NUMERICAL STUDIES OF DISCRETE VIBRATING STRINGS

1. Introduction

The motion of a vibrating elastic string has been studied [4] using a discrete computer-oriented approach which is applicable to both linear and nonlinear models. The purpose here is to extend the investigation by allowing two-dimensional particle motion, by improving upon the relatively naive nonlinear models used in [4], and by considering several types of problems not explored previously.

2. Geometrical Discription of the String

The discrete string may be modeled geometrically in the following way (see Figure 1). Let the string be composed of $n + 1$ ordered

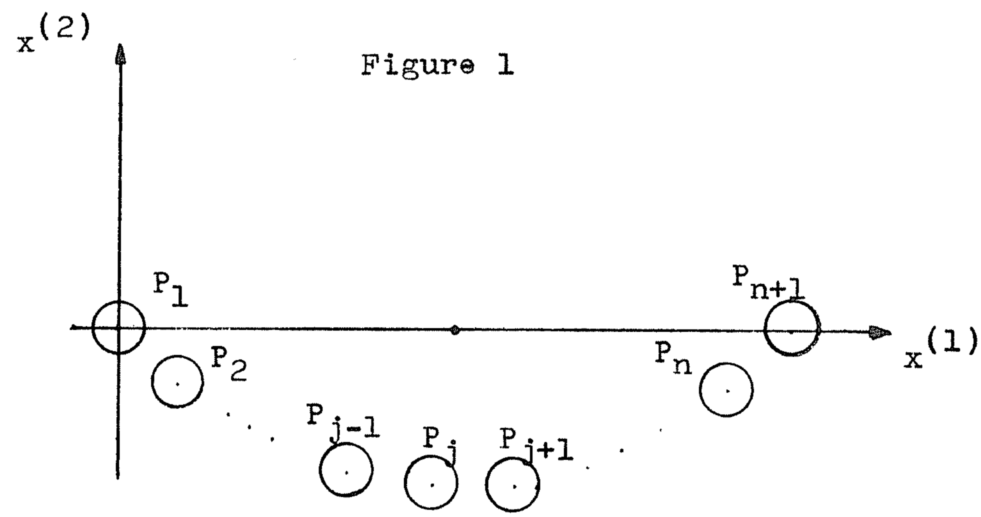


Figure 1

"particles" denoted by $P_1, P_2, \ldots, P_n, P_{n+1}$. Let each particle P_j have mass m which is assumed to be concentrated at its center

$$x_j = (x_j^{(1)}, x_j^{(2)}),\ j = 1, 2, \ldots, n+1 . \tag{2.1}$$

In all the problems to be considered, without loss of generality, the center of P_1 will be positioned initially at the origin, i.e. $x_1 = (0,0)$, while the center of P_{n+1} will be fixed permanently at $x_{n+1} = (2,0)$.

3. Definitions of Velocity and Acceleration of the String Particles

To write down equations of motion for the particles, we first need both notations for and definitions of their velocities and accelerations. Since these quantities are time-dependent we need some discrete description of time. Let discrete points of time be denoted by

$$t_k = k\Delta t,\ \ k = 0, 1, 2, \ldots, \tag{3.1}$$

where Δt is a positive parameter. For $j = 1, 2, \ldots, n+1$, and $k = 0, 1, \ldots$ let the velocity and acceleration of P_j at time t_k be denoted by

$$v_{j,k} = (v_{j,k}^{(1)}, v_{j,k}^{(2)}) \tag{3.2}$$

and

$$a_{j,k} = (a^{(1)}_{j,k}, a^{(2)}_{j,k}) , \tag{3.3}$$

respectively.

