m} c_j D R_j / R_j + DT \qquad (3.2)$$

where T , $R_1, \ldots, R_m$ are in $F(\theta)$ and each c_j is in C . Then T is in $F[\theta]$ except possibly when θ is exponential in which case θT is in $F[\theta]$.

Proof. By lemma 3.5 each $c_j D R_j/R_j$ can be written as A_j/B_j with A_j, B_j relatively prime in $F[\theta]$ and B_j square-free. Each term in equation (3.2), with the possible exception of DT, can be written as the quotient of relatively prime polynomials in $F[\theta]$ with a square-free denominator. Therefore there is a square-free polynomial L in $F[\theta]$ such that

$$L(DS - \sum_{j=1}^{m} A_j/B_j) = LDT \tag{3.3}$$

is in $F[\theta]$. When DT is written as a quotient P_T/Q_T of relatively prime polynomials in $F[\theta]$, Q_T divides L and thus is square-free. It follows from lemma 3.4 that T is in $F[\theta]$ or θ is exponential and θT is in $F[\theta]$. ∎

By strengthening the hypotheses of this lemma more can be said about T as is done in the following theorem. The above lemma with the weaker hypotheses is needed in some of the proofs of section 4.

Theorem 3.7 Suppose S is in U and for each D in Δ, $DS = P/Q$ where P and Q are relatively prime elements of $F[\theta]$ with Q square-free and $0 \leq \deg_\theta P < \deg_\theta Q$. Suppose further that

$$DS = \sum_{j=1}^{m} c_j DR_j/R_j + DT \tag{3.4}$$

where $T, R_1, \ldots, R_m$ are in $F(\theta)$ and each c_j is in C. Then if θ is primitive, T is either in C or $DT = \sum_{j=1}^{n} k_j Df_j/f_j$ where each f_j is in F and each k_j is in C . If θ is exponential, θT is in $F[\theta]$ and furthermore T in $F[\theta]$ implies that T is actually in F .

Proof. Proceeding as in the proof of lemma 3.6, it can be shown that T must be in $F[\theta]$ or θ must be exponential and θT must be in $F[\theta]$. If θ is exponential and T is in $F[\theta]$ then either $T = 0$ or $\deg_\theta T = 0$ since otherwise $\deg_\theta DT > 0$ and equation (3.3) implies that $\deg_\theta L(DS - \sum_{j=1}^{m} A_j/B_j) \leq \deg_\theta L < \deg_\theta LDT$ which is impossible. Thus the desired result is established for the exponential case.

If θ is primitive as in the proof of lemma 3.5 express each R_i as $f_i \Pi (\bar{R}_j)^{e_j}$ where f_i is in F , each e_j is a non-zero integer and each $\bar{R}_j$ is a monic irreducible member of $F[\theta]$. Then $DR_i/R_i = Df_i/f_i + \sum e_j D\bar{R}_j/\bar{R}_j$. Equation (3.4) becomes

$$P/Q = \sum k_j D\bar{R}_j/\bar{R}_j + \sum c_i Df_i/f_i + DT$$

Let $L = Q\Pi\bar{R}_j$. Then $L(P/Q - \sum k_j D\bar{R}_j/\bar{R}_j) = L(\sum c_i Df_i/f_i + DT)$. But the degree in θ of the left-hand side of this equation is less than the degree of L since each $\bar{R}_j$ is monic and the degree of the right-hand side is greater than or equal to L . This cannot be unless $P/Q - \sum k_j D\bar{R}_j/\bar{R}_j = 0$ and $DT + \sum c_i Df_i/f_i = 0$ which establishes the theorem. ∎

Note that if P/Q is a non-zero member of F the conclusions and the proof of the theorem still hold.

This theorem is a generalization of the well-known result [9] that $DT = 0$ when P, Q are in the ordinary differential field $F[z]$, F a field of constants and $Dz = 1$. It follows from the above proof that when θ is primitive and P/Q has an integral of the form (3.2) that we may take $DT = 0$.

Furthermore if the constant field of F is algebraically closed and P/Q has an elementary integral, it follows from Liouville's theorem [27] that the integral must have the form given in equation (3.2). The proof of the theorem shows that if θ is primitive $\int P/Q$ can always be expressed in the form $\sum k_j \log \bar{R}_j$ where each k_j is in the constant field of F and each $\bar{R}_j$ is a monic irreducible element of $F[\theta]$.

If θ is exponential, T can be more complicated as the following examples show.

A very simple example suffices to show that T may actually be a rational function in θ . Let $F = Q(z)$, the field of rational functions with rational number coefficients. Let $\theta = e^z$. In equation (3.4) take $DS = 1/\theta$, $m = 0$ and $T = -1/\theta$.

Theorem 3.7 says that if T is not actually a rational function in θ then in fact T must be in F . We want to show with an example that in this case T does not have to be in any subfield of F such as the field of constants of F . Let F_1 be a field of one variable z and suppose $F_1 = F_2(\psi)$ where F_2 is some subfield of F_1 and ψ is such that $\theta = e^{\psi}$ is a regular monomial over F_1 . For example F_2 could be Q and ψ could be z or F_2 could be $Q(z)$ and ψ could be e^z . We want to show that S can be chosen such that T will involve ψ . This will be the case if in equation (3.4), $DS = \psi'/(\theta+1)$, $m = 1$, $R_1 = \theta + 1$, $c_1 = -1$ and then $T = \psi$. (ψ' is the derivative of ψ with respect to z .)

The following lemma is a multivariate generalization of Proposition 1.2 of [27].

Lemma 3.8 Let ψ be an element of U the universal extension field of F . Suppose ψ is not a regular monomial over F .

(a) If ψ is primitive over F , then $\psi = g + c_1$, where g is in F and c_1 is in $C^{(U)}$.

(b) If ψ is exponential over F , then $\psi^m = c_2 h$, where m is a non-zero rational integer, h is in F and c_2 is in $C^{(U)}$.

Proof. If $\nabla\psi = 0$, the result is obvious so assume $\nabla\psi \neq 0$.

Since ψ is not a regular monomial over F there is a constant k in $C^{(F(\psi))}$ such that ψ is algebraic over $F(k)$. Let $P(x) = \sum_{j=0}^{m} p_j x^j$ be the monic polynomial of minimal degree m with coefficients in $F(k)$

such that $P(\psi) = 0$. If k is algebraic over F , then ψ is algebraic over F , since ψ is algebraic over $F(k)$. In this case one may assume $k = 1$ and then $F(k) = F$. Since $p_m = 1$, for each D in Δ ,

$$D(P(\psi)) = \sum_{j=0}^{m-1} D(p_j)\psi^j + \sum_{j=0}^{m-1} jp_j D(\psi)\psi^{j-1} = 0 \qquad (3.5)$$

Consider the case in which ψ is primitive. $D(\psi)$ is in $F \subset F(k)$ and so $D(P(\psi))$ is a polynomial of degree $< m$ with ψ as a root. This would contradict the minimality of m unless each coefficient of $DP(\psi)$ is 0 . Therefore, $D(p_{m-1}) + mD(\psi) = 0$. So $\nabla(p_{m-1}/m + \psi) = 0$ and $\psi = -p_{m-1/m} + c$ for some c in $C^{(U)}$. If k is algebraic the lemma is proved since p_{m-1} is in F . Otherwise, $-p_{m-1/m}$ can be written A/B with A,B relatively prime and B a monic polynomial in $F[k]$. Hence $\psi = A/B + c$ which implies that $(BDA - ADB)/B^2$ is in F . Thus $B|DB$ in $F[k]$. Since B is monic, $\deg_k B > \deg_k DB$ unless $\deg_k B = 0$. Thus B must be 1 . So $\psi = A + c$.

