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THE PERIHELION OF MERCURY

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Abstract. Local finite Taylor series expansion is shown to be an efficient numerical method for the study of the n-body problem. Application is made to the accurate determination of the motion of the perihelion of Mercury according to Newtonian celestial mechanics.

1. Introduction.

Let $\underline{x}_i$ $(i = 1, 2, \ldots, n)$ be the respective positions in three-dimensional Euclidean space of n bodies P_i with respective masses m_i. Let $\underline{v}_i$ be the velocity vector associated with P_i. Thus $d\underline{x}_i/dt = \underline{v}_i$. Newton's law of gravitation states that

$$d\underline{v}_i/dt = -\sum_{\substack{j=1\\ j\neq i}}^{n} \frac{Gm_j}{r_{ij}^3}(\underline{x}_i - \underline{x}_j), \quad (i = 1, 2, \ldots, n) \tag{1.1}$$

where $r_{ij} = \|\underline{x}_i - \underline{x}_j\|$, G is the gravitational constant, and r_{ij} is assumed to be non-zero.

Recursion formulas will be developed for the determination of Taylor series coefficients to be used in the numerical solution of the system of equations (1.1). A procedure will be given for the automatic selection of variable step size to be used in successive finite (truncated) Taylor series expansions and we will discuss the choice of order to be used. The resulting methods are applied to a detailed study of the motion of the perihelion of the planet Mercury over a 100 year period (the Julian century 1850.0-1950.0).

2. Recursion Formulas for Taylor Coefficients.

For scalar and vector valued functions of time we define the notation

$$(u)_k = \frac{1}{k!}\frac{d^k}{dt^k}u(t) .$$

Notice that $(u)_0 = u(t)$. Thus $(u)_k$ is the kth Taylor coefficient of u at t.

The following general recursion formulas are required (see: R. E. Moore, "Interval Analysis", Prentice-Hall, N.J., 1966, chap. 11)*.

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*The formula (11-14) on p. 114 of reference should read:

$$(P^a(x))_j = \frac{1}{P(x)} \sum_{i=0}^{j-1} \left(a - \frac{i(a+1)}{j}\right) (P(x))_{j-i} (P^a(x))_i .$$

$$
\begin{aligned}
(u+v)_k &= (u)_k + (v)_k \\
(u-v)_k &= (u)_k - (v)_k \\
(uv)_k &= \sum_{q=0}^{k} (u)_q (v)_{k-q} \\
(u^a)_k &= \frac{1}{u} \sum_{q=0}^{k-1} \left(a - \frac{q(a+1)}{k}\right)(u)_{k-q}(u^a)_q, \quad (u \neq 0,\ a \text{ is a real constant}) \\
((u)_1)_{k-1} &= k(u)_k \qquad (k = 1, 2, \dots) .
\end{aligned} \tag{2.1}
$$

We will use the notation (u,v) for the inner product

$$(u,v) = u_1 v_1 + u_2 v_2 + u_3 v_3 .$$

Using (2.1) we can derive the following recursion formulas for the Taylor coefficients of $\underline{x}_i$ satisfying (1.1).

$$
\begin{aligned}
(\underline{x}_i)_k &= \frac{1}{k} (\underline{v}_i)_{k-1} \\
(r_{ij})_k &= \frac{1}{r_{ij}} \left\{ \sum_{q=0}^{k-1} \left(1 - \frac{q}{k}\right)\left((\underline{x}_i)_q - (\underline{x}_j)_q,\ (x_i)_{k-q} - (x_j)_{k-q}\right) - \begin{pmatrix} 0, \text{ if } k = 1 \\ \sum_{q=1}^{k-1} \left(1 - \frac{q}{k}\right)(r_{ij})_q (r_{ij})_{k-q}, \text{ if } k > 1 \end{pmatrix} \right\} \\
(r_{ij}^{-3})_k &= \frac{1}{r_{ij}} \sum_{q=0}^{k-1} \left(-3 + \frac{2q}{k}\right)(r_{ij})_{k-q}(r_{ij}^{-3})_q \\
(\underline{v}_i)_k &= -\frac{1}{k} \sum_{\substack{j=1 \\ j \neq i}}^{n} Gm_j \left(\sum_{q=0}^{k-1} \left((\underline{x}_i)_q - (\underline{x}_j)_q\right)(r_{ij}^{-3})_{k-1-q} \right)
\end{aligned} \tag{2.2}
$$

$(k = 1, 2, \dots)$

Using (2.2) we fill out an array of values beginning with initial conditions for $\underline{x}_i$, $\underline{v}_i$ and calculate, for a given value of k, the quantities: $(\underline{x}_i)_k$, $(r_{ij})_k$, $(r_{ij}^{-3})_k$, $(\underline{v}_i)_k$ from stored values for all previous values of k beginning with $k = 0$. For the quantities $(r_{ij})_k$ and $(r_{ij}^{-3})_k$ we need only use (2.2) when $i < j$. We can fill out the remainder of the array by using the relations

$$(r_{ji})_k = (r_{ij})_k$$

$$(r_{ji}^{-3})_k = (r_{ij}^{-3})_k .$$

The quantities $(r_{ii})_k$ and $(r_{ii}^{-3})_k$ do not appear.

Using the coefficients obtained in this way, we have the following K^{th} order (finite) Taylor series expansions

$$\begin{aligned} \underline{x}_i(t+h) &= \sum_{q=0}^{K} (\underline{x}_i)_q \, h^q \\ \underline{v}_i(t+h) &= \sum_{q=0}^{K} (\underline{v}_i)_q \, h^q \\ r_{ij}(t+h) &= \sum_{q=0}^{K} (r_{ij})_q \, h^q . \end{aligned} \qquad (2.3)$$

Of course, we can find $r_{ij}^{-3}(t+h)$ from

$$r_{ij}^{-3}(t+h) = (r_{ij}(t+h))^{-3} .$$

The coefficients and series (2.3) are evaluated again at $t+h$

to carry the solution further. The method proceeds in this way in a step-by-step fashion.