If we extend the particle position notation given by (2.1) to include time dependence by appending a second subscript so that $x_{j,k} = (x^{(1)}_{j,k}, x^{(2)}_{j,k})$ denotes the position of the center of P_j at time t_k, then the <u>velocity of</u> P_j <u>at time</u> t_{k+1} is defined implicitly by

$$\frac{v_{j,k} + v_{j,k+1}}{2} = \frac{x_{j,k+1} - x_{j,k}}{\Delta t}, \; j = 1,2,\ldots,n+1 . \tag{3.4}$$

Let the <u>acceleration of</u> P_j <u>at time</u> t_0 be defined by

$$a_{j,0} = \frac{v_{j,1} - v_{j,0}}{\Delta t} , \; j = 1,2,\ldots,n+1 , \tag{3.5}$$

and for $k = 1,2,\ldots,$ let the <u>acceleration of</u> P_j <u>at time</u> t_k be defined by (see [6])

$$\frac{1}{2}[3a_{j,k} - a_{j,k-1}] = \frac{v_{j,k+1} - v_{j,k}}{\Delta t}, \; j = 1,2,\ldots,n+1. \tag{3.6}$$

Note that equation (3.5) is a one-step forward difference formula, whereas equation (3.6) is a two-step formula, making use of information at two previous time steps to determine quantities at the next successive time step. Formulas (3.4) and (3.6) were motivated by considering velocities and accelerations as continuous functions of time and by considering the point $t_{k+\frac{1}{2}} = (k + \frac{1}{2})\Delta t$ as the point where the average velocity and acceleration in the interval $[t_k, t_{k+1}]$

are attained. The truncation error in equation (3.6), when viewed as an approximation to acceleration as a continuous function of time in the interval $[t_k, t_{k+1}]$, is $0(\Delta t^2)$. When velocity is assumed to be a continuous function of time, the truncation error in equation (3.4) is $0(\Delta t^2)$. Equation (3.5), a starting formula for the two-step formula (3.6), has associated with it a truncation error which is $0(\Delta t)$.

4. The Equation of Motion

The motion of the string is assumed to be governed by Newton's second law when applied to each of the particles composing the string. Thus if $F_{j,k} = (F^{(1)}_{j,k}, F^{(2)}_{j,k})$ denotes the sum of the forces acting upon particle P_j at time t_k we have

$$(4.1) \qquad ma_{j,k} = F_{j,k}, \quad j = 1,2,\ldots,n+1, \; k = 0,1,2,\ldots .$$

We must next list and describe the various forces composing $F_{j,k}$. We assume the only possible forces acting upon each particle at each point in time are (1) viscosity, (2) gravity, and (3) tension, and these are defined as follows:

(1) As in [6] the force of viscosity $\mu_{j,k} = (\mu^{(1)}_{j,k}, \mu^{(2)}_{j,k})$ acting upon particle P_j at time t_k is assumed to be proportional to $v_{j,k}$ and oppositely directed. Let α denote the non-negative constant of propor-

tionality, so that

(4.2) $$\mu_{j,k} = -\alpha v_{j,k}, \quad j = 1,2,\ldots,n+1;\ k = 0,1,2,\ldots \ .$$

(2) The force G, due to gravity, is taken to be the weight of the particle, which acts in the negative vertical direction only. Thus, in English units,

(4.3) $$G = -mg = -m(0,32.2),$$

which is uniformly constant for all particles and all times.

(3) Tension is the force which couples the particles composing the string and is assumed to act between consecutive particles only. Thus if $\tau_{j,k} = (\tau^{(1)}_{j,k}, \tau^{(2)}_{j,k})$ denotes the net force of tension acting upon particle P_j at time t_k and $|T_{j,j+1,k}|$ denotes the magnitude of the tension between particles P_j and P_{j+1} we have as in [4] (see Figure 2)