Suppose $A = \sum_{j=0}^{\mu} a_j k^j$ with the a_j in F . Then for each D , $D\psi = DA = \sum_{j=0}^{\mu} (Da_j)k^j$ since $Dk = 0$. But $D\psi$ is in F and k is transcendental over F . Thus $Da_j = 0$ for $j = 1, \ldots, \mu$. Therefore $A = a_o + \sum_{j=1}^{\mu} a_j k^j$ with $\sum_{j=1}^{\mu} a_j k^j$ in $C^{(U)}$. Taking $g = a_o$ and $c_1 = c + \sum_{j=1}^{\mu} a_j k^j$ proves the lemma for this case.

When ψ is exponential, $\nabla\psi = \psi\nabla f$ for some f in F . Thus equation (3.5) becomes

$$D(P(\psi)) = \sum_{j=0}^{m} (D\, p_j + j\, p_j\, Df)\psi^j = 0$$

Let D be such that $Df \neq 0$. Divide $D(P(\psi))$ by mDf to yield a monic polynomial of degree m with ψ as a root. Therefore the trailing coefficients of $P(\psi)$ and $\frac{1}{mDf}\, D(P(\psi))$ must be equal. Thus $p_o = Dp_o/mDf$. So $Dp_o/p_o = D\psi/m\psi$ whenever $Df \neq 0$. When $Df = 0$, $D(P(\psi))$ can be written as a polynomial in $F(k)[\psi]$ of degree < m . Since $D(P(\psi)) = 0$, this would deny the minimality of m unless each coefficient is 0 . In particular, $D\,p_o + 0\,p_o\,D\,f = D\,p_o = 0$. Thus lemma (2.1) implies $\psi^{-m} = c_3\, p_o$ for some c_3 in $C^{(U)}$. If k is algebraic over F this completes the proof.

Otherwise p_o is in $F(k)$, and may be written A/B with A,B relatively prime elements in $F[k]$ with B monic. Thus $\psi^{-m}D(\psi^m) = -\,m\,D\,f = (B\,D\,A - A\,D\,B)/AB$ for each D . But Df is in F , so $B \mid DB$. As in the primitive case, this implies $B = 1$. Thus $-\,m\,D\,f = D\,A/A$. If $A = \sum_{j=0}^{\mu} a_j k^j$ with a_j in F , equating the leading coefficients of $-\,m\,A\,D\,f$ and DA yields $-mD(f)\,a_\mu = D(a_\mu)$. From lemma (2.1) it then follows that $\psi^{-m} = c_2\, a_\mu$ for some c_2 in $C^{(U)}$. ∎

4. Well-Structured Fields

Let F_o be a differential field with universal extension field U. For $\theta_1, \theta_2, \ldots, \theta_m$ in U let $F_j = F_o(\theta_1, \ldots, \theta_j)$, $1 \leq j \leq m$. If each θ_j is a monomial over F_{j-1} then F_m is what Kaplansky [11, p. 24] calls a Liouville extension of F_o. Kolchin [13, p. 408] calls such extensions Liouvillian extensions of type (2). If each θ_j is also regular over F_{j-1}, F_m will be called a regular Liouville extension of F_o.

For the remainder of this section the following assumptions will be made: F_o will denote the field $K(z_1, \ldots, z_n)$ of rational functions in the variables $z_1, \ldots, z_n$ with coefficients in K and K_o will denote $K[z_1, \ldots, z_n]$, the ring of multivariable polynomials over K, where K is a subfield of the complex numbers. For $\ell = 1, 2, \ldots, n$ let D_ℓ denote the derivation operator with the properties that $D_\ell z_j = \delta_{\ell j}$, the Kronecker delta, and $D_\ell k = 0$ for all k in K. Then F_o is the usual field of rational functions in n variables over the constant field K with the usual partial differentiation operators $D_1, \ldots, D_n$. Denote the set $\{D_1, \ldots, D_n\}$ by Δ.

Observe that each z_ℓ may be viewed as a regular monomial which is primitive over any subfield of $K(z_1, \ldots, z_{\ell-1}, z_{\ell+1}, \ldots, z_n)$.

Let F_m be a regular Liouville extension of F_o, $F_m = F_o(\theta_1, \ldots, \theta_m)$. Let K_m denote the polynomial ring $K[z_1, \ldots, z_n, \theta_1, \ldots, \theta_m]$. F_m is the quotient field of K_m and any element of F_m can be written as a quotient of polynomials in K_m . Suppose $R = P/Q$ is so written. A polynomial $\bar{P}$ in K_m is said to be a <u>factor</u> of the rational function R if $\bar{P}|P$ or $\bar{P}|Q$. In particular R is said to have an <u>exponential factor</u> if some exponential θ_j divides R . If the only common factors of R_1 and R_2 in F_m are elements of K then R_1 and R_2 are said to have no common factors.

Before defining well-structured fields, the concepts of index and distinguishing factors must be introduced. If R is a member of a Liouville extension F_m of F, the <u>index</u> of R , written index (R) , is the smallest j such that R is in F_j .

Suppose that $\theta_k = \log A_k$. Let f_k be a factor of A_k with the following properties:

(1) index (f_k) = index (A_k);

(2) f_k is a non-constant irreducible polynomial in K_m;

(3) f_k has no exponential factors;

and (4) if $\theta_j = \log A_j$ and $j \neq k$, f_k and A_j have no common factors.

Then f_k is called a <u>distinguishing factor</u> for θ_k . A regular monomial θ_j is said to be well-structured provided that it is exponential or $\theta_j = \log A_j$ has the properties:

(1) θ_j has a distinguishing factor;

and (2) A_j has no exponential factors.

If each θ_j is well-structured, F_m is said to be <u>well-structured</u>.

In the case of well-structured fields, the requirement that distinguishing factors be irreducible can be omitted. For, if some f in K_m lacked only irreducibility in order to be a distinguishing factor, any irreducible factor g of f with index(g) = index(f) would be a distinguishing factor. Thus a set of irreducible distinguishing factors could always be obtained from the non-irreducible factors.

The following notation will prove helpful.

Let $L(j,k) = \{\ell : 1 \leq \ell \leq k\ ,\ \theta_\ell = \log A_\ell \text{ and } \operatorname{index}(A_\ell) \leq j\}$.

Let $E(k) = \{j : 1 \leq j \leq k \text{ and } \theta_j = \exp(A_j)\}$. Thus $L(j,m)$ is an index set for all the logarithmic monomials whose arguments have index $\leq j$. $E(m)$ is an index set for the exponential monomials.

<u>Theorem 4.1 (Structured Liouville Theorem)</u> Suppose F_m is well-structured and f is in F_m . Suppose for each derivation operator D in Δ, index(Df) $\leq$ k . Then

$$f = \sum_{j \in L(k,m)} c_j \theta_j + R_k \tag{4.1}$$

where R_k is in F_k and c_j is in K .

Proof. The proof will be by induction on m-k . When m-k = 0 , letting each $c_j = 0$ and $R_k = f$ gives the desired result. Suppose inductively that the theorem is true whenever $0 \leq m - k < m$. Then since index(Df) $\leq$ $k < k + 1$

$$f = \sum_{j \in L(k+1,m)} c_j \theta_j + R_{k+1} \qquad (4.2)$$

where R_{k+1} is in F_{k+1} , and each c_j is in K . Since F_m is well-structured, for each j in $L(k+1,m)$ there is a distinguishing factor f_j such that $A_j = f_j^{e_j} \bar{A}_j$ where f_j and $\bar{A}_j$ have no common factors and e_j is a non-zero integer. Thus

$$Df = \sum_{j \in L(k+1,m)} c_j e_j Df_j/f_j + \sum_{j \in L(k+1,m)} c_j D \bar{A}_j/\bar{A}_j + DR_{k+1} \qquad (4.3)$$

Each term in equation (4.3) can be written as a quotient of two relatively prime polynomials in K_{k+1} . Let each term in (4.3) be represented as a quotient of two such polynomials in K_{k+1} and let Q be the product of the denominators. Suppose index(A_i) = $k+1$ and $c_i \neq 0$. Then because f_i is a distinguishing factor for θ_i , lemma 3.3 implies that f_i divides the denominator of $c_i e_i Df_i/f_i$ and thus $f_i | Q$ in K_{k+1} .