3. Step Size and Order.

Numerous theoretical and experimental studies* have shown that, in general, an approximately "optimal" choice for successive step sizes h_t in successive applications of a step-by-step K^{th} order method such as that given by (2.3) is as follows

$$h_t = \left(\frac{N_0}{N_t}\right)^{\frac{1}{K}} h_0$$

where N_t is some functional representing an estimate of the coefficient of a leading error term of interest and h_0 is the initial step size. This has the effect of maintaining nearly constant "local truncation error."

In "Interval Analysis," (chap. 12) it is shown that an efficient choice for the order K in using K^{th} order Taylor series to achieve approximately d decimal digit accuracy locally is $K = d$.

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*See R. E. Moore: "Interval Analysis," pp. 101-102; also J. W. Daniel & R. E. Moore: "Computation & Theory in Ordinary Differential Equations", W. H. Freeman & Co., S.F., 1970, chap. 9.

Thus the most efficient order K for <u>maximum</u> possible accuracy in single-precision on the UNIVAC 1108 (8 decimal digits) is about $K = 8$. In double-precision on the UNIVAC 1108 (18 decimal digits) the most efficient choice of K for <u>maximum</u> possible accuracy is about $K = 18$. For $p \leq 18$ decimal digit accuracy (locally; that is, at each step) we can take (in double precision computation on the UNIVAC 1108)

$$h_t = \frac{1}{10(N_t(\underline{x}_i))^{1/p}} \tag{3.1}$$

in order to maintain a local truncation error of about $N_t(\underline{x}_i)\, h_t^p = 10^{-p}$ in the term $\underline{x}_i(t)$. We can select, in this way, one of the n bodies in (1.1) for our particular attention. For $N_t(\underline{x}_i)$, we can use

$$N_t(\underline{x}_i) = \| (\underline{x}_i)_p \| \; . \tag{3.2}$$

After a number of trial calculations of the perihelion motion of Mercury using $K = 8$, $K = 12$, and $K = 18$ we settled upon the use of double precision (18 decimals) and $K = 12$ as a reasonable compromise between speed and accuracy for the main calculations to be reported on in the next section.

4. The Perihelion of Murcury

In 1947, G. M. Clemence* announced agreement of a prediction of Einstein's general theory of relativity concerning the motion of the perihelion of Mercury with long, existing discrepancies between observation and calculations based on Newtonian celestial mechanics.

We have undertaken carefully to verify or repudiate that claim by carrying out highly accurate computations based on a Newtonian model of motions of the solar system, using the exceptional power of modern computers. Indeed, the method we have used, based on the formulas presented in the previous sections, is particularly well suited for stored program computers, but hardly ideal for hand computation or desk computers, since it requires the intermediate storage of a rather large amount of data and the use of a fairly complicated set of formulas. The probject to be described now was carried out using FORTRAN programming for the UNIVAC 1108 at the University of Wisconsin.

According to Chebotarev≠ accelerations relative to the center of the galaxy on objects in the solar system due to the motion of the solar system about the center of the galaxy amount to about $2 \cdot 10^{-9}$

*G. M. Clemence, "The Relativity Effect in Planetary Motions," Reviews of Modern Physics, Vol. 19, No. 4, Oct. 1947, pp. 361-64.

≠G. A. Chebotarev, "Analytical and Numerical Methods of Celestial Mechanics," Elsevier, N. Y., 1967, p. 28.

in the units we will use: A. U., "Astronomical Unit" for a unit of distance, and 10 days (= $\frac{1}{36.525}$ Julian years) for a unit of time. As will be seen from comparison with numbers to follow, we can neglect these accelerations and consider the solar system to be in an "inertial frame of reference" with its center of mass at the origin.

For our first computation, we started at 1951.0 and ran backwards in time to 1850.0. As can be seen from (1.1) the solutions are invariant with respect to a simultaneous change in sign of t and $\underline{v}_i$.

In what follows we will take X, Y, Z to be an inertial coordinate system with origin at the center of mass of the "solar system" (which we will take to include only: the sun, Mercury, Venus, Earth, and Jupiter since these are the major influences on the perihelion motion of Mercury). The X-axis will point toward the vernal equinox of 1951.0. The X,Y plane will be parallel to the plane of the Earth's orbit around the sun at 1951.0. The Z-axis will be normal to the X,Y plane to make X, Y, Z a "right-handed" coordinate system. The equations of motion (1.1) are assumed to hold in the X, Y, Z coordinate system.

In a second, "heliocentric", x, y, z coordinate system - with the sun at the origin and with the axes parallel to the corresponding axes in the X, Y, Z system - the Earth is initially (at

1951.0) moving in the x,y plane. Again the x axis is pointed toward the vernal equinox of 1951.0. In this coordinate system the orbital elements of the (relevant) planets are given by Krogdahl* as

PLANET	a	e	i	Ω	ω	τ
Mercury	0.387	0.2056	$7^\circ 0' 14''$	$47^\circ 45'$	$28^\circ 57'$	1950.993
Venus	0.723	0.0068	$3^\circ 23' 39''$	$76^\circ 14'$	$54^\circ 39'$	1950.700
Earth	1.000	0.0167	0°	0°	$120^\circ 6'$	1951.008
Jupiter	5.203	0.0484	$1^\circ 18' 21''$	$99^\circ 57'$	$273^\circ 35'$	1951.891

The "elements" given are: a, the semi-major axis; e, the eccentricity; i, the inclination of the planets orbital plane to the "ecliptic" (Earth's orbital plane); Ω, the longitude of the ascending node; ω, the longitude (in the planet's orbital plane from the ascending node) of the perihelion point; and τ, the time of perihelion passage.