(4.4) $$\tau_{j,k} = |T_{j,j+1,k}| \frac{x_{j+1,k} - x_{j,k}}{r_{j,j+1,k}}$$

$$- |T_{j-1,j,k}| \frac{x_{j,k} - x_{j-1,k}}{r_{j-1,j,k}}, \quad j = 2,3,\ldots,n \ .$$

where $r_{j,j+1,k}$ is the distance between the centers

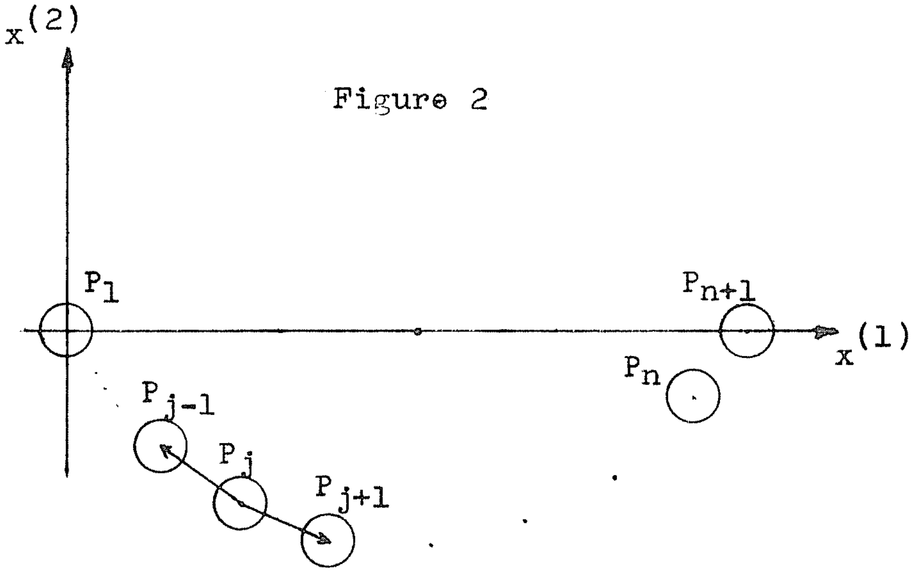


Figure 2

of P_j and P_{j+1} and is given explicitly by

$$r_{j,j+1,k} = [(x^{(1)}_{j,k} - x^{(1)}_{j+1,k})^2 + (x^{(2)}_{j,k} - x^{(2)}_{j+1,k})^2]^{1/2} .$$

Note that $\tau_{1,k}$ and $\tau_{n+1,k}$ are not defined by (4.4). These values must be defined for each special type of problem being considered, e.g., for a string fixed at both ends or for a free-end string. (See Sections 6 and 7.) Note that each term of equation (4.4) consists of the magnitude of a tensile force and a factor indicating the direction in which the force acts. The magnitude of the tension between P_j and P_{j+1} is defined in the following way:

$$(4.5)\qquad |T_{j,j+1,k}| = T_0\left[(1-\varepsilon)\left(\frac{r_{j,j+1,k}}{\Delta x}\right) + \varepsilon\left(\frac{r_{j,j+1,k}}{\Delta x}\right)^2\right],\ j = 1,2,\ldots,n,$$

where $0 \leqq \varepsilon \leqq 1$, $\Delta x = 2/n$, and T_0 is a reference tension acting between P_j and P_{j+1} when they are separated by a reference distance Δx, which is seen easily by inserting Δx for $r_{j,j+1,k}$ in (4.5).

Note that if $\varepsilon = 0$ in (4.5) we have that

$$(4.6)\qquad |T_{j,j+1,k}| = \frac{T_0}{\Delta x}(r_{j,j+1,k}),$$

which is the linear tension formula known as Hooke's law and which, when inserted in (4.4), yields

$$(4.7)\qquad \tau_{j,k} = \frac{T_0}{\Delta x}\left[x_{j+1,k} - 2x_{j,k} + x_{j-1,k}\right].$$

From (4.7) it can be seen how our formulation of the problem is related to the continuous wave equation, for the right-hand side of (4.7) includes a second difference in the spatial variable, while the left-hand side of equation (4.1) has acceleration.