Since index$(A_i) = k+1$, index$(f_i) = k+1$. For each term in either sum on the right of equation (4.3), each factor of the denominator having index $k+1$ must be a factor of some f_j or $\bar{A}_j$. Since f_i is relatively prime to f_j when $i \neq j$ and f_i is relatively prime to each A_j , f_i is not a factor of the denominator of any of the terms in the sums other than $c_i e_i Df_i/f_i$. Furthermore, since Df is in F_k lemma 3.6 implies that f_i is not a factor of the denominator of DR_{k+1} . Thus $f_i | Q$ but f_i^2 does not divide Q .

Multiply equation (4.3) by Q to obtain

$$QDf = \sum_{j\in L(k+1,m)} c_j e_j QDf_j/f_j + \sum_{j\in L(k+1,m)} c_j QD\bar{A}_j/\bar{A}_j + QDR_{k+1} . \quad (4.4)$$

Since f_i is not a factor of the denominator of any term in equation (4.3) other than $c_i e_i Df_i/f_i$, f_i divides each term in equation (4.4) except possibly the term $c_i e_i QDf_i/f_i$. Thus f_i must divide this polynomial also. Since f_i is not a factor of Df_i nor of Q/f_i , it must be that $c_i = 0$.

This shows that whenever $c_j \neq 0$ in equation (4.2) it must be the case that $\theta_j = \log A_j$ and $\text{index}(A_j) \leq k$. Thus each term in equation (4.3) except possibly DR_{k+1} has index $\leq k$. Therefore $\text{index}(DR_{k+1}) \leq k$. By theorem 3.7 either R_{k+1} is in F_k or $\theta_{k+1} R_{k+1} = \bar{R}_{k+1}$ is in $F[\theta_{k+1}]$ and θ_{k+1} is not a factor of $\bar{R}_{k+1}$. In the second case differentiate $\bar{R}_{k+1}$ to obtain $D\bar{R}_{k+1} = \theta_{k+1} DR_{k+1} + R_{k+1} D\theta_{k+1}$.

$$\text{Thus } \theta_{k+1} DR_{k+1} = D\bar{R}_{k+1} - R_{k+1} D\theta_{k+1}$$

$$= D\bar{R}_{k+1} - (D\theta_{k+1}/\theta_{k+1}) \bar{R}_{k+1} .$$

Hence θ_{k+1} divides $D\bar{R}_{k+1} - (D\theta_{k+1}/\theta_{k+1}) \bar{R}_{k+1}$, a polynomial in θ_{k+1} over F_k , and hence the trailing coefficient of $\bar{R}_{k+1}$ satisfies the same differential equation as θ_{k+1} . By lemma 2.1 θ_{k+1} is not a regular monomial. This contradiction implies that the second case cannot occur and that R_{k+1}

is in F_k . ∎

By applying lemma 3.6 to equation (4.1) the following corollary is obtained.

Corollary 4.2 Let the notation be as in the preceding theorem. Suppose for each D in Δ there exists P,Q in $F_{k-1}[\theta_k]$ with Q square-free such that $Df = P/Q$. Then R_k is in $F_{k-1}[\theta_k]$ except possibly when θ_k is exponential in which case $\theta_k R_k$ is in $F_{k-1}[\theta_k]$.

The next lemma is actually the statement of the initial part of the induction argument used to prove the main result of this section, theorem 4.5. It is presented separately because it has some intrinsic interest and because the proof of theorem 4.5 is thereby shortened.

Lemma 4.3 Let F_m be a well-structured differential field. Suppose A_o is an element of F_m with index(A_o) = k . Suppose that θ_k is not exponential or is not a factor of A . Let $L(k-1,m) = L(k,m)$. Then if A_o is not a constant, $\log A_o$ is a regular monomial over F_m .

Proof. Suppose $\log A_o$ is not a regular monomial over F_m . Then lemma 3.8 implies there is an R in F_m and a constant c in U such that

$$\log A_o = R + c . \tag{4.5}$$

Then the Structured Liouville Theorem implies that

$$R = \sum_{j \in L(k,m)} c_j \theta_j + R_k$$

where R_k is in F_k since $k \geq \text{index}(DA_o/A_o) = \text{index}(DR)$ for each derivation operator D. Thus

$$DA_o/A_o = DR = \sum_{j \in L(k,m)} c_j DA_j/A_j + DR_k \ . \qquad (4.6)$$

Let $A_o = P/Q$ where P, Q are relatively prime elements of K_k. If $k = 0$, let $\theta_o = z_\ell$ where $\deg_{z_\ell} P > 0$ or $\deg_{z_\ell} Q > 0$. Then substituting P/Q for A_o and multiplying (4.6) by PQ yields

$$(QDP - PDQ) = PQ(\sum_{j \in L(k,m)} c_j DA_j/A_j + DR_k) \ . \qquad (4.7)$$

Since $L(k-1,m) = L(k,m)$, for each j in $L(k,m)$ $\text{index}(A_j) < k$ and lemma 3.6 implies that for $k > 0$ either R_k is in $F_{k-1}[\theta_k]$ or θ_k is exponential and $\theta_k R_k$ is in $F_{k-1}[\theta_k]$ and for $k = 0$ R_k is in $K(z_1, \ldots, z_{\ell-1}, z_{\ell+1}, \ldots, z_m)[z_\ell]$. Since θ_k cannot be an exponential factor of A_o, equation (4.7) implies $P \mid (QDP - PDQ)$ and $Q \mid (QDP - PDQ)$. Since P and Q are relatively prime, this implies that $Q \mid DQ$ and $P \mid DP$. But lemma 3.3 implies that this can happen only if P and Q have exponential factors or are of degree zero in θ_k. Since both P and Q cannot be of degree zero in θ_k, this is a contradiction proving the lemma. ∎

The next theorem is an easy consequence of the preceding lemma and is a generalization of the well-known result [9, p. 14] that says, roughly speaking,that the logarithm of a non-constant rational function is not a rational function.

<u>Theorem 4.4</u> Let F_m be a well-structured differential field. Suppose A is algebraic over F_m, the index(norm(A)) = m, norm(A) is not a constant and $\log A$ is not a regular monomial over $F_m(A)$. Then θ_m is exponential and there is an f in F_m and a positive integer ℓ such that norm(A) $= f\theta_m^{\ell}$.

Proof. Let $A_1, A_2, \ldots, A_k$ denote all the conjugates of A over F_m. Since A is algebraic over F_m and $\log A$ is not a regular monomial there exists $g(A)$ in $F_m[A]$ and some constant c in U such that $\log A =$ $g(A) + c$ which implies $DA/A = Dg(A)$. Applying the trace function to this latter equation yields: trace $(DA/A) = \sum_{j=1}^{k} DA_j/A_j = D(\prod_{j=1}^{k} A_j)/\prod_{j=1}^{n} A_j =$ $D(\text{norm}(A))/\text{norm}(A) = \text{trace } D(g(A)) = D(\text{trace}(g(A)))$. (The fact that D can be extended to $F_m(A)$ and that D commutes with the trace and norm follows from the uniqueness of the extension of D to $F_m(A)$ as is shown in [31].) Hence $\log(\text{norm}(A)) = \text{trace}(g(A)) +$ a constant. Thus by the preceding lemma the theorem follows since $\log(\text{norm}(A))$ is not a regular monomial by lemma 3.8. ■

If the hypothesis "norm(A) is not a constant" is removed theorem 4.4

is a generalization of the result that says that the logarithm of a non-constant algebraic function is not an algebraic function. However if this hypothesis is removed no simple algebraic proof is known for the resulting statement.

