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DISCRETE SOLITARY WAVES

by

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Appendix: FORTRAN Program for Discrete
Solitary Waves by S. T. Jones

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1. Introduction

Wave motion is of fundamental interest to mathematicians, physicists, and engineers. Besides experimental methods, both analytical and numerical mathematical techniques have been, and are being, applied to the study of related linear and nonlinear problems (see, e.g., refs. [1-5, 8-14] and the numerous additional references contained therein).

In this paper we will develop a discrete numerical approach for the study of a special class of solitary waves, that is, those which can be generated by applying a whiplike motion to an elastic string. Our strings will consist only of a finite number of particles, each of whose motion is determined by a dynamical difference equation [4,5]. The primary advantage of such an approach is that we will be able to solve both linear and nonlinear initial value problems with equal ease.

2. Discrete Strings

A discrete string is one which consists of a finite number of ordered particles $P_0, P_1, P_2, \ldots, P_n$, each of mass m. It is assumed, without loss of generality, that P_0 is fixed at $(0,0)$ and P_n is fixed at $(2,0)$. If x_i denotes the x coordinate of P_i, then let $x_i = i\Delta x = \frac{2i}{n}$, $i = 0,1,2,\ldots,n$.

For positive time step Δt, set $t_k = k\Delta t$, $k = 0,1,2,\ldots$, and, assuming that $P_1, P_2, \ldots, P_{n-1}$ can move in the y-direction only, let us denote the y coordinate, the velocity, and the acceleration of P_i at t_k by $y_{i,k}$, $v_{i,k}$, and $a_{i,k}$, respectively. Let $|T_{i-1,i,k}|$ be the tensile force between P_{i-1} and P_i at t_k. For $g \geq 0$ and $\alpha \geq 0$, then, each $a_{i,k}$ is defined by the dynamical difference equation

$$(2.1)\qquad m a_{i,k} = |T_{i,i+1,k}| \frac{y_{i+1,k} - y_{i,k}}{[(\Delta x)^2 + (y_{i+1,k} - y_{i,k})^2]^{1/2}} - |T_{i-1,i,k}| \frac{y_{i,k} - y_{i-1,k}}{[(\Delta x)^2 + (y_{i,k} - y_{i-1,k})^2]^{1/2}} - \alpha v_{i,k} - mg;\quad i = 1,2,3,\ldots,n-1 .$$

From (2.1), each $v_{i,k+1}$ is then determined from the special formulas [6]:

$$(2.2)\qquad \begin{cases} v_{i,1} = v_{i,0} + (\Delta t)\, a_{i,0} \\ v_{i,k+1} = v_{i,k} + (\Delta t)\left(\frac{3}{2} a_{i,k} - \frac{1}{2} a_{i,k-1}\right), \quad k = 1,2,\ldots . \end{cases}$$

while, with the aid of (2.2), each $y_{i,k+1}$ is determined from

$$(2.3)\qquad y_{i,k+1} = y_{i,k} + \frac{\Delta t}{2}(v_{i,k+1} + v_{i,k}),\quad k = 0,1,2,\ldots .$$

From initial data $y_{i,0}$ and $v_{i,0}$, $i = 1,2,\ldots,n-1$, then, the motions of $P_1, P_2, \ldots, P_{n-1}$ are determined recursively at each time step by (2.1)-(2.3). The particular value of (2.2) with regard to conservation and stability is discussed in [6].

It only remains then to discuss the class of tension formulas to be explored in the computer examples which follow. In this paper we will limit attention to tensile forces of the particular form

$$(2.4) \qquad |T_{i,i+1,k}| = T_0 \left[(1-\varepsilon)\left(\frac{r_{i,i+1,k}}{\Delta x}\right) + \varepsilon\left(\frac{r_{i,i+1,k}}{\Delta x}\right)^2\right],$$

where $0 \leq \varepsilon \leq 1$, where T_0 is a reference tension, and where

$$(2.5) \qquad r_{i,i+1,k} = [(\Delta x)^2 + (y_{i+1,k} - y_{i,k})^2]^{1/2}.$$

Formula (2.4) is nonlinear if and only if $\varepsilon \neq 0$. If $\varepsilon = 0$, (2.1) takes the particular form

$$m a_{i,k} = \frac{T_0}{\Delta x}[y_{i+1,k} - 2y_{i,k} + y_{i-1,k}] - \alpha v_{i,k} - mg,$$

which we will call Hooke's law. If $y_{i,k} = 0$ for $i = 1,2,\ldots,n-1$, so that all particles are on the X-axis, then (2.4) yields

$$|T_{i,i+1,k}| = T_0, \quad i = 0,1,2,\ldots,n-1.$$

3. Examples.

In all the examples which are given now, the parameters n, m, T_0, α, g, and Δx are fixed as follows: $n = 100$, $m = .01$, $T_0 = 10$, $\alpha = g = 0$, $\Delta x = 0.02$. Since the effects of viscosity and gravity have been explored in some detail in [5], they are being neglected here in order to facilitate our study of the nature of wave interaction. The motions resulting from the choices $\varepsilon = 0, 0.02, 0.1$ will be called linear, mildly nonlinear, and nonlinear, respectively. All particles are fixed initially on the X-axis.

In order to generate a solitary wave, one need only prescribe nonzero velocities to particles near either end of the string and zero velocities to the remaining particles. In Figure 1 is shown the linear motion of a solitary wave at t_{15}, t_{100}, t_{200}, t_{300} and t_{400} for $\Delta t = 0.001$, $v_{1,0} = 60$, $v_{2,0} = 50$, $v_{3,0} = 40$, $v_{4,0} = 30$, $v_{5,0} = 20$, $v_{6,0} = 10$ and $v_{i,0} = 0$, $7 \leq i \leq 100$. The development in time of numerous trailing waves is accompanied by a relatively small decrease in the amplitude of the solitary wave.

In Figure 2 is shown the linear motion of a solitary wave at t_{15}, t_{100}, t_{200}, t_{300}, and t_{400} for $\Delta t = 0.001$, $v_{1,0} = v_{2,0} = v_{3,0} = v_{4,0} = v_{5,0} = 40$, and $v_{i,0} = 0$, $6 \leq i \leq 100$. The trailing waves in this example are more erratic in nature than those shown in Figure 1 and the amplitude of the solitary wave does, again, decrease moderately with time.