At this point it should be noted that formulas other than (4.5) can be chosen for the magnitude of the tension. In fact the method used to obtain (4.5) can be extended to a polynomial of arbitrary degree in the variable $\frac{r_{j,j+1,k}}{\Delta x}$ as follows. (For convenience we drop the subscripts on r and T, with the understanding that these quantities are associated with two consecutive particles in the discrete string).

Assume the magnitude of the tension $|T(r)|$ has the form

$$(4.8) \qquad |T(r)| = T_0[c_0 + c_1(\frac{r}{\Delta x}) + c_2(\frac{r}{\Delta x})^2 + \cdots + c_p(\frac{r}{\Delta x})^p] ,$$

where T_0 and Δx are positive parameters, c_i, $i = 0,1,\ldots,p$, are non-negative coefficients, and p is a positive integer. Two of the coefficients c_i, $i = 0,1,\ldots,p$, are determined by imposing the following two conditions.

(1) $|T(0)| = 0$ (This is equivalent to assuming that zero tension between consecutive particles corresponds to their centers being coincident.)

(2) $|T(\Delta x)| = T_0$. (This condition allows one the ability to assign a reference tension of T_0 when two particles are separated by a reference distance Δx.)

From (4.8) we see that condition (1) implies that $c_0 = 0$, whereas condition (2) implies

$$(4.9) \qquad c_1 + c_2 + \cdots + c_p = 1 .$$

So one can specify arbitrarily any $p-1$ of the non-negative coefficients c_i, $i = 1, \ldots, p$, and the remaining coefficient is then determined by equation (4.9).

Let us now rewrite equation (4.1) with the right-hand side expanded into its constituent forces, as follows,

$$\text{(4.10)} \qquad m a_{j,k} = \tau_{j,k} + \mu_{j,k} + G .$$

The right side of (4.10) can be expanded further with the aid of (4.2), (4.3), (4.4), and (4.5) as follows:

$$m a_{j,k} = T_0 \left[(1-\varepsilon)\left(\frac{r_{j,j+1,k}}{\Delta x}\right) + \varepsilon \left(\frac{r_{j,j+1,k}}{\Delta x}\right)^2 \right] \frac{x_{j+1,k} - x_{j,k}}{r_{j,j+1,k}}$$

$$- T_0 \left[(1-\varepsilon)\left(\frac{r_{j-1,j,k}}{\Delta x}\right) + \varepsilon \left(\frac{r_{j-1,j,k}}{\Delta x}\right)^2 \right] \frac{x_{j,k} - x_{j-1,k}}{r_{j-1,j,k}}$$

$$- \alpha v_{j,k} - m g .$$

This last equation simplifies further to

$$\text{(4.11)} \qquad a_{j,k} = \frac{T_0}{m \Delta x} \{ (1-\varepsilon)(x_{j+1,k} - 2x_{j,k} + x_{j-1,k})$$

$$+ \frac{\varepsilon}{\Delta x} [r_{j,j+1,k}\, x_{j+1,k} - (r_{j,j+1,k} + r_{j-1,j,k})\, x_{j,k}$$

$$+ r_{j-1,j,k}\, x_{j-1,k}] \} - \frac{\alpha}{m} v_{j,k} - g, \quad j = 2, 3, \ldots, n;$$

$$k = 0, 1, 2, \ldots \quad .$$

Equation (4.11) was so written to show how the linear and nonlinear terms in the tension separate.

5. The Computational Procedure

From (3.4), (3.5), (3.6), and (4.11), we are able now to describe the following general method to be used in computing the positions of the string particles at all points in time, given only the initial positions and velocities of the particles.

Step 1: Specify all parameters and initial data, i.e., n, Δt, T_0, m, α, ε, $x_{j,0}$, $v_{j,0}$, $j = 1,\ldots,n+1$.

Step 2: Compute $\Delta x = 1/n$.

Step 3: From equation (4.11), with $k = 0$, compute $a_{j,0}$, $j = 2,3,\ldots,n$.