A set of transformations from the orbital elements to the rectangular heliocentric coordinates x,y,z are given (and were programmed) as follows≠ (the subscript i denotes the planet: i = 1,

*W. S. Krogdahl, "The Astronomical Universe", The Mac Millan Co., N. Y., 1952, p. 95.

≠See, e.g., A. E. Roy, "The Foundations of Astrodynamics", MacMillan, N.Y., 1965, p. 106.

sun; i = 2, Mercury; i = 3, Venus; i = 4, Earth; i = 5, Jupiter -- to avoid confusion, we will denote "inclination" by "I"). For i = 2,3,4,5, we have

$$n_i = (Gm_1 + Gm_i)^{1/2}\, a_i^{-3/2} \tag{4.1}$$

$$E_i - e_i \sin E_i = n_i(t - \tau_i) \tag{4.2}$$

$$\ell_{1,i} = \cos\Omega_i \cos\omega_i - \sin\Omega_i \sin\omega_i \cos I_i \tag{4.3}$$

$$m_{1,i} = \sin\Omega_i \cos\omega_i + \cos\Omega_i \sin\omega_i \cos I_i \tag{4.4}$$

$$n_{1,i} = \sin\omega_i \sin I_i \tag{4.5}$$

$$\ell_{2,i} = -\cos\Omega_i \sin\omega_i - \sin\Omega_i \cos\omega_i \cos I_i \tag{4.6}$$

$$m_{2,i} = -\sin\Omega_i \sin\omega_i + \cos\Omega_i \cos\omega_i \cos I_i \tag{4.7}$$

$$n_{2,i} = \cos\omega_i \sin I_i \tag{4.8}$$

$$b_i = a_i(1 - e_i^2)^{1/2} \tag{4.9}$$

$$x_i = a_i\, \ell_{1,i} \cos E_i + b_i\, \ell_{2,i} \sin E_i - a_i e_i \ell_{1,i} \tag{4.10}$$

$$y_i = a_i m_{1,i} \cos E_i + b_i m_{2,i} \sin E_i - a_i e_i m_{1,i} \tag{4.11}$$

$$z_i = a_i n_{1,i} \cos E_i + b_i n_{2,i} \sin E_i - a_i e_i n_{1,i} \tag{4.12}$$

$$r_i = (x_i^2 + y_i^2 + z_i^2)^{1/2} \tag{4.13}$$

$$\dot{x}_i = \frac{n_i a_i}{r_i} (b_i\, \ell_{2,i} \cos E_i - a_i\, \ell_{1,i} \sin E_i) \tag{4.14}$$

$$\dot{y}_i = \frac{n_i a_i}{r_i} (b_i\, m_{2,i} \cos E_i - a_i\, m_{1,i} \sin E_i) \tag{4.15}$$

$$\dot{z}_i = \frac{n_i a_i}{r_i} (b_i\, n_{2,i} \cos E_i - a_i\, n_{1,i} \sin E_i) \tag{4.16}$$

A few comments are in order concerning the transformation formulas. Equations (4.1) through (4.9) introduce auxiliary variables. Equation (4.2) defines E_i, the "eccentric anomaly" of planet i, implicitly. We use "Newton's method" to find E_i; putting $E_i^o = n_i(t-\tau_i)$ where $t = 1951.0$ (after converting time to units of 10 days) we iterated the formula

$$E_i^{(p+1)} = E_i^{(p)} - \frac{E_i^{(p)} - e_i \sin E_i^{(p)} - n_i(t-\tau_i)}{1 - e_i \cos E_i^{(p)}}$$

$p = 0,1,2,\ldots$, until* $|E_i^{(p+1)} - E_i^{(p)}| < 10^{-6}$, and then took $E_i^{(p+1)}$ for E_i. The resulting heliocentric initial conditions were (from computer print-out).

i	x_i	y_i	z_i
2	$-.14271822.10^{-1}$	$.30803245$	$.26742133.10^{-1}$
3	$.43322016$	$-.58400374$	$-.33198056.10^{-1}$
4	$-.15591831$	$.97088167$	0
5	$.47831822.10^{1}$	$-.13874603.10^{1}$	$-.10192802$

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* giving $|E_i^{(p+1)} - E_i| \approx 10^{-12}$.

i	$\dot{x}_i$	$\dot{y}_i$	$\dot{z}_i$
2	$-.33735568$	$-.20254059.10^{-2}$	$.30511267.10^{-1}$
3	$.16112397$	$.11982741$	$-.75903491.10^{-2}$
4	$-.17267800$	$-.27882252.10^{-1}$	0
5	$.20172708.10^{-1}$	$.76103702.10^{-1}$	$-.75267269.10^{-3}$

The gravitational constant was combined with the masses of the planets and the following values were used (in accordance with Clemence, Rev. Mod. Phys., 1947, p. 363.)

Sun: $Gm_1 = .29591220.10^{-1}$ (A.U.^{3}/(10 days)^2)

Mercury: $Gm_2 = .4931870.10^{-8}$

Venus: $Gm_3 = .7252750.10^{-7}$

Earth: $Gm_4 = .898364.10^{-7}$

Jupiter: $Gm_5 = .2825234.10^{-4}$

To find the X, Y, Z coordinates with respect to the center of mass of the solar system, the following transformations are used

$$GM = \sum_{i=1}^{5} Gm_i$$

$$\underline{x}_c = \frac{1}{GM} \sum_{i=2}^{5} Gm_i \, \underline{x}_i$$

$$\dot{\underline{x}}_c = \frac{1}{GM} \sum_{i=2}^{5} Gm_i \, \dot{\underline{x}}_i$$

$$\underline{X}_i = \underline{x}_i - \underline{x}_c$$

$$(i = 1,2,3,4,5)$$

$$\underline{V}_i = \dot{\underline{x}}_i - \dot{\underline{x}}_c$$

Let $\underline{X}_i = (X_i, Y_i, Z_i)$ and $\underline{V}_i = (VX_i, VY_i, VZ_i)$.