In Figure 3 is shown the linear motion of a solitary wave at t_{15}, t_{100}, t_{200}, t_{300} and t_{400} for $\Delta t = 0.001$, $v_{1,0} = 10$, $v_{2,0} = 20$, $v_{3,0} = 30$, $v_{4,0} = 40$, $v_{5,0} = 50$, $v_{6,0} = 60$ and $v_{i,0} = 0$, $7 \le i \le 100$. The trailing waves, again, are more erratic than those shown in Figure 1 while the solitary wave, this time, shows various qualitative changes as it moves. Most noticable are a widening of its base with time and the development, and then the disappearance, of a small well in the peak at t_{100}.

Figure 4 shows the effect of changing ε from 0 to 0.02 while leaving unchanged the other initial conditions prescribed for the motion shown in Figure 2. The solid curve is that of the mildly nonlinear motion, while the dotted curve is that of the linear motion, both at t_{350}. The resulting decrease in amplitude with the increase in ε followed also to other choices of ε. Figure 5 shows at t_{350} this same effect when ε is again changed from 0 to .02, while the other initial conditions are those of the motion shown in Figure 1. The solid curve corresponds to the mildly nonlinear motion, while the dotted curve corresponds to linear motion.

To study the interaction of solitary waves, one need only generate a wave at each end of the string and observe the resulting motion. Figure 6 shows the results of the symmetric choices $v_{1,0} = v_{99,0} = 60$,

$v_{2,0} = v_{97,0} = 50$, $v_{3,0} = v_{96,0} = 40$, $v_{4,0} = v_{95,0} = 30$, $v_{5,0} = v_{94,0} = 20$, $v_{6,0} = v_{93,0} = 10$, and $v_{i,0} = 0$, $7 \le i \le 92$. The motion is mildly nonlinear, only the middle fifty particles are shown, and the time step is $\Delta t = 0.001$. At t_{195} one sees the two solitary waves begin to form the single wave shown at t_{215}. At t_{220}, this large wave begins to drop in amplitude and widen at the base. This continues through t_{225} and at t_{230} one sees that the apex point has dropped sufficiently and the base widened sufficiently to yield, again, two solitary waves. However, after this interaction, in which the original waves have passed through each other, the resulting motions are no longer as uniform as before the interaction. As shown at t_{245}, t_{255} and t_{300}, the resulting solitary waves oscillate a small amount in amplitude and in base width as they continue their motion.

Figure 7 shows that the same qualitative results are valid as in the above example when the wave on the right is replaced by a smaller wave. This motion was generated, as above, merely by the change of input parameters: $v_{99,0} = 40$, $v_{98,0} = 30$, $v_{97,0} = 20$, $v_{96,0} = 10$, $v_{95,0} = v_{94,0} = 0$.

In Figure 8 is shown the effect of changing ε to 0.1, changing Δt to $.0005$, and leaving unchanged all other conditions for the motion shown in Figure 7. At t_{350}, then, one sees the larger wave approaching from the left and the smaller wave approaching from the right. The trailing waves, however, are no longer insignificant due to the relatively large decrease in amplitudes of both solitary waves. At t_{380} is shown the merger of the two solitary waves. A very complex behavior is shown at t_{410} in which the larger wave has emerged, but the smaller one is still not discernable, due to interaction with the relatively large trailing waves. However, at t_{480}, one sees again the reappearance of the smaller of the solitary waves, and the interchange is complete.

Last, in Figure 9 is shown the interaction of two solitary waves which are generated like those shown in Figures 1 and 3, but whose motions are nonlinear. The choice of initial velocities was $v_{1,0} = v_{94,0} = 60$, $v_{2,0} = v_{95,0} = 50$, $v_{3,0} = v_{96,0} = 40$, $v_{4,0} = v_{97,0} = 30$, $v_{5,0} = v_{98,0} = 20$, $v_{6,0} = v_{99,0} = 10$, and $v_{i,0} = 0$, $7 \leq i \leq 93$. For $\Delta t = 0.0005$, Figure 9 shows at t_{250} the standing waves approaching each other. At t_{340} the interaction has begun and at t_{380} they have formed a single wave. At t_{430} is shown their initial separation and distortion immediately after the interaction. A deep well has formed in the peak of the larger wave and, as they continue to separate, this well decreases in depth while the motion of the trailing waves between the solitary waves becomes very complex.

Finally, note that no example described in this section used more than five minutes of Univac 1108 running time, and that, for the convenience of those who wish to verify the computations, the FORTRAN program used is given in [7].