The accelerations of the end particles, $a_{1,0}$ and $a_{n+1,0}$, will be determined in a special way, in accordance with the particular problem being solved.

Step 4: Compute $v_{j,1}$, $j = 1,\ldots,n+1$ from the following equivalent form of equation (3.5).

$$v_{j,1} = v_{j,0} + (\Delta t)\, a_{j,0},\ j = 1,\ldots,n+1 .$$

<u>Step 5</u>: Compute $x_{j,1}$, $j = 1,\ldots,n+1$ by rewriting (3.4) with $k = 0$ as follows:

$$x_{j,1} = x_{j,0} + \frac{1}{2}\Delta t(v_{j,0} + v_{j,1})$$

<u>Step 6</u>: For each $k = 1,2,\ldots,$ execute the following 3 steps in the order given.

a) From (4.11) compute $a_{j,k}$, $j = 2,\ldots,n$. Compute $a_{0,k}$ and $a_{n+1,k}$ by special formulas to be given with each problem being considered.

b) From (3.6) compute

$$v_{j,k+1} = v_{j,k} + \frac{1}{2}\Delta t(3a_{j,k} - a_{j,k-1}),\ j = 1,\ldots,n+1.$$

c) From (3.4) compute

$$x_{j,k+1} = x_{j,k} + \frac{1}{2}\Delta t(v_{j,k+1} + v_{j,k}),\ j = 1,\ldots,n+1\ .$$

The iteration described in step 6 is continued up to some arbitrarily specified stopping point, or until the particles stop moving, e.g., when for some time step q and specified tolerance δ

$$\max_{1 \le j \le n+1} \| x_{j,q} - x_{j,q-1} \| \le \delta\ .$$

We will next describe a variety of examples and results. All of the examples fall into two major categories:

(1) a string fixed at both ends throughout;

and

(2) a string fixed initially at both ends, but with the end at $x = 0$ released at $t = 0^+$.

The examples of type (1) will be discussed first.

For convenience, $n + 1$ will always be chosen to be odd, so that there will always be a center particle.

6. Studies of a String Fixed at Both Ends

Example 6.1

In this example of a string fixed at both ends we consider the case of a symmetric, linear initial displacement of particles with zero initial velocities. See Figure 3 for a graph of a line passing through the centers of the particles at $t = 0$.

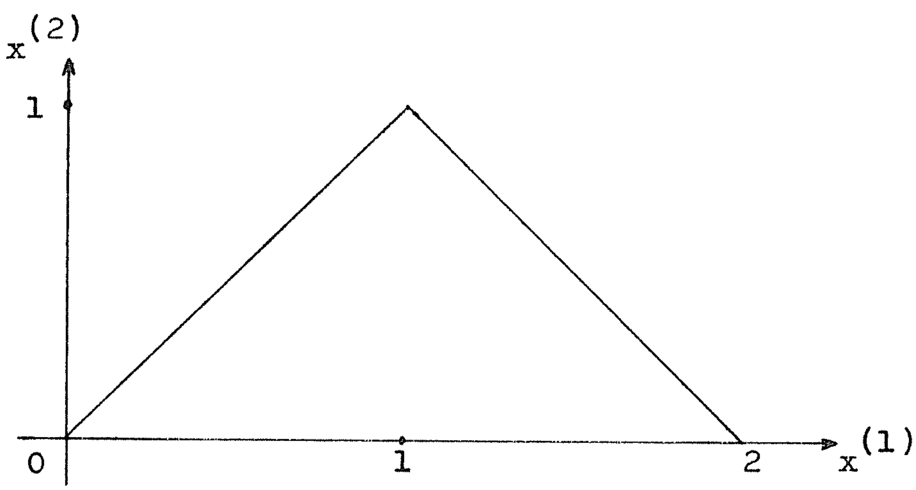


Figure 3

All of the forces described in Section 4 are included. To define completely the computational procedure for this and all examples of strings with both ends fixed throughout all times, we refer to the general procedure detailed in Section 5 and will indicate, in addition, how the end particles are treated in steps 3 and 6a. Thus we have that, for all $k = 0,1,2,\ldots$,

$$(6.1)\qquad \begin{cases} x_{1,k} = (0,0), \\ x_{n+1,k} = (2,0), \\ v_{1,k} = v_{n+1,k} = (0,0), \\ a_{1,k} = a_{n+1,k} = (0,0) . \end{cases}$$