A set of functions $\{A_1, A_2, \ldots, A_m\}$ is said to be pseudo-linearly dependent over the rational numbers if there exist rational numbers $r_1, \ldots, r_m$, not all zero, such that $\sum_{i=1}^{m} r_i A_i$ is a constant. If there do not exist such r_i, $\{A_1, \ldots, A_m\}$ is said to be pseudo-linearly independent. $\{A_1, \ldots, A_m\}$ is said to be pseudo-multiplicatively dependent if there exist integers $e_1, \ldots, e_m$, not all zero, such that $\prod_{i=1}^{m} A_i^{e_i}$ is a constant. $\{A_1, \ldots, A_m\}$ is called pseudo-multiplicately independent if it is not pseudo-multiplicatively dependent. It is easy to see that $\{A_1, \ldots, A_m\}$ is pseudo-multiplicatively independent if and only if $\{\log A_1, \ldots, \log A_m\}$ is pseudo-linearly independent.

The next theorem is our primary objective for well-structured fields.

Theorem 4.5 Let F_m be a well-structured differential field. Suppose A_o is an element of F_m with $\text{index}(A_o) = k$ and assume that A_o has no exponential factors. In this case $\log A_o$ is a regular monomial over F_m if and only if $\{\log A_o\} \cup \{\theta_j\}_{j \in L(k,m)}$ is pseudo-linearly independent or equivalently if and only if $\{A_o\} \cup \{A_j\}_{j \in L(k,m)}$ is pseudo-multiplicatively independent where $\theta_j = \log A_j$ for $j \in L(k,m)$.

The proof of this theorem is somewhat long but the basic ideas are quite simple. The hardest part of the proof is to prove the pseudo-linear dependence relationship when $\log A_o$ is not a regular monomial. Before giving a detailed proof, this part of the proof will be sketched.

For this case first employ lemma 3.8 to conclude that $\log A_o$ + a constant is in F_m. Now use lemma 4.2 to get

$$\log A_o + \text{constant} = \sum_{j \in L(k,m)} c_j \theta_j + R_k \qquad (4.8)$$

where each c_j is in K and R_k is in $F_{k-1}[\theta_k]$ except possibly when θ_k is exponential in which case $\theta_k R_k$ must be in $F_{k-1}[\theta_k]$. At this point two things remain to be shown: (1) each c_j is rational and (2) R_k is a constant. To do this we show that if c_ℓ in equation (4.8) is not zero then the distinguishing factor of θ_ℓ must be a factor of A_o. From this fact it follows without much difficulty that c_ℓ must be rational. Knowing that c_ℓ is rational, equation (4.8) may be slightly rewritten with the sum on the right-hand side taken over $L(k,m) - \{\ell\}$ and then by induction on the number of elements in $L(k,m) - L(k-1,m)$ the desired result will follow. Lemma 4.3 is used to get the induction started. Now for the detailed proof.

Proof. (of theorem 4.5). If $\{\log A_o\} \cup \{\theta_j\}_{j \in L(k,m)}$ is pseudo-linearly dependent then $\log A_o$ is obviously not a regular monomial. So assume that $\log A_o$ is not a regular monomial. It must be shown that $\{\log A_o\} \cup \{\theta_j\}_{j \in L(k,m)}$ is pseudo-linearly dependent. The proof is by induction on

t , the cardinality of $L(k,m) - L(k-1,m)$.

When $t = 0$ the desired result follows from lemma 4.4. Suppose the result holds for integers less than $t > 0$. As noted earlier it follows from lemma 3.8 and the Structured Liouville Theorem that equation (4.8) holds. Let ℓ be the largest integer in $L(k,m)$ not in $L(k-1,m)$. Of course, $\ell > k$ since F_m is well-structured and $\text{index}(A_\ell) = k$. If $c_\ell = 0$, form the subfield, F_m^* of F_m obtained by adjoining only those θ_j to F_0 where j satisfies one of the following properties

(1) $j \leq k$

or

(2) j is in $L(k,m)$ and $j \neq \ell$.

Then A_0 is in F_m^* since $F_m^* \supset F_k$. Also, R is in F_m^* since R_k is, as is each θ_j for which $c_j \neq 0$. Thus $\log A_o$ is not a monomial over F_m^* . Since F_m is well-structured it follows that F_m^* is. Applying the induction hypothesis to $\log A_o$ in the field F_m^* now implies the theorem in this case.

If $c_\ell \neq 0$, let f be the distinguishing factor of θ_ℓ so that $A_\ell = f^{p_\ell} \bar{A}_\ell$ where f is not a factor of $\bar{A}_\ell$ and p_ℓ is a non-zero integer. Since f is a distinguishing factor there is a derivation operator D such that $Df \neq 0$. Substitute $f^{p_\ell} \bar{A}_\ell$ for A_ℓ in (4.8) and apply D to obtain

$$D\,A_o/A_o = \sum_{\substack{j \in L(k,m) \\ j \neq \ell}} c_j\, DA_j/A_j + c_\ell\, p_\ell\, Df/f + c_\ell\, D\bar{A}_\ell/\bar{A}_\ell + DR_k \,. \quad (4.9)$$

If $k = 0$, let $\theta_k = z_\ell$ where $\deg_{z_\ell} f \neq 0$ and $F_{k-1} = K(z_1, \ldots, z_{\ell-1}, z_{\ell+1}, \ldots, z_n)$ in the discussion which follows.

Then from equation (4.9) and lemmas 3.5 and 3.6 it follows that R_k is in $F_{k-1}[\theta_k]$ or θ_k is exponential and $\theta_k R_k$ is in $F_{k-1}[\theta_k]$. Each term in (4.9) can be written as a quotient of two relatively prime polynomials in $F_{k-1}[\theta_k]$. The proof of lemma 3.5 implies that for each j, any factor of the denominator of DA_j/A_j is a factor of A_j; similarly, any factor of the denominator of $D\bar{A}_\ell/\bar{A}_\ell$ is a factor of $\bar{A}_\ell$. Let M be the product of the denominators of all the terms in (4.9) other than $c_\ell p_\ell \, Df/f$. Then f is not a factor of M since it is not a factor of any A_j for $j \neq \ell$, and it is not a factor of $\bar{A}_\ell$ and is not equal to θ_k when θ_k is exponential. Multiply equation (4.9) by fM and transpose the term $c_\ell p_\ell \, MDf$ to obtain

$$-c_\ell p_\ell \, MDf + fMDA_o/A_o = \sum_{\substack{j \in L(k,m) \\ j \neq \ell}} fMc_j \, DA_j/A_j + fM \, D\bar{A}_\ell/\bar{A}_\ell + fMDR_k \quad (4.10)$$

where each term in this equation is in $F_{k-1}[\theta_k]$.

Now f divides the right-hand side of equation (4.10). Thus, f must divide the left-hand side of equation (4.10). However, f does not divide $-c_\ell p_\ell \, MDf$ since f has no exponential factors. Therefore, f cannot divide $fMDA_o/A_o$. Note that $fMDA_o/A_o$ is in $F_{k-1}[\theta_k]$ since every other term in equation (4.10) is. Thus f must be a factor of A_o. Therefore, $A_o = f^{n_\ell} \bar{A}_o$ where n_ℓ is a non-zero integer and f is not a factor of $\bar{A}_o$. Thus $DA_o/A_o = n_\ell Df/f + D\bar{A}_o/\bar{A}_o$. Using this relationship in (4.10) one sees that f must divide $-c_\ell p_\ell \text{ MDf} + n_\ell \text{ MDf}$. But this cannot be unless $c_\ell p_\ell \text{ MDf} = n_\ell \text{ MDf}$. Since $\text{Df} \neq 0$, $c_\ell = n_\ell / p_\ell$. Therefore c_ℓ is a rational number.

Now let $A^* = A_o^{p_i} / A_\ell^{n_\ell}$. Then

$$\log A^* = p_\ell \log A_o - n_\ell \theta_\ell + \text{constant} . \tag{4.11}$$

Thus $\log A^* = \sum_{\substack{j \in L(k,m) \\ j \neq \ell}} p_\ell c_j \theta_j + p_\ell R_k + \text{constant}$. An argument similar to the one for the case in which $c_\ell = 0$ now shows that

$$\log A^* = \sum_{j \in L(k,m)} d_j \theta_j + \text{constant} \tag{4.12}$$

where each d_j is a rational number. The theorem now follows from equations (4.11) and (4.12). ∎

5. The Structure Theorem

In this section we begin by showing that every regular Liouville extension field is differentially isomorphic to a well-structured differential field.