The initial conditions in the inertial (X,Y,Z) coordinate system which result are as follows: (at t = 1951.0)

planet	i	X_i	Y_i	Z_i
sun	1	$-.45629668.10^{-2}$	$.13218465.10^{-2}$	$.97299666.10^{-4}$
Mercury	2	$-.18834788.10^{-1}$	$.30935429$	$.26839433.10^{-1}$
Venus	3	$.42865720$	$-.58268189$	$-.33100756.10^{-1}$
Earth	4	$-.16048128$	$.97220352$	$.97299666.10^{-4}$
Jupiter	5	$.47786192.10^{1}$	$-.13861385.10^{1}$	$-.10183072$

planet	i	VX_i	VY_i	VZ_i
sun	1	$.19056125.10^{-4}$	$.72799117.10^{-4}$	$-.73143349.10^{-6}$
Mercury	2	$.33737474$	$.20982050.10^{-2}$	$-.30511998.10^{-1}$
Venus	3	$-.16110492$	$-.11975461$	$.75896176.10^{-2}$
Earth	4	$.17269706$	$.27955051.10^{-1}$	$-.73143349.10^{-6}$
Jupiter	5	$-.20153652.10^{-1}$	$-.76030902.10^{-1}$	$.75194126.10^{-3}$

The signs of the initial velocities were changed, replacing $\underline{V}_i$ by $-\underline{V}_i$ $(i = 1,2,3,4,5)$ and dt was replaced by $-dt$. The system (1.1) then was solved, using the method described in Sections 2 and 3, for the period from 1951.0 back to 1850.0. Numerical results are given in the section following this.

In order to study the motion of the perihelion of Mercury during the century in question, the following items were computed and printed for each of 419 orbits of Mercury:

time at perihelion,	T
position of Mercury at perihelion	(X_2, Y_2, Z_2)
longitude of perihelion,	THETA
(relative to the vernal equinox of 1951.0),	
perihelion distance,	$PD(= \Vert \underline{X}_2 - \underline{X}_1 \Vert)$

As a check on accuracy, all ten known integrals to the n-body problem were computed and printed at each perihelion. These quantities, which remain constant for an exact solution, are as follows:

$$c_1 = \sum_{i=1}^{5} Gm_i X_i$$

$$c_2 = \sum_{i=1}^{5} Gm_i Y_i$$

$$c_3 = \sum_{i=1}^{5} Gm_i Z_i$$

$$c_4 = \sum_{i=1}^{5} Gm_i VX_i$$

$$c_5 = \sum_{i=1}^{5} Gm_i VY_i$$

$$c_6 = \sum_{i=1}^{5} Gm_i VZ_i$$

$$c_7 = \sum_{i=1}^{5} Gm_i(X_i \cdot VY_i - Y_i \cdot VX_i)$$

$$c_8 = \sum_{i=1}^{5} Gm_i(Y_i \cdot VZ_i - Z_i \cdot VY_i)$$

$$c_9 = \sum_{i=1}^{5} Gm_i(Z_i \cdot VX_i - X_i \cdot VZ_i)$$

$$c_{10} = \sum_{i=1}^{5} \frac{1}{2} Gm_i \{VX_i^2 + VY_i^2 + VZ_i^2\} - \sum_{i<j} \frac{Gm_i Gm_j}{R_{ij}}$$

where $R_{ij} = \|\underline{X}_i - \underline{X}_j\|$ as in (1.1).

The quantities c_1, c_2, c_3 are the coordinates of the center of mass and should remain zero. Similarly c_4, c_5, c_6 are the velocity components of the center of mass and should also remain zero. Now, c_7, c_8, c_9 are the components of total angular momentum of the system and should remain constant and equal to their initial values. Finally c_{10} is the total energy of the system and this should remain constant also and equal to its initial value.

The <u>time at perihelion</u> - when the planet Mercury is closest to the sun during an orbit -- is determined as follows. We keep track of $(R_{12})_1$, which is the time derivative of the distance between Mercury and the sun. During the evaluation of the Taylor coefficients (2.2), $(R_{12})_q$, $(q = 1,2,\ldots,K)$, are computed, say at time t. Then we also evaluate

$$(R_{12})_1(t+h) = \sum_{q=1}^{K} (R_{12})_q \cdot q \cdot h^{q-1} .$$

We are going to use Newton's method to approximate h_1 such that $(R_{12})_1(t+h_1) = 0$. Therefore we also need $\frac{d}{dt}(R_{12})_1(t+h)$ or

$$2(R_{12})_2(t+h) = \sum_{q=2}^{K} (R_{12})_q (q)(q-1)\, h^{q-2} .$$

We then find h_1 as follows. First, to distinguish perihelion from aphelion (farthest distance from the sun on Mercury's orbit) we test to see whether the following two conditions are both met (where h is the step size h_t determined by the method of section 3, using

$$N_t(\underline{X}_2) = |(X_2)_K| + |(Y_2)_K| + |(Z_2)_K|):$$

1) $(R_{12})_1 < 0$

2) $(R_{12})_1(t+h) > 0$.

When 1) and 2) occur together, there is an h_1 such that $(R_{12})_1$ $(t+h_1) = 0$ and such that Mercury is at perihelion at time $t + h_1$. When 1) and 2) occur together, we put

$$h_1^{(0)} = h$$

and iteratively determine $h_1^{(p+1)}$ $(p = 0, 1, 2, \ldots)$ from

$$h_1^{(p+1)} = h_1^{(p)} - \frac{(R_{12})_1 (t+h_1^{(p)})}{2(R_{12})_2 (t+h_1^{(p)})}$$

until* $|h_1^{(p+1)} - h_1^{(p)}| < 10^{-8}$. Then we take $h_1 = h_1^{(p+1)}$ (the final iterate).