Further, we have that, for $\Delta x = 1/n$,

$$(6.2)\qquad \begin{cases} x^{(2)}_{j,0} = x^{(1)}_{j,0} = (j-1)\Delta x, \; j = 2,3,\ldots,\frac{n+2}{2} , \\ x^{(2)}_{j,0} = 2 - x^{(1)}_{j,0}, \; \frac{n+4}{2},\ldots,n , \\ v_{j,0} = (0,0), \;\; j = 2,3,\ldots,n . \end{cases}$$

The following parameter values were chosen for this example:

$$(6.3)\qquad \begin{cases} n = 40 \;\; (\text{hence}, \;\; \Delta x = .05) \\ m = 1/n = .025 \\ T_0 = 12.5 \\ \Delta t = .001 \\ \alpha = .15 \\ \varepsilon = .01 . \end{cases}$$

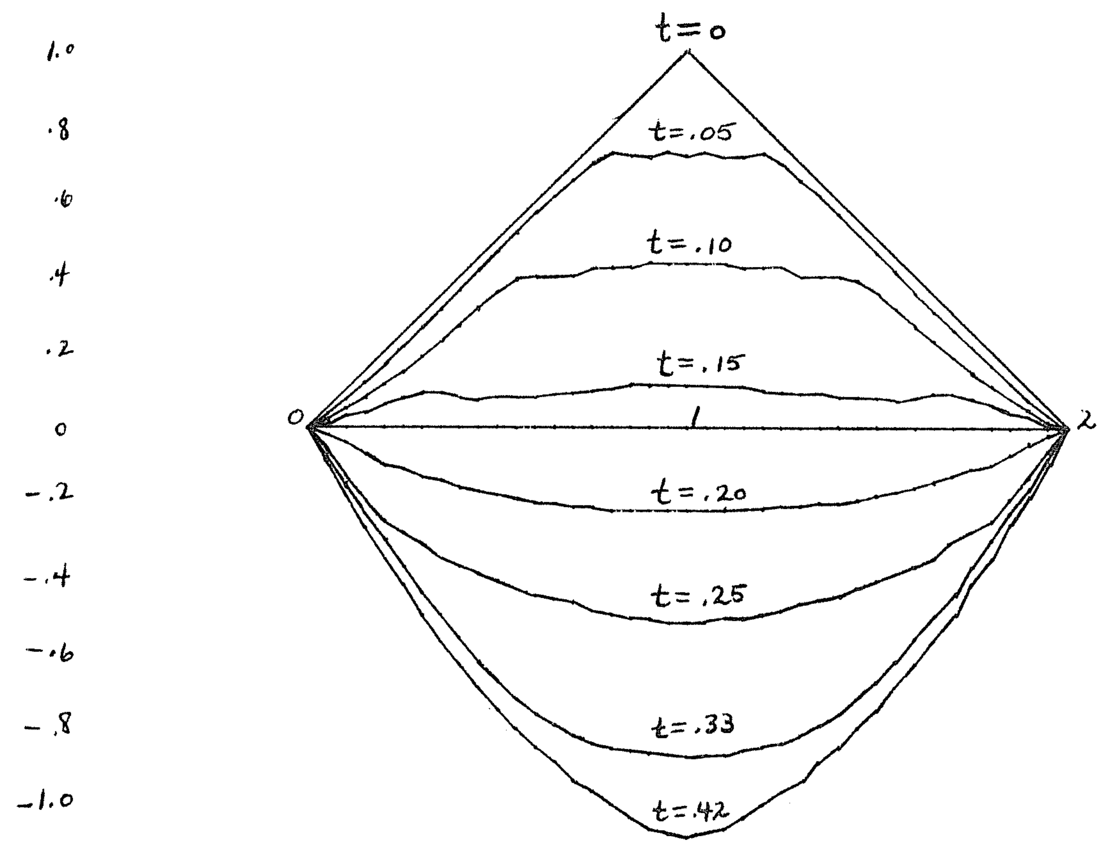
Figures 4a and 4b yield a graphical illustration of the positions of the string at various times during its first downward swing and its first upward swing, respectively. Figure 4b also shows the steady-state, or equilibrium, position of the string which was reached, to 3 decimal places, at $t \doteq 2.8$ in approximately 20 secs of computing time on the Univac 1108. The motion of the string particles in the horizontal $(x^{(1)})$ direction was so small for the choice of parameters in this case that it cannot be seen in Figure 4. In fact, from the numerical printout it was seen that, in the $x^{(1)}$ direction, the displacement of the particles from the equally spaced positions $x_j^{(1)} = (j-1)\Delta x$, $j = 1,2,\ldots,n+1$, was of order 10^{-4}, or about 1% of Δx, uniformly in time for all particles. Thus, the popular assumption of negligible motion in the horizontal direction is verified to be reasonable to 3 decimal place accuracy for this choice of parameters.