FIGURE 1

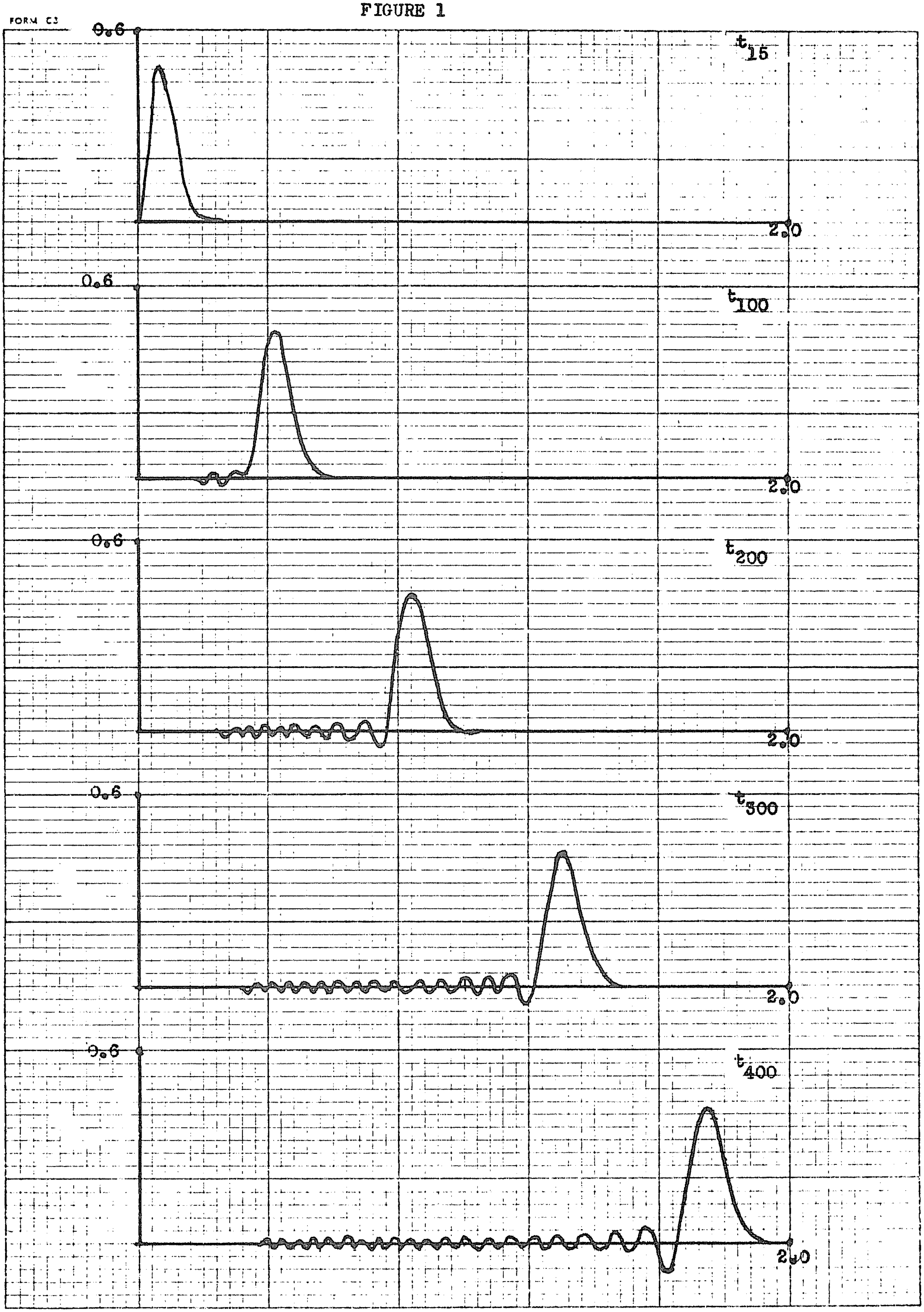


FIGURE 2

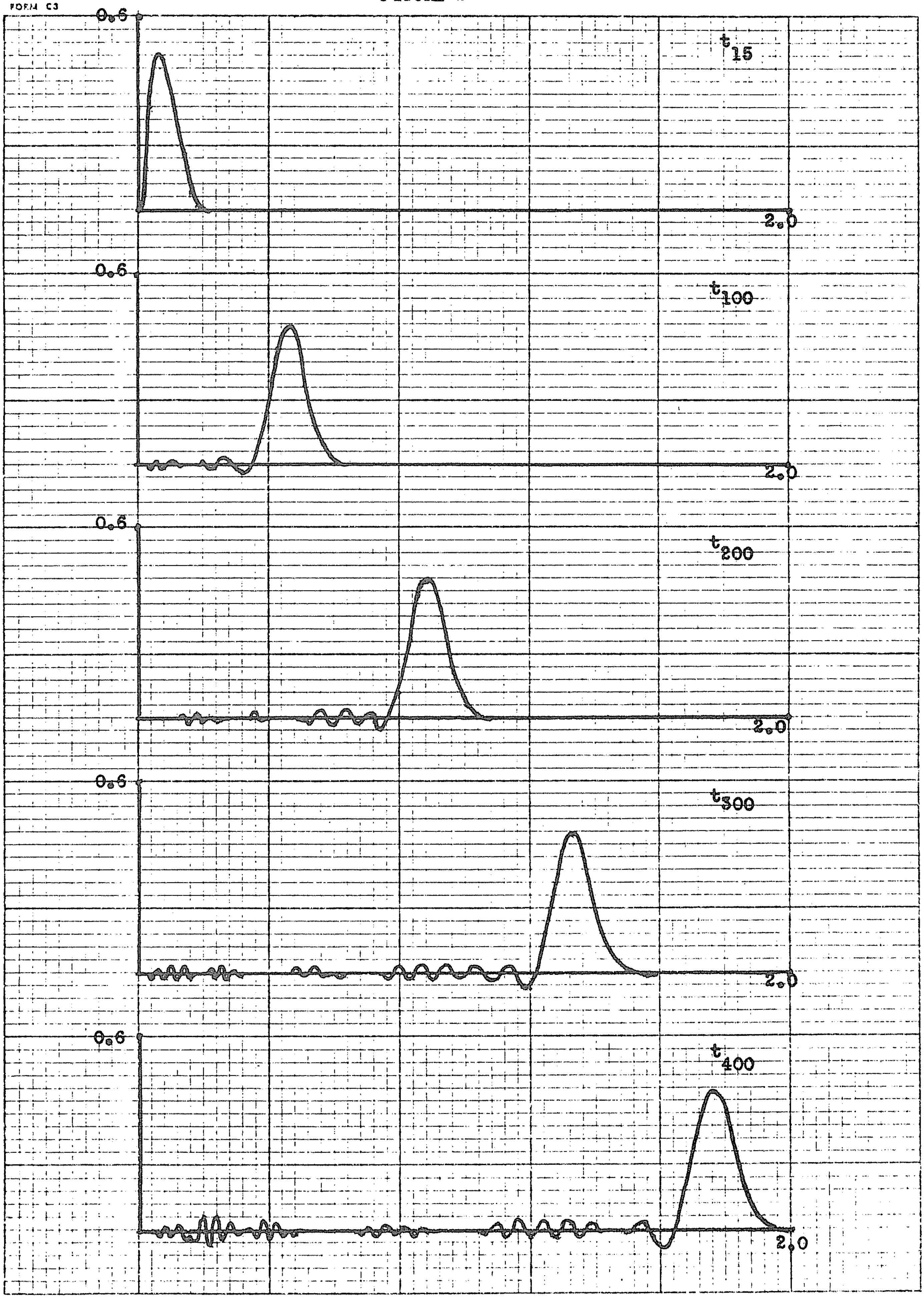


FIGURE 3