To find the <u>position of Mercury at perihelion</u>, we use the h_1 just determined and evaluate the finite series with vector coefficients

$$\underline{X}_2(t+h_1) = \sum_{q=0}^{K} (\underline{X}_2)_q \, h_1^q ,$$

or, in component form,

*In the single precision version we use $|h_1^{(p+1)} - h_1^{(p)}| < 10^{-6}$.

$$X_2(t+h_1) = \sum_{q=0}^{K} (X_2)_q \, h_1^q$$

$$Y_2(t+h_1) = \sum_{q=0}^{K} (Y_2)_q \, h_1^q$$

$$Z_2(t+h_1) = \sum_{q=0}^{K} (Z_2)_q \, h_1^q \quad .$$

To get the perihelion distance, we evaluate the three series for the components of

$$\underline{X}_1(t+h_1) = \sum_{q=0}^{K} (\underline{X}_1)_q \, h_1^q$$

and compute

$$\begin{aligned} PD &= \|\underline{X}_2(t+h_1) - \underline{X}_1(t+h_1)\| \\ &= ((X_2(t+h_1) - X_1(t+h_1))^2 + (Y_2(t+h_1) - Y_1(t+h_1))^2 \\ &\quad + (Z_2(t+h_1) - Z_1(t+h_1))^2)^{1/2} \quad . \end{aligned}$$

The longitude of perihelion (relative to the vernal equinox of 1951.0) is defined as the sum of the two angles ω and Ω mentioned near the beginning of this section (in connection with the table of orbital elements of the planets). Actually, at any given time the "vernal equinox is defined as a directed half-line in the heliocentric coordinate system from the sun toward the "fixed stars" along which the plane of the Earth's orbit (the "ecliptic") and the plane of the Earth's equator intersect. In reality, the vernal equinox is not fixed but moves relative to the heliocentric coordinate system in a slow arc due to the precession of the Earth's axis of rotation.

We calculate ω, the angle in the plane of Mercury's orbit from the ascending node (where the plane of Mercury's orbit intersects the plane of the Earth's orbit) and Ω, the angle between the x-axis and the ascending node.*

Strictly speaking this is not quite the difference between the correctly defined "heliscentric longitude of the perihelion of Mercury relative to the moving equinox" and the angle of precession of the equinox. But the difference between $\omega + \Omega$ as we have defined it above and the correct version: $\omega + \Omega + (\Omega_t - \Pi_t - \Omega)$ due to motion of the ecliptic relative to the inertial frame is $\Omega_t - \Pi_t - \Omega$, where Ω_t is the angle in the ecliptic plane from the moving equinox to the ascending node and Π_t is the angle from the moving equinox to the x-axis (equinox of 1951.0). This difference is of the order $1.6 \cdot 10^{-8}$ radians and does not affect our interpretation of the numberical results.

*See, e.g., Krogdahl, Chebotarev, or Roy.

We turn finally to a description of our method for computing the angles ω and Ω. At any fixed time t the plane of the Earth's orbit around the sun is defined by the vectors

$$XE = \underline{X}_4 - \underline{X}_1$$

and

$$VXE = \underline{V}_4 - \underline{V}_1 .$$

Similarly the plane of the orbit of Mercury is defined by

$$XM = \underline{X}_2 - \underline{X}_1$$

and

$$VXM = \underline{V}_2 - \underline{V}_1 .$$

Thus a point V is in the intersection of these two planes (the line of nodes) if for some α, β, γ, δ

$$V = \alpha XE + \beta VXE = \gamma XM + \delta VXM .$$

Let us define a unit vector on the line of nodes. If we choose t, XM, VXM when Mercury is at perihelion, then $(XM, VXM) = 0$ and we will have $\|V\| = 1$ if

$$\gamma^2 \|XM\|^2 + \delta^2 \|VXM\|^2 = 1 .$$

We will assume that XM, XE, and VXE are linearly independent, then

$$VXM = \bar{c}_1 XM + \bar{c}_2 XE + \bar{c}_3 VXE$$

for some $\bar{c}_1, \bar{c}_2, \bar{c}_3$ (since we are in a three-dimensional space). Then

$$(\gamma + \delta\bar{c}_1)\,XM + (\delta\bar{c}_2 - \alpha)\,XE + (\delta\bar{c}_3 - \beta)\,VXE = 0$$

and so $\gamma = -\delta\bar{c}_1$ (and also $\alpha = \delta\bar{c}_2$, $\beta = \delta\bar{c}_3$). We find, then, that

$$\delta = \left(\frac{1}{\|VXM\|^2 + \bar{c}_1^2\,\|XM\|^2}\right)^{1/2},$$

and

$$V = \delta(VXM - \bar{c}_1 XM) .$$

We need now to find $\bar{c}_1$. We can do this by taking the inner product of both sides of the equation

$$\bar{c}_1 XM + \bar{c}_2 XE + \bar{c}_3 VXE = VXM$$

with each of the three vectors XM, XE, and VXE. There results a system of three linear algebraic equations in $\bar{c}_1, \bar{c}_2, \bar{c}_3$ which we can solve for $\bar{c}_1$. A formal solution for $\bar{c}_1$, which we used in the computations, is

$$\bar{c}_1 = \frac{-\left(\frac{A_{12}}{A_{22}}\right) A_{24} - RR\left(A_{34} - \left(\frac{A_{23}}{A_{22}}\right) A_{24}\right)}{A_{11} - \left(\frac{A_{12}}{A_{22}}\right) A_{12} - RR\left(A_{13} - \left(\frac{A_{23}}{A_{22}}\right) A_{12}\right)}$$

where
$$RR = \frac{A_{13} - \left(\frac{A_{12}}{A_{22}}\right) A_{23}}{A_{33} - \left(\frac{A_{23}}{A_{22}}\right) A_{23}}$$

and where the A_{ij} are inner products which are defined as follows:

$$A_{11} = (XM, XM)$$

$$A_{12} = (XM, XE)$$

$$A_{13} = (XM, VXE)$$

$$A_{22} = (XE, XE)$$

$$A_{23} = (XE, VXE)$$

$$A_{24} = (XE, VXM)$$

$$A_{33} = (VXE, VXE)$$

$$A_{34} = (VXE, VXM) .$$

Finally, we have

$$\omega = \arc\cos \left\{ \frac{(V, XM)}{\| V \| \cdot \| XM \|} \right\}$$

$$= \arc\cos \left\{ \frac{-\bar{c}_1 \| XM \|}{\| VXM - \bar{c}_1 XM \|} \right\}$$