Figure 5 shows a magnified view of the motion of the center seven particles in the interval $t = 0$ to $t = .043$. It is of interest to observe that adjacent particles fall with a "flapping" type of motion as was described in [4].

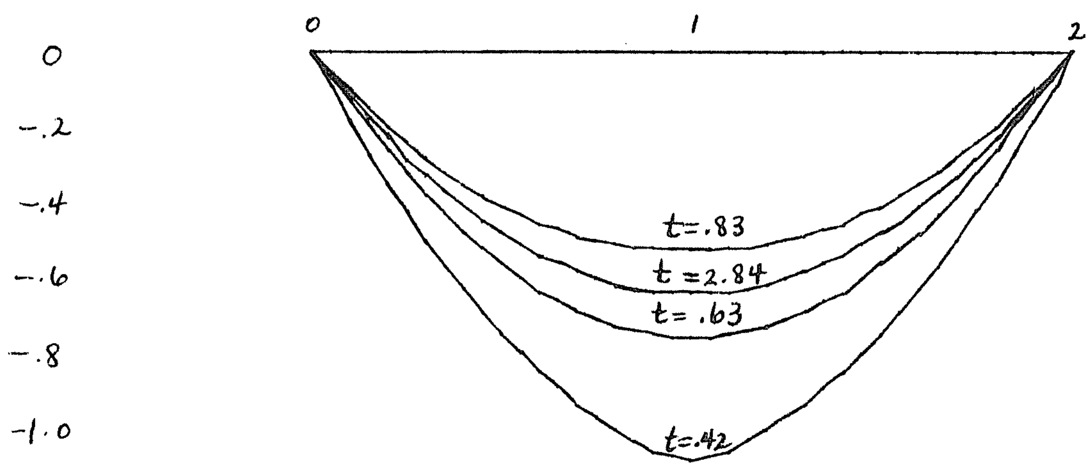
Note, finally, that the average execution time for this example was estimated to be approximately 240 time-steps per second of real Univac 1108 time.