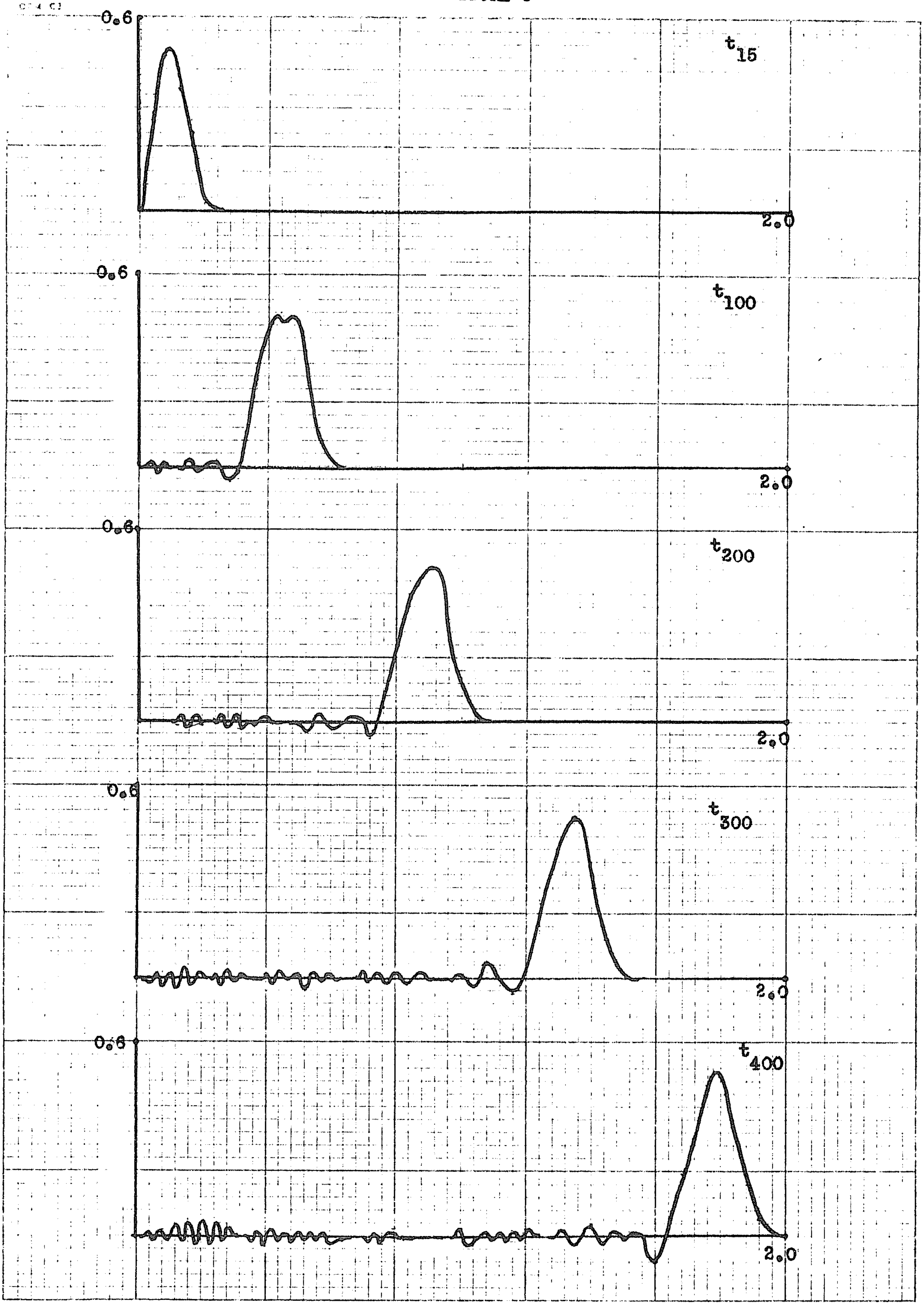


FIGURE 4

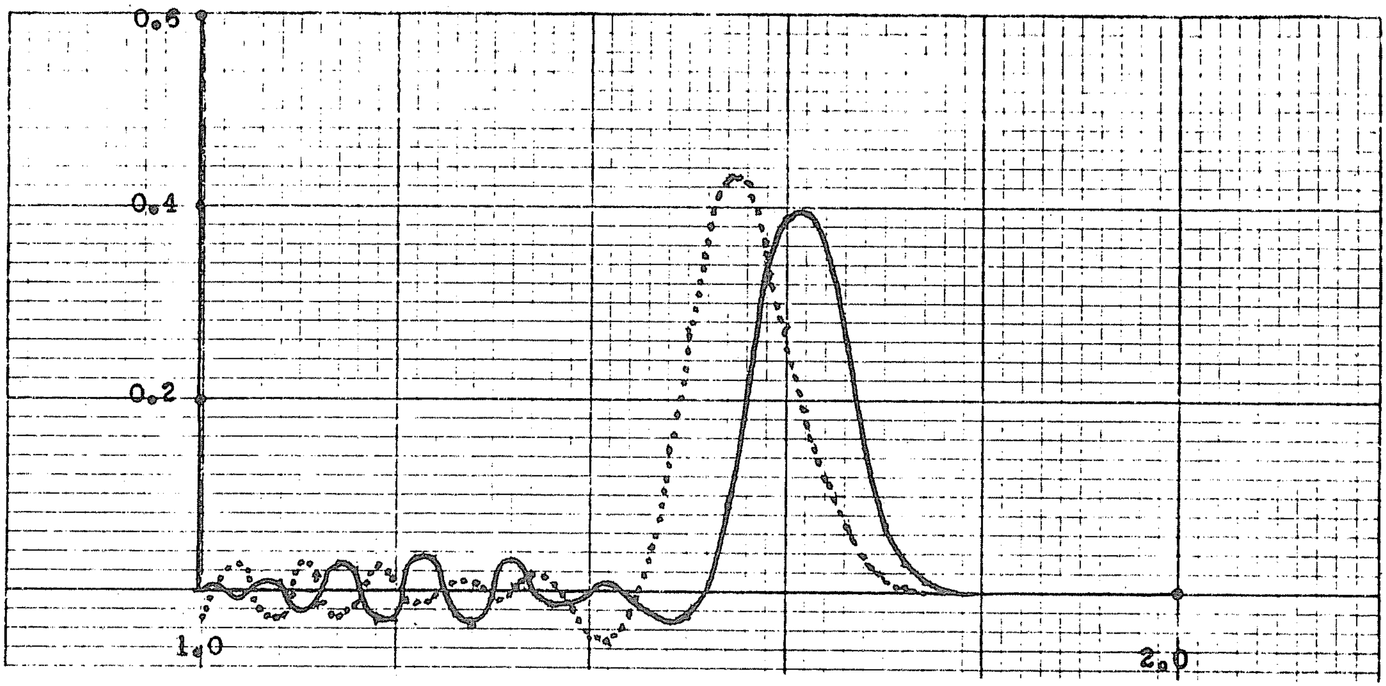


FIGURE 5