Actually we used the arctangent function. If

$$\omega = \text{arc cos } Q, \text{ then } \omega = \text{arc tan}\left(\frac{\sqrt{1-Q^2}}{Q}\right)$$

so we take

$$\omega = \text{arc tan}\left(\frac{\sqrt{1-Q^2}}{Q}\right)$$

where

$$Q = \frac{-\bar{c}_1 \, \|XM\|}{\|VXM - \bar{c}_1 XM\|} \quad .$$

The minus sign gives us the ascending node on the line of nodes.

For Ω we have

$$\Omega = \text{arc tan}\left(\frac{\sqrt{1-U^2}}{U}\right) \quad ,$$

where

$$U = \frac{VXM_1 - \bar{c}_1 XM_1}{\|VXM - \bar{c}_1 XM\|} \quad ,$$

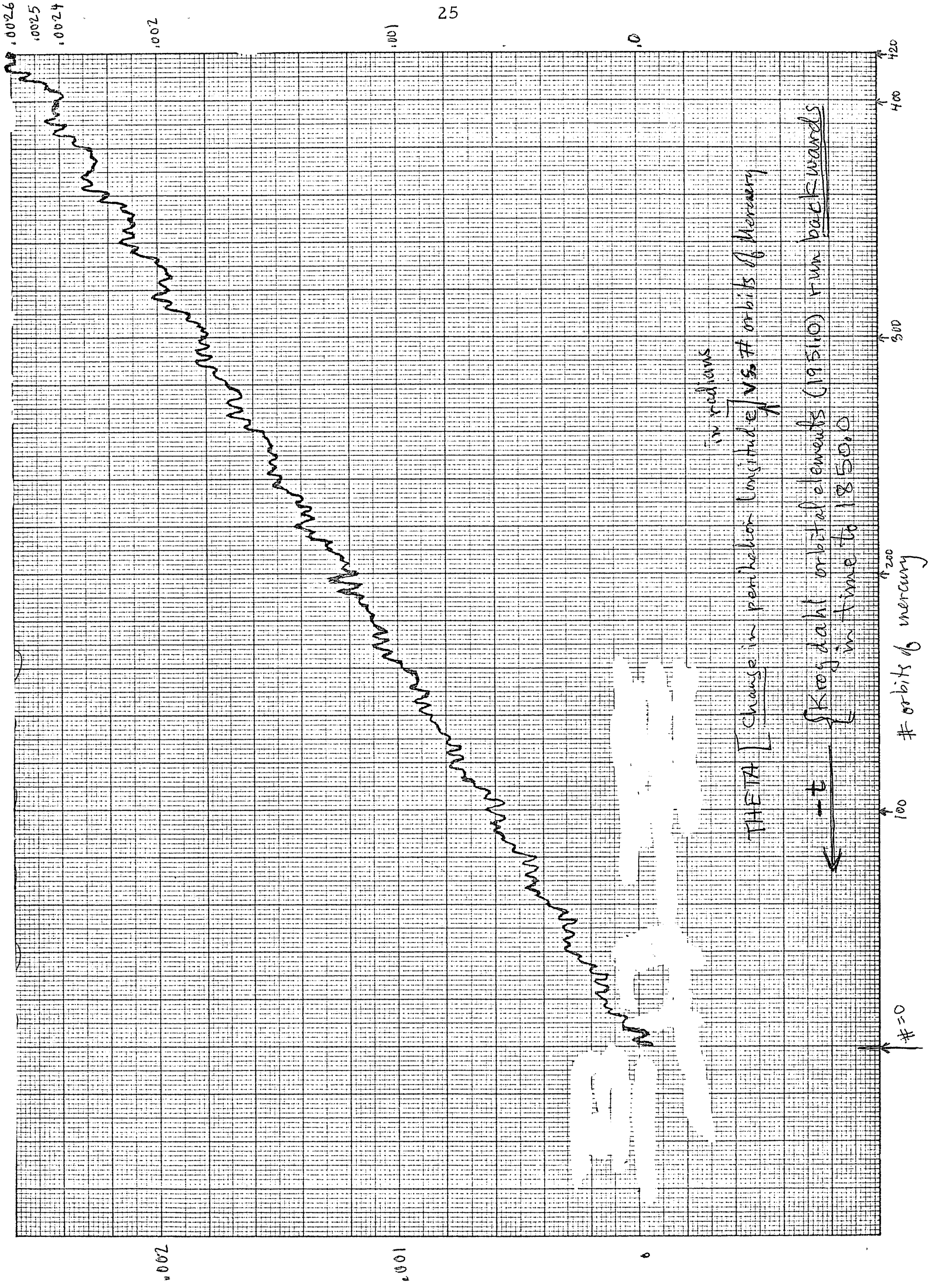
and VXM_1, XM_1 are the <u>first components</u> of the vectors VXM and XM.

To study the motion of the perihelion of Mercury, we compute and print out the <u>change</u> in <u>longitude of perihelion</u>

$$\text{THETA} = \omega + \Omega - (\omega_0 + \Omega_0)$$

where ω_0 and Ω_0 are the initial values of ω and Ω for Mercury.