Lemma 5.1 Let F be a differential field and let θ and ψ be in U, the universal extension of F. Suppose for each derivation operator D that $D\theta = D\psi$ is in F or that each $(D\theta)/\theta = (D\psi)/\psi$ is in F. Then the mapping σ from $F(\theta)$ to $F(\psi)$ defined by $\sigma(f) = f$ for f in F, $\sigma(\theta) = \psi$ and $\sigma(RS + T) = \sigma(R)\sigma(S) + \sigma(T)$ for R, S, T in $F(\theta)$ is a differential isomorphism.

Proof. It is clear that σ is a isomorphism from $F(\theta)$ to $F(\psi)$ and $\sigma(Df) = D(\sigma(f))$ for each f in F. If each $D\theta = D\psi$ is in F, $D(\sigma(\theta)) = D(\psi) = D(\theta) = \sigma(D(\theta))$ since σ is the identity mapping on F. If each $D\theta/\theta = D\psi/\psi$ is in F, $D(\sigma(\theta)) = D(\psi) = (D\psi/\psi)\psi = (D\theta/\theta)\psi = \sigma(D\theta/\theta) \cdot \sigma(\theta) = \sigma D\theta$. Thus σ is a differential isomorphism. ∎

Lemma 5.2 Suppose F^* and F are differential fields and σ is a differential isomorphism from F^* to F. Let A be in F^*. Then $\log A$ is a regular monomial over F^* if and only if $\log \sigma(A)$ is a regular monomial over F and $\exp A$ is a regular monomial over F^* if and only if $\exp \sigma(A)$ is a regular monomial over F.

Proof. If $\log A$ is not a regular monomial over F^*, lemma 3.8 implies that there is an f in F^* such that $\nabla \log A = \nabla f$. But this implies $\nabla \log \sigma(A) = \nabla(\sigma f)$ (where here ∇ refers to the gradient operator in F). Thus $\log \sigma(A)$

is not a regular monomial over F. Reversing the roles of A and $\sigma(A)$ in the above argument shows that $\log A$ is a regular monomial over F^* precisely when $\log \sigma(A)$ is a regular monomial over F.

If $\exp A$ is not a regular monomial over F^*, lemma 3.8 implies that for some f in F^* and for some integer $m \neq 0$, $m \nabla A = \nabla f/f$. Thus $m \nabla(\sigma A) = \nabla \sigma(f)/\sigma(f)$ (where ∇ is the gradient operator in F in this equation.) Thus $\exp(\sigma(A))^m = k \sigma f$ where k is in $C^{(F)}$ which implies that $\exp(\sigma A)$ is not a regular monomial over F. Again, reversing the roles of A and $\sigma(A)$ shows that $\exp A$ is a regular monomial if and only if $\exp(\sigma(A))$ is. ∎

<u>Lemma 5.3</u> Let F and F^* be differentially isomorphic fields. Suppose that $F(\theta_1, \theta_2, \ldots, \theta_m)$ is a regular Liouville extension of F. Then one can construct a regular Liouville extension $F^*(\psi_1, \ldots, \psi_m)$ of F^* which is differentially isomorphic to $F(\theta_1, \ldots, \theta_m)$ and which has the property that if $\psi_j = \log B_j$, B_j has no exponential factors.

Proof. The proof is constructive, showing explicitly how each ψ_j is obtained. Proceed by induction on m. The lemma is trivially true for $m = 0$. Suppose it is true for non-negative integers less than m when $m > 0$.

By the induction hypothesis there exists a differential isomorphism σ from $F(\theta_1, \ldots, \theta_{m-1})$ onto $F^*(\psi_1, \ldots, \psi_{m-1})$. To define ψ_m and extend σ to an isomorphism from $F(\theta_1, \ldots, \theta_m)$ onto $F^*(\psi_1, \ldots, \psi_m)$ two cases must be considered. First, suppose $\theta_m = \exp A$ where A is in $F(\theta_1, \ldots, \theta_{m-1})$. Let $\psi_m = \exp(\sigma A)$ and $\sigma\theta_m = \psi_m$. If σ obeys the usual laws of a homomorphism (i.e. commutes with the operations of addition and multiplication), then it is easy to see that σ is a differential isomorphism from $F(\theta_1, \ldots, \theta_m)$ onto $F^*(\psi_1, \ldots, \psi_m)$. Thus, by the previous lemma it follows that ψ_m is a regular monomial.

For the other case, assume $\theta_m = \log A$. Then A can be written $A = \bar{A} \prod_{j \in E(m-1)} \theta_j^{n_j}$ where each n_j is an integer and $\bar{A}$ has no exponential factors. Let $\psi_m = \log(\sigma\bar{A})$. By lemma 5.1, ψ_m is a regular monomial if and only if $\log(\sigma A)$ is since $\log(\sigma A) = \psi_m + \sum_{j \in E(m-1)} n_j \sigma A_j$ + a constant where $\theta_j = \exp A_j$ for each j in $E(m-1)$. By the previous lemma $\log(\sigma A)$ is a regular monomial and hence so is ψ_m.

Now extend σ by defining $\sigma(\theta_m) = \psi_m + \sum_{j \in E(m-1)} n_j \sigma A_j$ and letting σ obey the usual homomorphism laws. Then $(\theta_m - \sum_{j \in E(m-1)} n_j A_j)^n$ is the unique polynomial which is mapped into ψ_m^n by σ from which it follows easily that σ is an isomorphism from $F(\theta_1, \ldots, \theta_m)$ onto $F^*(\psi_1, \ldots, \psi_m)$.

Thus it remains only to show that σ commutes with each D in Δ. This will follow easily for all elements of $F(\theta_1, \ldots, \theta_m)$ once it is shown that $D(\sigma\theta_m) = \sigma(D\theta_m)$. But

$$D(\sigma\theta_m) = D\psi_m + \sum_{j \in E(m-1)} n_j D(\sigma A_j)$$

$$= D\sigma\bar{A}/\bar{A} + \sum_{j \in E(m-1)} n_j D(\sigma A_j) = D\sigma A/\sigma A$$

$$= \sigma DA/\sigma A = \sigma DA/A = \sigma(D\theta_m). \quad \blacksquare$$

Lemma 5.4 Let F_o and F_o^* be differentially isomorphic fields. Let $F_m = F_o(\theta_1, \ldots, \theta_m)$ be a regular Liouville extension of F_o such that for each j in $L(m,m)$, A_j has no exponential factors. Then there are $\psi_1, \ldots, \psi_m$ such that $F_m^* = F_o^*(\psi_1, \ldots, \psi_m)$ is well-structured and there is a differential isomorphism σ from F_m to F_m^* such that

(1) $\sigma(\theta_i) = \psi_i$ when θ_i is exponential

and (2) $\sigma(\theta_i) = \sum_{j \in L(m,m)} r_{i,j} \psi_j$ when θ_i is logarithmic

where each $r_{i,j}$ is a rational number. Furthermore, ψ_i is logarithmic precisely when θ_i is logarithmic.

Proof. The proof is by induction on m. The lemma is trivially true when $m = 0$. Assume inductively that the lemma is true for non-negative integers less than m when $m > 0$. Thus there are $\phi_1, \ldots, \phi_{m-1}$ and a differential isomorphism σ from $F_o(\theta_1, \ldots, \theta_{m-1})$ to the well-structured $F_o^*(\phi_1, \ldots, \phi_{m-1})$ satisfying (1) and (2). If $\theta_m = \exp A_m$, let $\phi_m = \exp(\sigma(A_m))$ and the lemma follows easily from lemma 5.2 with $\psi_j = \phi_j$, $1 \leq j \leq m$.