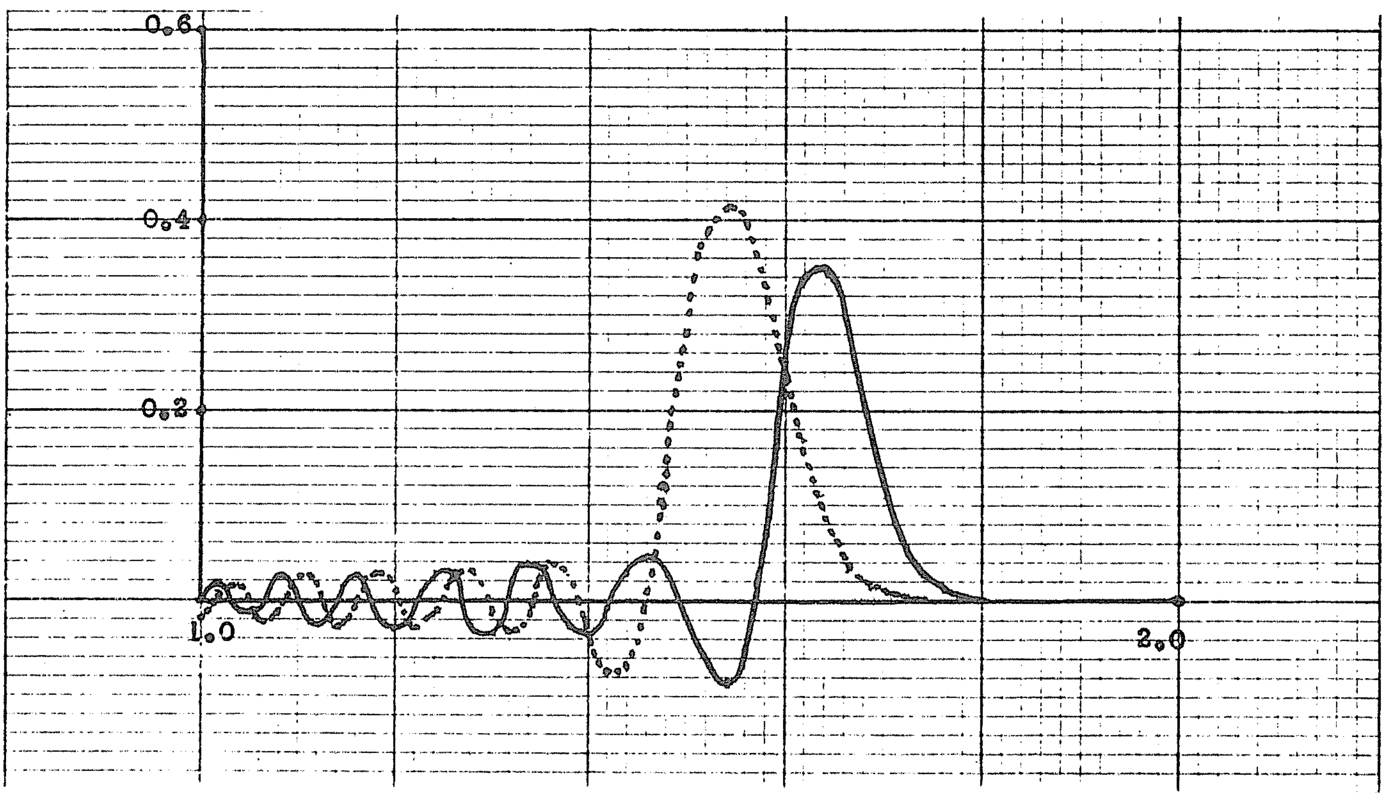


FIGURE 6

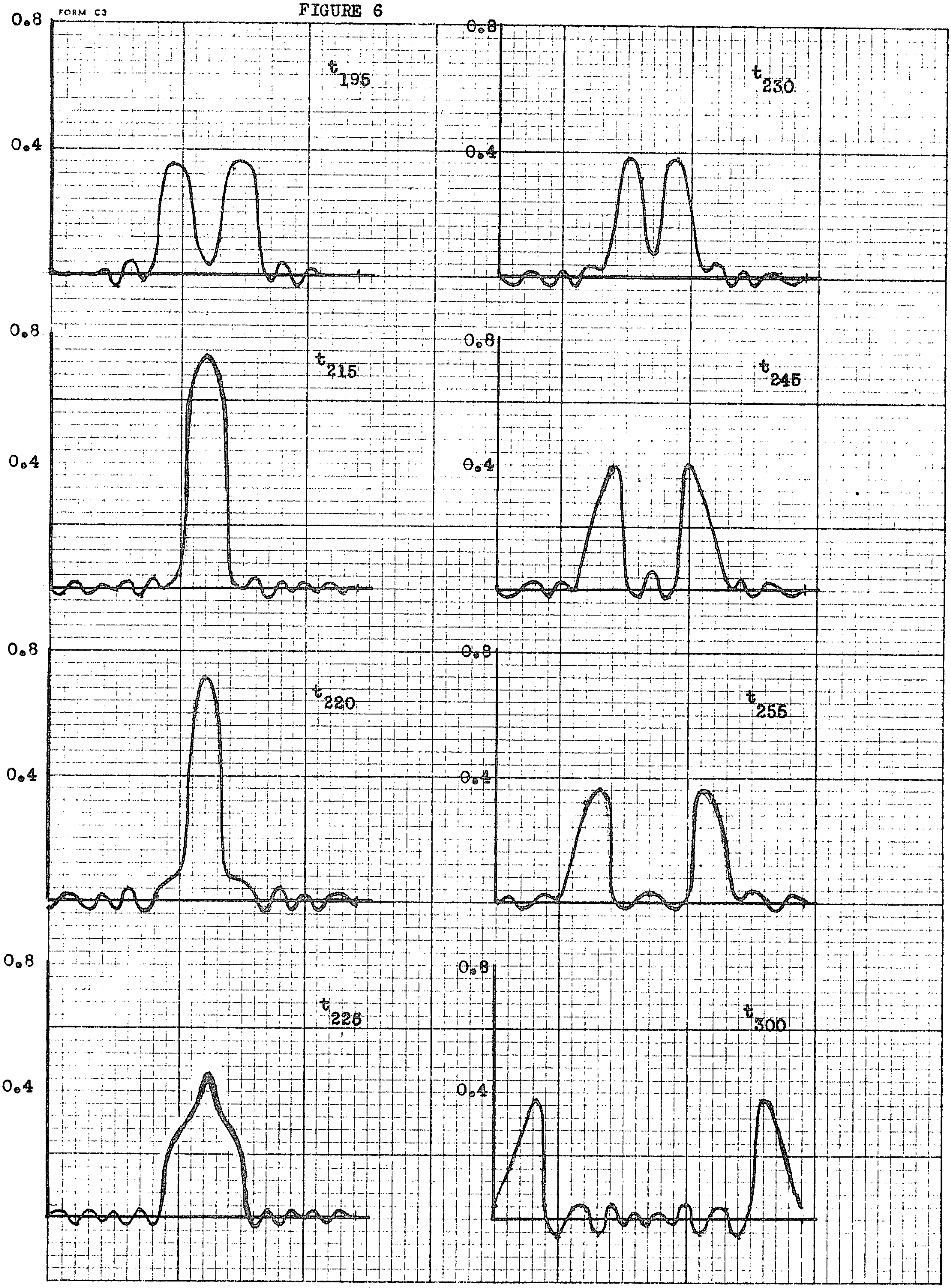


FIGURE 7