The results obtained for THETA vs. # orbits of Mercury (perihelion to perihelion) are shown in the following figure. A 12^{th} order expansion was used at each step of the numerical integration. Thus we used $K = 12$ in formulas (2.3). Double precision was used (18 decimals). The computing time was 53 minutes, 25 seconds. The ten integrals $c_1, c_2, \ldots, c_{10}$ all remained constant throughout to <u>at least</u> 8 decimals.


THETA [Change in perihelion longitude] in radians VS. # orbits of Mercury
−t ← Krogdahl orbital elements (1951.0) run backwards in time to 1850.0
orbits of mercury
#=0
100
200
300
400
420
0
.001
.002
.0024
.0025
.0026

A second set of initial conditions was then tried. This time we took the orbital elements given for the planets at 1850.0 by G. M. Clemence ["First order theory of Mars," Astronomical Papers: American Ephemeris, Vol. XI, Part II, United States Government Printing Office, Washington, 1949, pp. 231-232].

Planet	i	a	e	i
Mercury	2	0.3870 986713	0.2056 0396	$7^\circ\ 0'\ 7''$
Venus	3	0.7233 322169	0.0068 4458	$3^\circ\ 23'\ 35.26''$
Earth	4	1.0000 00021	0.0167 7126	0°
Jupiter	5	5.2028 03945	0.0482 5382	$1^\circ\ 18'\ 41.81''$

Planet	i	$\theta[=\Omega]$	$\Pi[=\omega+\Omega]$
Mercury	2	$46^\circ\ 33'\ 12.24''$	$75^\circ\ 7'\ 19.37''$
Venus	3	$75^\circ\ 19'\ 47.41''$	$129^\circ\ 27'\ 34.5''$
Earth	4	0°	$100^\circ\ 21'\ 36.30''$
Jupiter	5	$98^\circ\ 55'\ 58.16''$	$11^\circ\ 54'\ 26.72''$

Planet	i	ϵ_0	$n\ (\frac{\text{seconds}}{\text{year}})$
Mercury	2	$323^\circ\ 11'\ 23.53''$	538 1016.3893
Venus	3	$243^\circ\ 57'\ 44.19''$	210 6641.4171
Earth	4	$99^\circ\ 48'\ 18.56''$	129 5997.4496
Jupiter	5	$159^\circ\ 56'\ 25.05''$	109 256.6395

The quantity ϵ_0 given here is defined by

$$\epsilon_0 - \omega - \Omega = n(t - \tau),$$

in terms of our previous variables. We had to convert the given values for n to our units of $(\frac{\text{radians}}{10 \text{ days}})$. Calling the values in the table $\bar{n}$, we put

$$n = \frac{\bar{n}}{(36.525)(57.295779)(3600)} .$$

Actually we did not need to solve for τ_i, but took, instead $\epsilon_{0,i} - \omega_i - \Omega_i$ for the right hand side of equation (4.2), which is used to determine E_i. We needed n_i as well for equations (4.14) - (4.16) to get $\dot{x}_i$, $\dot{y}_i$, $\dot{z}_i$.

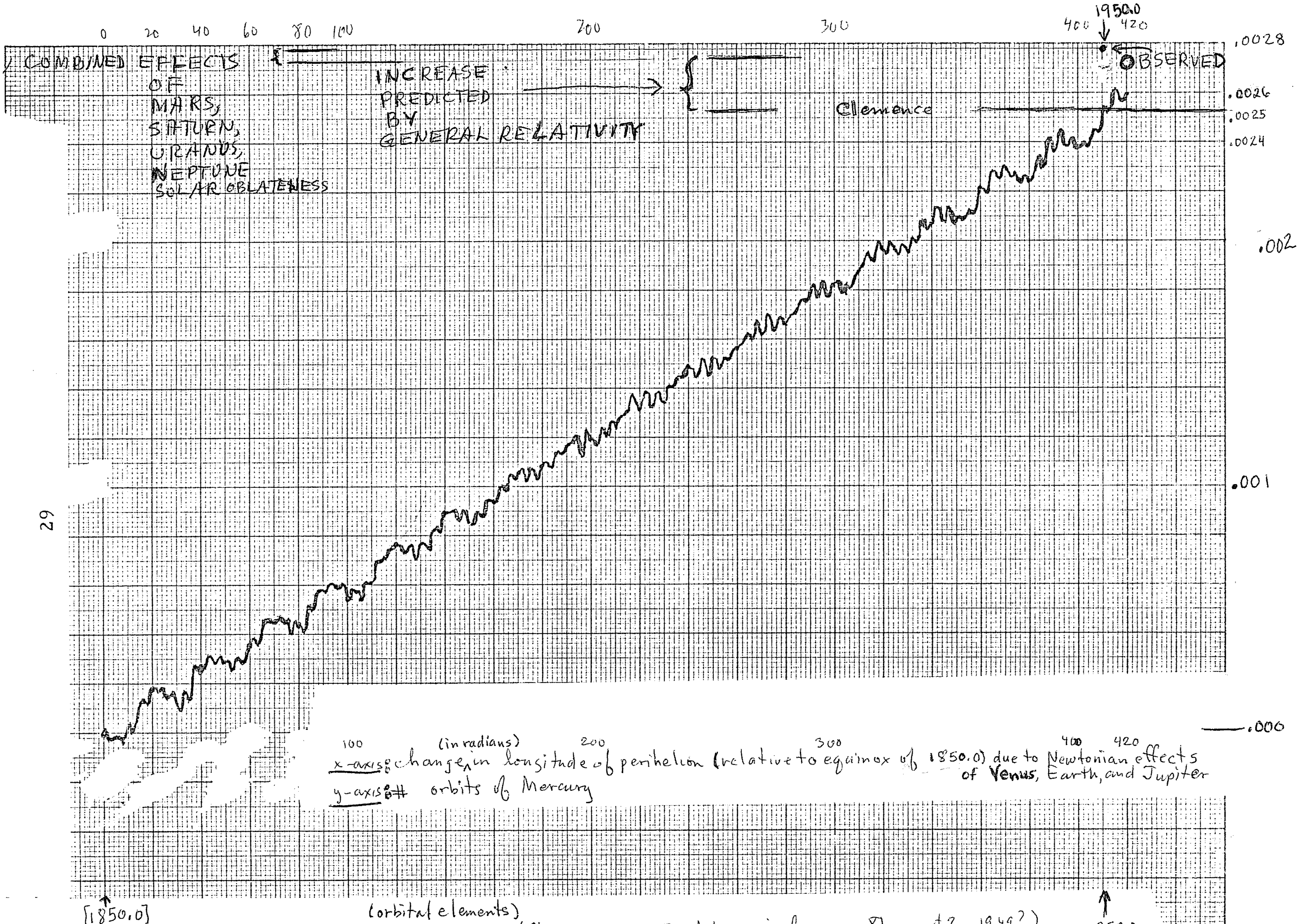
This time we integrated forward in time from 1850.0 to 1950.0. Notice that we have, in this case, an inertial frame which puts the x,y plane in the ecliptic of 1850.0 and the x axis is pointed toward the vernal equimax of 1850.0.