When $\theta_m = \log A_m$, the situation is somewhat more difficult. Suppose then that the differential isomorphism σ from $F_o(\theta_1, \ldots, \theta_{m-1})$ to $F_o^*(\phi_1, \ldots, \phi_{m-1})$ has been defined by

$$\sigma(\theta_i) = \begin{cases} \sum_{j \in L(m-1,m-1)} r_{i,j} \phi_j & \text{for } i \text{ in } L(m-1,m-1) \\ \phi_i & \text{otherwise .} \end{cases}$$

If ϕ_m were defined to be $\log(\sigma A_m)$ where $\theta_m = \log A_m$, problems could arise. For it may be that $\sigma A_m = \bar{A}_m \prod_{j \in L(m-1,m-1)} f_j^{p_j}$ where f_j is the distinguishing factor of ϕ_j in the field $F_o^*(\phi_1,\ldots,\phi_{m-1})$, and p_j is an integer. Then, if $p_j \neq 0$, f_j is not a distinguishing factor for ϕ_j in the field $F_o^*(\phi_1,\ldots,\phi_{m-1},\log(\sigma A_m))$. To avoid this problem, suppose that for each j in L(m-1, m-1), $\phi_j = \log A_j^*$ with $A_j^* = f_j^{n_j} \bar{A}_j$ where f_j is not a factor of $\bar{A}_j$. Let $n = \prod_{j \in L(m-1,m-1)} n_j$ and define

$$A_m^* = (\sigma A_m)^n / \prod_{j \in L(m-1,m-1)} (A_j^*)^{p_j \frac{n}{n_j}} . \qquad (5.1)$$

Then no f_j is a factor of A_m^* and theorem 4.5 implies that $\phi_m = \log A_m^*$ is a regular monomial over $F_o^*(\phi_1,\ldots,\phi_{m-1})$ since $\log(\sigma A_m)$ is a regular monomial over $F_o^*(\phi_1,\ldots,\phi_{m-1})$. Extend σ to a differential isomorphism from F_m to $F_o^*(\phi_1,\ldots,\phi_m)$ by letting

$$\sigma(\theta_m) = n\,\phi_m - \sum_{j \in L(m-1,m-1)} (np_j/n_j)\,\phi_j$$

$$= \log[(A_m^*)^n / \prod_{j \in L(m-1,m-1)} (A_j^*)^{np_j/n_j}] + \text{a constant} \qquad (5.2)$$

and by letting σ obey the usual laws of homomorphisms.

In a manner similar to that used in the preceding lemma to show the extension of σ was indeed a differential isomorphism, it can be shown that this extension of σ is a differential isomorphism from F_m to $F_o^*(\phi_1, \ldots, \phi_m)$. Clearly, σ satisfies (1) and (2).

However, it may be that $F_o^*(\phi_1, \ldots, \phi_m)$ is still not well-structured in case ϕ_m has no distinguishing factor. So let f_m be any irreducible factor of A_m^* with index equal to index(A_m^*). Then for each logarithmic ϕ_j suppose f_m is a factor of A_j^* precisely e_j times. If $e_j = 0$, $j = m$ or ϕ_j is exponential, and in this case let $\psi_j = \phi_j$. Otherwise, let

$$\psi_j = e_m \phi_j - e_j \phi_m = \log((A_j^*)^{e_m} / (A_m^*)^{e_j}) + \text{a constant} . \qquad (5.3)$$

Then each ψ_j is either exponential or f_j is a distinguishing factor of ψ_j so that $F_o^*(\psi_1, \ldots, \psi_m)$ is well-structured.

Define $\sigma_1(\phi_j) = \psi_j$ if $e_j = 0$, $m = j$ or j is in $E(m)$; $\sigma_1(\phi_j) = e_m^{-1}[\psi_j + e_j \psi_m]$ otherwise, and let $\sigma_1(1) = 1$. If σ_1 obeys the usual laws of homomorphism, then σ_1 is a differential isomorphism from $F_o^*(\phi_1, \ldots, \phi_m)$ to $F_o^*(\psi_1, \ldots, \psi_m)$. For it follows from the definition of σ_1 that there is a unique element of $F_o^*(\phi_1, \ldots, \phi_m)$ which is mapped onto each ψ_j. Thus σ_1 is an isomorphism. That σ_1 commutes with the derivation operators follows immediately from equation (5.3) and the relations defining σ_1.

It is immediate that $\sigma_1 \circ \sigma$ is a differential isomorphism satisfying (1) and (2) and $F_m^* = F_o(\psi_1, \ldots, \psi_m)$ is the required well-structured field.

Combining the last two lemmas yields the following theorem.

Theorem 5.5 Let $F_m = F_o(\theta_1, \ldots, \theta_m)$ be a regular Liouville extension of $F_o = K(z_1, \ldots, z_n)$. Then there is a well-structured differential field $F_o(\psi_1, \ldots, \psi_m)$ and a differential isomorphism from F_m to $F_o(\psi_1, \ldots, \psi_m)$ such that

(1) $\sigma(\theta_j) = \psi_j$ if j is in $E(m)$,

and (2) $\sigma(\theta_j) = \sum_{k \in L(m,m)} r_{k,j}\, \psi_k + \sum_{k \in E(m)} r_{k,j}\, \sigma(A_k)$

when j is in $L(m,m)$ where each $r_{k,j}$ is a rational number.

Proof. The isomorphism σ is the isomorphism $\sigma_2 \circ \sigma_1$ obtained by first applying lemma 5.3 to F_m to obtain the isomorphism σ_1 and a field $F_o(\phi_1, \ldots, \phi_m)$ and then applying lemma 5.4 to $F_o(\phi_1, \ldots, \phi_m)$ to obtain $F_o(\psi_1, \ldots, \psi_m)$ and an isomorphism σ_2. ▮

Lemma 5.6 Let $F_m = F_o(\theta_1, \ldots, \theta_m)$ be a regular Liouville extension of F_o. If A is in F_m and $\log A$ is not a regular monomial over F_m

$$\log A = \sum_{j \in L(m,m)} p_j\, \theta_j + \sum_{j \in E(m)} q_j\, A_j + c \qquad (5.4)$$

where c is in $C^{(U)}$, each p_j and q_j is a rational number, and the A_j are the arguments of the exponential monomials. Equivalently, if $\log A$ is not a regular monomial over F_m, $\{A\} \cup \{A_j : j \in L(m,m)\} \cup \{\theta_j : j \in E(m)\}$ is a pseudo-multiplicatively dependent set.

Proof. By theorem 5.5 there is a well-structured field $F_o(\psi_1, \ldots, \psi_m)$ and a differential isomorphism σ from F_m to $F_o(\psi_1, \ldots, \psi_m)$ satisfying (1) and (2) of theorem 5.5. Suppose A in F_m satisfies

$$A = \left(\prod_{j \in E(m)} \theta_j^{n_j} \right) \bar{A} \tag{5.5}$$

where $\bar{A}$ has no exponential factors and each n_j is an integer. Then lemma 3.8 implies that $\log \bar{A}$ is not a regular monomial over F_m since $\log A$ is not a regular monomial over F_m. Thus, lemma 5.2 implies that $\log \sigma(\bar{A})$ is not a regular monomial over $F_o(\psi_1, \ldots, \psi_m)$. Since $\bar{A}$ has no exponential θ_j as a factor, $\sigma(\bar{A})$ has no exponential ψ_j as a factor. Theorem 4.3 implies

$$\log \sigma(\bar{A}) = \sum_{j \in L(m,m)} r_j \psi_j + \text{a constant} \tag{5.6}$$

where each r_j is a rational number. Let σ^{-1} denote the inverse of σ so that $\sigma\sigma^{-1}(f) = f$ for each f in $F_o(\psi_1, \ldots, \psi_m)$. Then (5.6) and (2) of theorem 5.5 imply there are rational numbers $\bar{q}_j$ and p_j such that

$$\begin{aligned} \log \sigma(\bar{A}) &= \sum_{j \in L(m,m)} r_j \, \sigma(\sigma^{-1} \psi_j) + \text{a constant} \\ &= \sum_{j \in L(m,m)} p_j \, \sigma(\theta_j) + \sum_{j \in E(m)} \bar{q}_j \, \sigma(A_j) + \text{a constant} \quad . \end{aligned} \tag{5.7}$$

Therefore

$$\log \bar{A} = \sum_{j \in L(m,m)} p_j \theta_j + \sum_{j \in E(m)} \bar{q}_j A_j + \text{a constant} . \qquad (5.8)$$

The lemma now follows easily from equations (5.5) and (5.8). ∎

Theorem 5.7 (Structure Theorem) Let $F_m = F_o(\theta_1, \ldots, \theta_m)$ be a regular Liouville extension of F_o . Suppose $\theta_j = \log A_j$ for j in $L(m,m)$, $\theta_j = \exp A_j$ for j in $E(m)$, and A_{m+1} is in F_m .