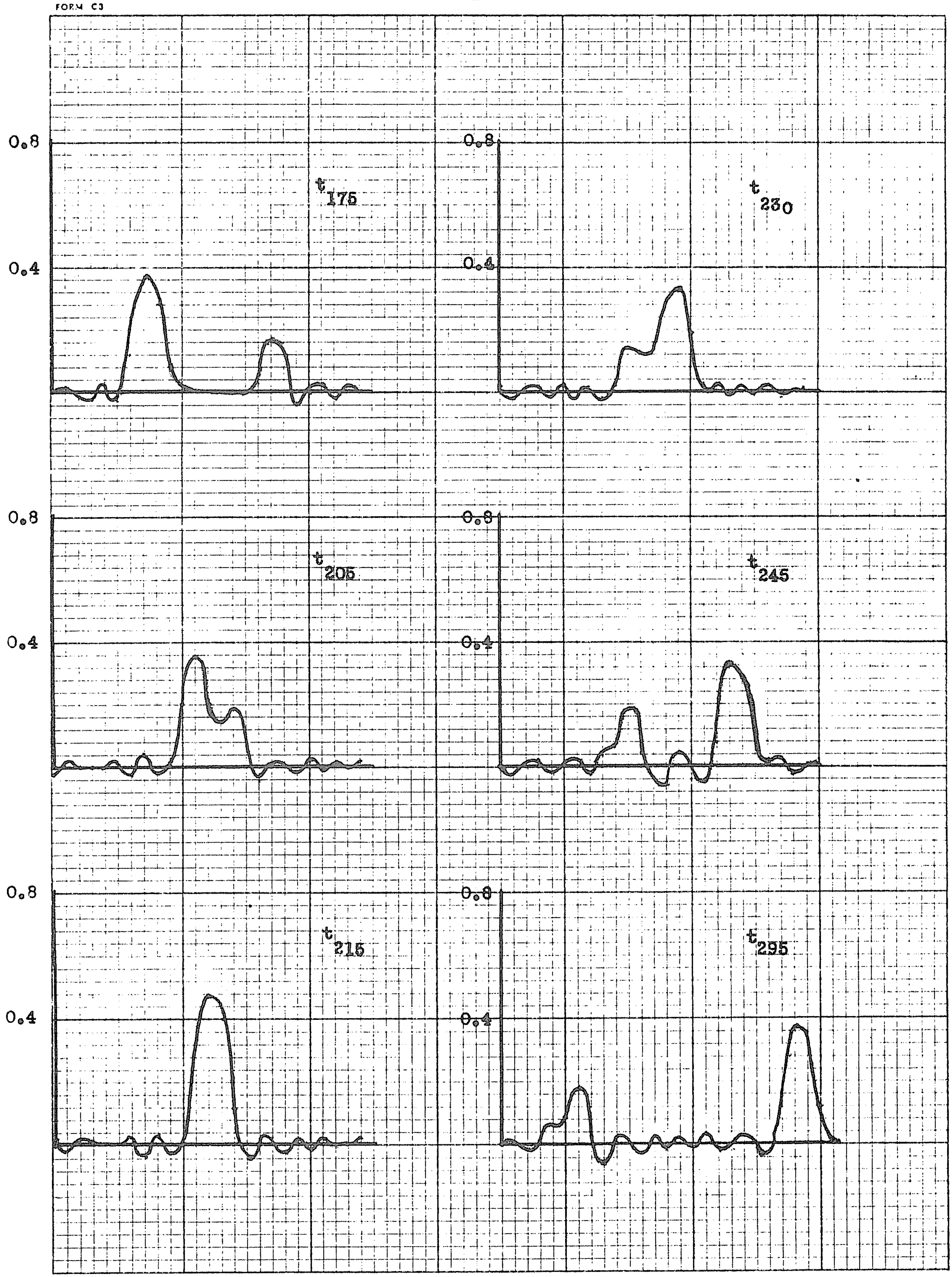

FORM C3
0.8
0.4
t175
0.8
0.4
t230
0.8
0.4
t205
0.8
0.4
t245
0.8
0.4
t215
0.8
0.4
t295