Again using double precision (18 decimals) and K = 12 (12th order Taylor expansions at each integration step), we computed the perihelion motion for 1850.0 - 1950.0. The results are shown in the next figure. There are only very minor differences between these and the previous results (using the 1951.0 elements and running backwards to 1850.0). The total computing time was 54 minutes 27 seconds.

This time we are putting in orbital elements to several more decimal places and so the results are based on more accurate initial conditions. Again the integrals $c_1, c_2, \ldots, c_{10}$ remained constant to at least 8 decimals. Also shown on the figure is the value .002528, reported by Clemence (1947, p. 363), for the motion of the perihelion due to the Newtonian gravitational effects of Venus, Earth, and Jupiter. Clemence (1947, p. 363) also reports a value for the "observed" motion relative to a fixed equinox (leaving out precession of the equinox) of .0027833. The effects of the other planets and solar oblateness (due to their Newtonian gravitational influence) adds up (Clemence (1947, p. 363)) to about .0000485. The _general theory of relativity_ predicts an increase of .000209 over what is predicted by Newtonian celestial mechanics. The sum of the Newtonian effects as computed by Clemence plus the relativistic term almost exactly equals the "observed" value.

All that _we_ have computed, of course, is the motion of the perihelion due to the combined Newtonian influence of Venus, Earth, and Jupiter. The value .002528, given by Clemence for this contribution, is in excellent agreement with our results.

From our computed _times of perihelion_, we can derive values for the sidereal period of Mercury. The _average_ period for the 415 orbits (perihelion to perihelion) during 1850.0 - 1950.0 according to



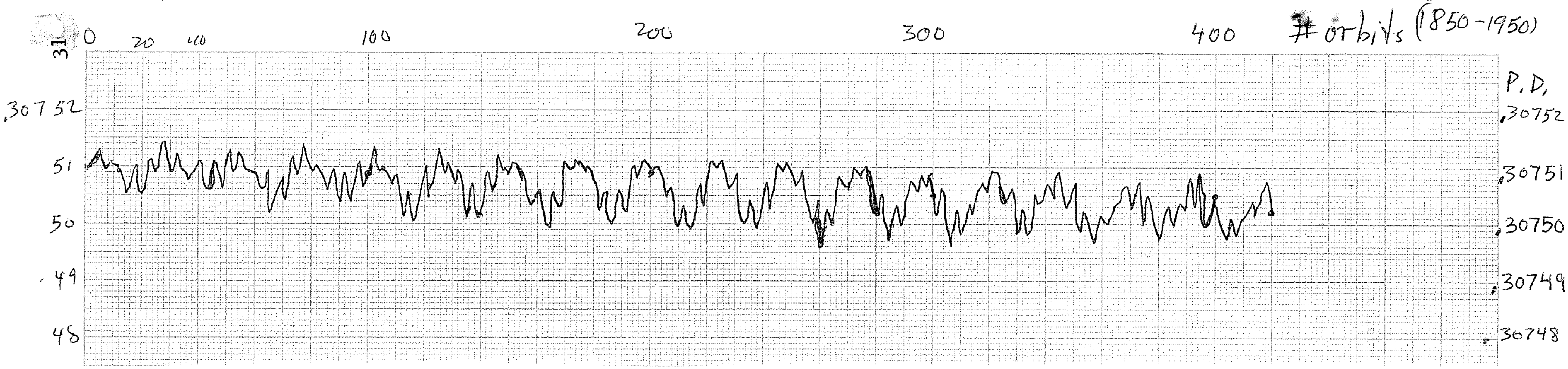
INITIAL DATA (orbital elements) AT 1850.0 (Clemence, p232, Astronomical papers XI, part 2, 1949?)

[Clemence (1947, p.363) predicts .002528 at 1950.0 for the quantity plotted in the graph]; general relativity theory predicts an increase of .000209 our calculations is 87.9697213 days (taking 1 year = 365.25 days).

Finally, in the last figure, the perihelion distance (Mercury to sun at perihelion of Mercury) is shown (vs. # of orbits of Mercury). There seems to be a small (secular) rate of decrease in addition to the fluctuations from orbit to orbit.

A rough estimate from the figure of this downward trend is $\frac{.000008 \text{ A.U.}}{100 \text{ years}}$. If this were to continue, Mercuy would, perhaps, collide with the sun within about $3.8 \cdot 10^6$ years!

On the other hand, there are some extremely long period phenomena involved here too. For instance at the average rate of about $.0025 \frac{\text{radians}}{100 \text{ years}}$, it will take about 252,000 years for the longitude of Mercury's perihelion to make a complete revolution of 2π radians. Thus the perihelion distance may also have some very long period fluctuations.



Perihelion distance of Mercury vs. # orbits of Mercury (1850.0-1950.0)