(a) Let $\theta_{m+1} = \log A_{m+1}$, then θ_{m+1} is a regular monomial over F_m if and only if $\{\theta_j\}_{j \in E(m)} \cup \{A_j\}_{j \in L(m,m)} \cup \{A_{m+1}\}$ is pseudo-multiplicatively independent.

(b) Let $\theta_{m+1} = \exp A_{m+1}$, then θ_{m+1} is a regular monomial over F_m if and only if $\{\theta_j\}_{j \in L(m,m)} \cup \{A_j\}_{j \in E(m)} \cup \{A_{m+1}\}$ is pseudo-linearly independent over Q .

Proof. (a) If $\log A_{m+1}$ is not a regular monomial over F_m , lemma 5.6 implies that $\log A_{m+1}$ satisfies an equation of the form (5.4) which implies that there exist rational numbers p_j and q_j such that

$$A_{m+1} = c_1 \Big(\prod_{j \in L(m,m)} A_j^{p_j} \Big) \Big(\prod_{j \in E(m)} \theta_j^{q_j} \Big)$$

where c_1 is a constant. Thus there exist integers n_j such that

$$\Big(\prod_{j\in E(m)} \theta_j^{\,n_j}\Big)\Big(\prod_{j\in L(m,m)\cup\{m+1\}} A_j^{\,n_j}\Big) = c_2 \tag{5.9}$$

where c_2 is a constant. c_2 must be in F_m since the left-hand side of (5.9) is in F_m. Conversely if (5.9) holds $\log A_{m+1}$ is algebraic over $F_m(c_3)$ for a constant c_3 and so is not a regular monomial. Thus (a) is true.

(b) If $\theta_{m+1} = \exp A_{m+1}$ is not a monomial, lemma 3.8 implies that there is an integer $k \neq 0$, an R in F_m and a constant c_4 such that $\theta_{m+1}^{k} = c_4 R$. Therefore $k\nabla A_{m+1} = \nabla R/R$ from which it follows that $k A_{m+1} = \log R +$ a constant. Applying lemma 5.6 to $\log R$ yields

$$\sum_{j\in L(m,m)} p_j\,\theta_j + \sum_{j\in E(m)\cup\{m+1\}} q_j\,A_j = c_5 \tag{5.10}$$

where p_j and q_j are rational numbers. Conversely if A_{m+1} satisfies (5.10) then

$$\prod_{j\in L(m,m)} A_j^{\,p_j} \prod_{j\in E(m)\cup\{m+1\}} \theta_j = c_6$$

where c_6 is a constant. Thus θ_{m+1} is algebraic over $F_m(c_6)$ and is not a regular monomial. ∎

In [6] Epstein gives algorithms for determining pseudo-linear and pseudo-multiplicative independence for the case that $F_o = Q(i)(z_1, \ldots, z_n)$.

6. Ostrowski's Theorem

In this section several generalizations due to Kolchin [12] of a theorem of Ostrowski [23] are presented. The results are of a weaker character than the Structure Theorem of the previous section but are proved directly here to further illustrate the usefulness of the simple lemmas of section 3. A multivariate version of Ostrowski's original theorem is an easy consequence of the following lemma.

Lemma 6.1 Let $\theta_1, \ldots, \theta_m$ be primitive over the differential field F. Then $\theta_1, \ldots, \theta_m$ are algebraically independent over F unless there are $c_1, \ldots, c_m$, not all zero, in $C^{(F)}$ and a c in $C^{(U)}$ such that

$$\sum_{j=1}^{m} c_j \theta_j + c \text{ is in } F .$$

Proof. The proof is by induction on m. When m is 1 the result is just lemma 3.8 (a). Suppose the theorem is true for integers less than m. There are two cases:

Case I: $C^{(F)} = C^{(F(\theta_1))}$.

In this case, use the induction hypothesis to conclude that

$$\sum_{j=2}^{m} c_j \theta_j + c \text{ is in } F(\theta_1)$$

where each c_j is in $C^{(F(\theta_1))} = C^{(F)}$ and c is in $C^{(U)}$ and some $c_j \neq 0$. Thus there are P, Q in $F[\theta_1]$ which are relatively prime with Q monic such that

$$P/Q = \sum_{j=2}^{m} c_j \theta_j + c \quad . \tag{6.1}$$

Because each θ_j is primitive over F, so is $\sum_{j=2}^{m} c_j \theta_j + c$. Therefore, lemma 3.4 implies $Q = 1$. Now, lemma 3.1 implies that either P is in F or $P = c_1 \theta_1 + f$ where c_1 is in $C^{(F)}$ and f is in F, thus proving the lemma in this case.

CaseII: $\qquad C^{(F)} \neq C^{(F(\theta_1))}$.

Since $C^{(F)} \neq C^{(F(\theta_1))}$, θ_1 is not a monomial over F. Lemma 3.8 implies there is a c in $C^{(U)}$ and an f in F such that $\theta_1 = f - c$ from which the lemma follows easily. ∎

Theorem 6.2 (Ostrowski) Let $\theta_1, \ldots, \theta_m$ be primitive over the differential field F. Let $F_m = F(\theta_1, \ldots, \theta_m)$ and suppose that $C^{(F)} = C^{(F_m)}$. Then $\theta_1, \ldots, \theta_m$ are algebraically dependent over F if and only if there are constants $c_1, \ldots, c_m$ in $C^{(F)}$, not all zero, such that

$$\sum_{j=1}^{m} c_j \theta_j \text{ is in } F .$$

Proof. Certainly, when $\sum_{j=1}^{m} c_j \theta_j = f$ where some $c_j \neq 0$, each c_j is in $C^{(F)}$ and f is in F, $\theta_1, \ldots, \theta_m$ are algebraically dependent over F.

Conversely, suppose $\theta_1, \ldots, \theta_m$ are algebraically dependent. Then by the preceding lemma, there are constants $c_1, \ldots, c_m$ in $C^{(F)}$ and c in $C^{(U)}$ with some $c_j \neq 0$ and an f in F such that

$$f = c + \sum_{j=1}^{m} c_j \theta_j \tag{6.2}$$

Thus, c is in F_m. Since $C^{(F)} = C^{(F_m)}$ c must be in F. So

$$\sum_{j=1}^{m} c_j \theta_j \text{ is in } F$$

which implies the theorem. ▮

The following is the exponential analog of Ostrowski's Theorem which appears in [12].

Theorem 6.3 Let $\theta_1, \ldots, \theta_m$ be exponential over the differential field F. Suppose F and $F(\theta_1, \ldots, \theta_m)$ have the same constant fields. Then $\theta_1, \ldots, \theta_m$ are algebraically dependent over F if and only if there are integers $e_1, \ldots, e_m$ not all zero such that

$$\prod_{j=1}^{m} \theta_j^{e_j} \text{ is in } F .$$

Proof. When there are integers $e_1, \ldots, e_m$ not all zero such that $\prod_{j=1}^{m} \theta_j^{e_j}$ is in F, $\theta_1, \ldots, \theta_m$ are algebraically dependent. The converse will be proven by induction on m.