FIGURE 8

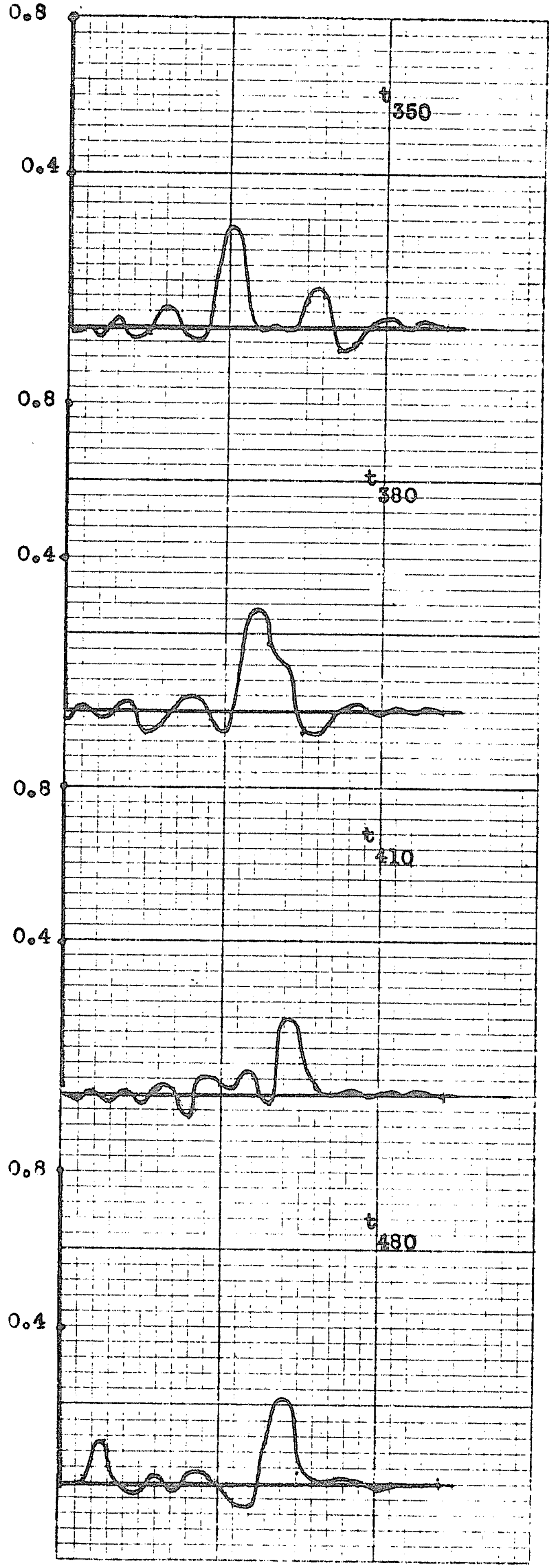
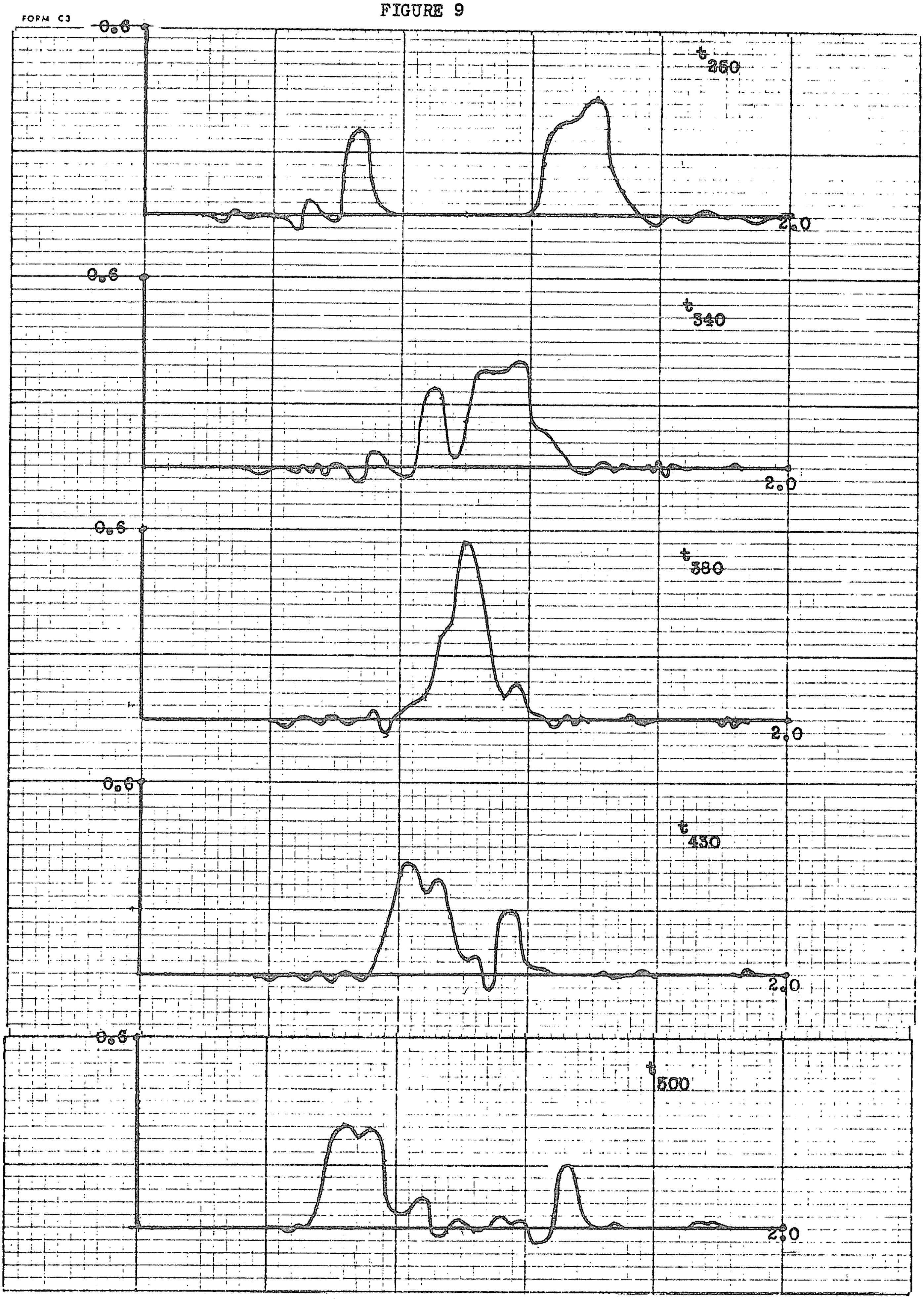


FIGURE 9



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Appendix

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      REAL MASS
      DIMENSION X(510),Y(510),V(510,2),A(510,2),R(510),T(510)
 1001 FORMAT(16I5)
 1002 FORMAT(2F10.5,3I5)
 1003 FORMAT(8F10.5)
 1004 FORMAT(16F5.0)
 2000 FORMAT(1H1)
 2001 FORMAT(5X,'N= ',I3,5X,'TENSION FORMULA= ',I1,/,5X,'MASS= ',F5.3,
     15X,'TO= ',F4.1,5X,'ALPHA= ',F6.3,/,5X,'G= ',F4.1,5X,'EPS= ',F4.2,/
     2,5X,'DT= ',F7.5,5X,'DX= ',F5.3)
 2002 FORMAT(' TIMESTEP= ',I5)
 2003 FORMAT(10F10.5)
 2004 FORMAT(' VELOCITIES= ',6F5.0)
C  SPECIFY INPUT PARAMETERS
      READ 1003,MASS,G,ALPHA,DX,TO
    5 READ 1001,N,ITEN,IEND
      PRINT 2000
      NM1=N-1
      DXSQ=DX**2
    9 READ 1002,DT,EPS,ITEPS,KMAX,IPRINT
      DT2=DT/2.0
C  SPECIFY INITIAL POSITIONS,VELOCITIES,ACCELERATIONS
   15 DO 10 I=1,N
      X(I)=(I-1)*DX
      Y(I)=0.0
      V(I,2)=0.0
      A(I,2)=0.0
   10 CONTINUE
      READ 1001,IV,JV,JV2
      READ 1004,(V(I,2),I=2,JV)
      READ 1004,(V(I,2),I=JV2,100)
      K=0
      ISTEP=1
C  PRINT INITIAL CONDITIONS,PARAMETERS
      PRINT 2001,N,ITEN,MASS,TO,ALPHA,G,EPS,DT,DX
      PRINT 2004,(V(I,2),I=2,JV)
      PRINT 2004,(V(I,2),I=JV2,100)
      PRINT 2002,K
      PRINT 2003,(Y(I),I=1,N)
C  UPDATE VELOCITIES,ACCELERATIONS
   39 K=K+1
      DO 40 I=1,N
      V(I,1)=V(I,2)
      A(I,1)=A(I,2)
   40 CONTINUE
C  COMPUTE TENSION BETWEEN NEIGHBORING PARTICLES
      DO 50 I=1,NM1
      IP1=I+1
      R(I)=SQRT(DXSQ+(Y(IP1)-Y(I))**2)
   50 CONTINUE
      GO TO(99,110),ITEN
   99 OME=1.0-EPS
      DO 100 I=1,NM1
      RDX=R(I)/DX
```

```
      T(I)=TO*(OME*RDX+EPS*RDX**2)
  100 CONTINUE
      GO TO 130
  110 DO 120 I=1,NM1
      IP1=I+1
      DYDX=ABS((Y(IP1)-Y(I))/DX)
      T(I)=TO*(1.0+DYDX+EPS*DYDX**2)
  120 CONTINUE
C   COMPUTE ACCELERATION
  130 DO 140 I=2,NM1
      IP1=I+1
      IM1=I-1
      A(I,2)=T(I)*(Y(IP1)-Y(I))/R(I)-T(IM1)*(Y(I)-Y(IM1))/R(IM1)-ALPHA*
     1V(I,2)-MASS*G
      A(I,2)=A(I,2)/MASS
  140 CONTINUE
C   COMPUTE VELOCITIES,POSITIONS,CURRENT TIMESTEP
      GO TO(150,160),ISTEP
  150 DO 155 I=2,NM1
      V(I,2)=V(I,1)+DT*A(I,2)
  155 CONTINUE
      ISTEP=2
      GO TO 170
  160 DO 165 I=2,NM1
      V(I,2)=V(I,1)+DT*(1.5*A(I,2)-0.5*A(I,1))
  165 CONTINUE
  170 DO 175 I=2,NM1
      Y(I)=Y(I)+DT2*(V(I,2)+V(I,1))
  175 CONTINUE
      IF(MOD(K,IPRINT).NE.0)GO TO 180
C   PRINT
      PRINT 2002,K
      PRINT 2003,(Y(I),I=1,N)
  180 IF(K.LT.KMAX)GO TO 39
      IF(IV.EQ.0)GO TO 15
      IF(ITEPS.EQ.0)GO TO 9
      IF(IEND.EQ.0)GO TO 5
      STOP
      END
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15. Supplementary Notes

16. Abstracts

It is shown how to generate solitary waves by means of discrete string vibrations. Both linear and nonlinear models are studied with equal ease. Computer examples are given of the interaction of solitary waves, and their passage through each other.

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