When $m = 1$, the hypothesis of the theorem is that θ_1 is algebraic over F. So lemma 3.8 implies there is an integer $e_1 \neq 0$ and a c in $C^{(U)}$ such that $\theta_1^{e_1} = cg$ for some g in F. Thus c is in $F(\theta_1)$. Since $C^{(F)} = C^{(F(\theta_1))}$, c is in F and the desired result holds.

Suppose $m > 1$ and the theorem is true for positive integers less than m. If $\theta_1, \ldots, \theta_m$ are algebraically dependent over F, $\theta_2, \ldots, \theta_m$ are algebraically dependent over $F(\theta_1)$. If θ_1 is algebraic over F, the theorem follows from the case $m = 1$. Otherwise, by the induction hypothesis there are integers $e_2, \ldots, e_m$ not all zero such that $\prod_{j=2}^{m} \theta_j^{e_j}$ is in $F(\theta_1)$. Suppose

$$\prod_{j=2}^{m} \theta_j^{e_j} = P/Q \tag{6.3}$$

where P, Q are relatively prime polynomials in $F[\theta_1]$ and Q is monic. Then for each derivation operator D on F

$$\left(\prod_{j=2}^{m} \theta_j^{e_j}\right)\left(\sum_{j=2}^{m} e_j \, D\theta_j/\theta_j\right) = (QDP - PDQ)/Q^2 \ . \tag{6.4}$$

Since each θ_j is exponential, $f_1 = \sum_{j=2}^{m} e_j D\theta_j/\theta_j$ is in F. From (6.3) and (6.4) it follows that

$$f_1 PQ = QDP - PDQ .$$

Thus P divides DP and Q divides DQ in $F[\theta_1]$, for each derivation operator D. Lemma 3.3 implies, $P = p\,\theta_1^{n_1}$ and $Q = \theta_1^{n_2}$ (since Q is monic) where p is in F. Thus,

$$\theta_1^{n_2 - n_1} \prod_{j=2}^{m} \theta_j^{e_j} = p$$

proving the theorem. ∎

The following result, which combines the previous two results, also appears in [12].

<u>Theorem 6.4</u> Let F be a differential field. Suppose each of $\theta_1, \ldots, \theta_r$ is primitive over F and each of $\psi_1, \ldots, \psi_s$ is exponential over F. Suppose also that $F(\theta_1, \ldots, \theta_r, \psi_1, \ldots, \psi_s)$ has the same constant field as F. Then $\theta_1, \ldots, \theta_r, \psi_1, \ldots, \psi_s$ are algebraically dependent over F if and only if

(1) there are constants $c_1, \ldots, c_r$ in $C^{(F)}$ not all zero such that $\sum_{j=1}^{r} c_j \theta_j$ is in F;

or, (2) there are integers $e_1, \ldots, e_s$ not all zero such that $\prod_{j=1}^{s} \psi_j^{e_j}$ is in F.

Proof. It is immediate that if (1) or (2) is true $\theta_1,\ldots,\theta_r,\psi_1,\ldots,\psi_s$ are algebraically dependent. The converse will be proved by induction on $r+s$. When $r+s=1$, the theorem follows from the two preceding theorems since either $r=0$ or $s=0$ in this case.

Suppose, now that $r+s>1$ and the theorem is true for positive integers less than $r+s$. If either $r=0$ or $s=0$, the theorem follows from the two preceding theorems. So assume that $r>0$ and $s>0$. If $\theta_1,\ldots,\theta_r,\psi_1,\ldots,\psi_s$ are algebraically dependent over F, $\theta_2,\ldots,\theta_r,\psi_1,\ldots,\psi_s$ are algebraically dependent over $F(\theta_1)$. Then by the induction hypothesis either there are integers $e_1,\ldots,e_s$ not all zero such that $\prod_{j=1}^{s}\psi_j^{e_j}$ is in $F(\theta_1)$ or there are constants $c_2,\ldots,c_r$ not all zero such that $\sum_{j=2}^{r} c_j\,\theta_j$ is in $F(\theta_1)$. When the latter is the case, the proof follows from Ostrowski's Theorem. For, if $\sum_{j=2}^{r} c_j\,\theta_j$ is in $F(\theta_1)$, $\theta_2,\ldots,\theta_r$ are algebraically dependent over $F(\theta_1)$. Thus $\theta_1,\ldots,\theta_r$ are algebraically dependent over F and the conclusion of Ostrowski's theorem provides the desired result.

So, suppose there are integers $e_1,\ldots,e_s$ not all zero such that $\prod_{j=1}^{s}\psi_j^{e_j}$ is in $F(\theta_1)$. If θ_1 is algebraic over F the desired result follows immediately from Ostrowski's theorem. So assume that θ_1 is transcendental over F. Thus there are P, Q relatively prime polynomials in $F[\theta_1]$ with Q monic such that

$$\prod_{j=1}^{s}\psi_j^{e_j} = P/Q \quad . \tag{6.6}$$

For each derivation operator D ,

$$\left(\prod_{j=1}^{s} \psi_j^{e_j}\right)\left(\sum_{j=1}^{s} e_j \, D\psi_j/\psi_j\right) = (QDP - PDQ)/Q^2 \ . \qquad (6.7)$$

Since each ψ_j is exponential, $\sum_{j=1}^{s} e_j \, D\psi_j/\psi_j$ is in F . Thus writing f for $\sum_{j=1}^{s} e_j \, D\psi_j/\psi_j$, (6.6) and (6.7) imply

$$PQf = QDP - PDQ \ .$$

So P divides DP and Q divides DQ (in $F[\theta_1]$) for each derivation operator D . Since θ_1 is primitive, lemma 3.3 implies that $Q = 1$ and P is in F which completes the proof. ∎

The following result, which is apparently new, follows in the same spirit as the other results of this section.

<u>Theorem 6.5</u> Let $\theta_1, \ldots, \theta_m$ be primitive over the differential field F . Suppose the constant field of $F(\theta_1, \ldots, \theta_n, \log \theta_1, \ldots, \log \theta_n)$ is the same as the constant field of F . Then the following two statements are equivalent:

(1) $\theta_1, \ldots, \theta_m$ are algebraically independent over F ;

(2) $\theta_1, \ldots, \theta_m$, $\log \theta_1, \ldots, \log \theta_m$ are algebraically independent over F .

Proof. It is clear that (2) implies (1). The proof will be complete when it is shown that not (2) implies not (1). So, suppose (2) is false. Then $\log \theta_1, \ldots, \log \theta_m$ are algebraically dependent over $F(\theta_1, \ldots, \theta_m)$ which has the same constant field as F does since $F(\theta_1, \ldots, \theta_m, \log \theta_1, \ldots, \log \theta_m)$ does. Then Ostrowski's Theorem implies there are constants $c_1, \ldots, c_m$, not all zero, in $F(\theta_1, \ldots, \theta_m)$ and therefore in F and a g in $F(\theta_1, \ldots, \theta_m)$ such that

$$\sum_{j=1}^{m} c_j \log \theta_j - g = 0 . \tag{6.8}$$

Without loss of generality suppose $c_1 \neq 0$, since otherwise the θ_j may be renumbered. If for each derivation operator $D, D\theta_1 = 0$, θ_1 is a constant, and θ_1 must be in F. So $\theta_1, \ldots, \theta_m$ are algebraically dependent over F when each $D\theta_1 = 0$. Suppose, then, that there is a derivation operator D such that $D\theta_1$ is not 0. Equation (6.8) implies

$$\sum_{j=1}^{m} c_j D\theta_j/\theta_j - Dg = 0 . \tag{6.9}$$

Since each θ_j is primitive over F_1 each $D\theta_j$ is in F. Also Dg is in $F(\theta_1, \ldots, \theta_m)$ and hence (6.9) is an algebraic relationship over F for $\theta_1, \ldots, \theta_m$. ∎

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