Figure 4

$\Delta t = .001$
$n = 40$
$T_o = 12.5$
$\alpha = .15$
$\varepsilon = .01$



(a)



(b)

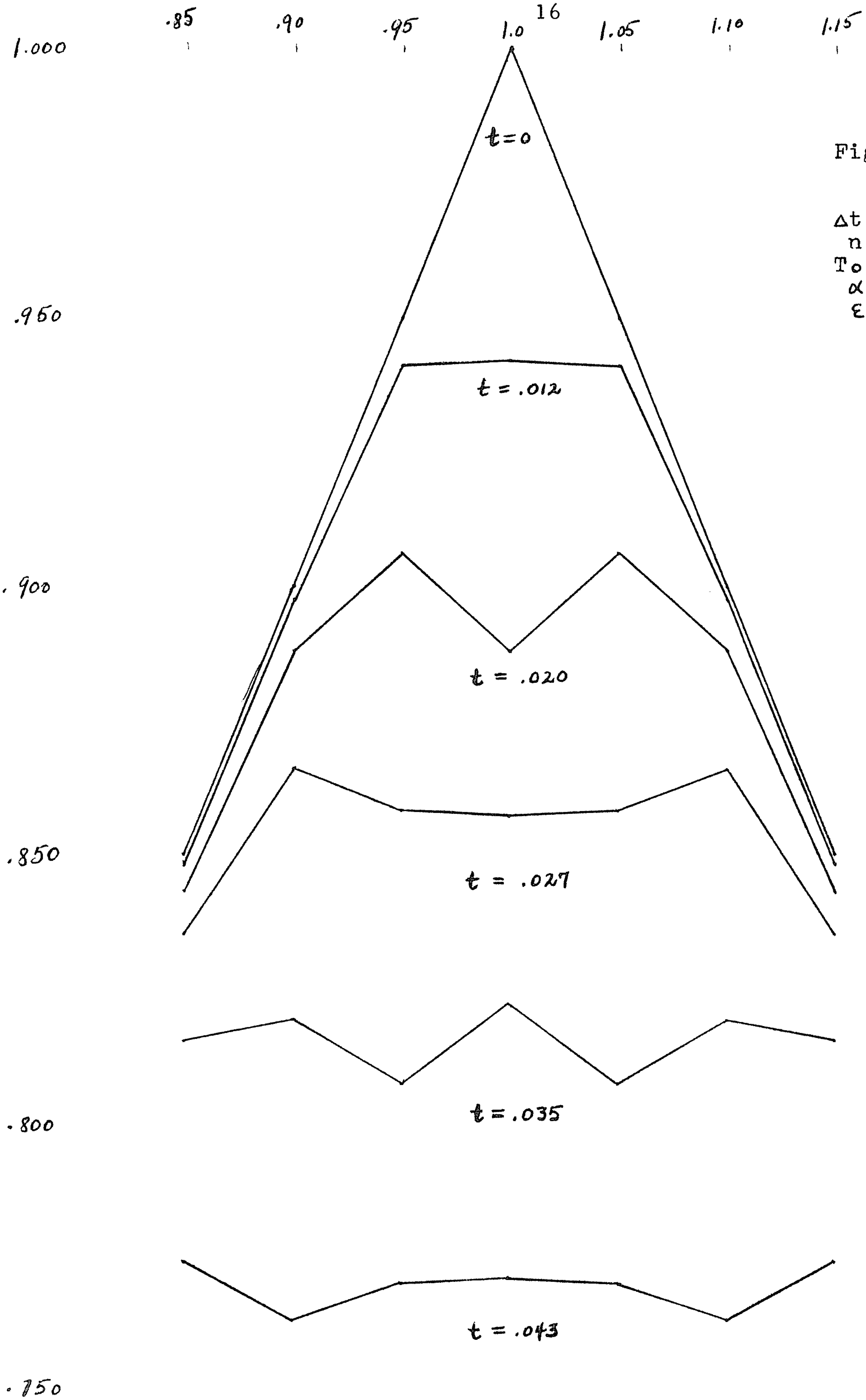


Figure 5

$\Delta t = .001$
$n = 40$
$T_o = 12.5$
$\alpha = .15$
$\varepsilon = .01$

Example 6.2

In this example we consider a string whose component particles are initially held at rest in the asymmetric position illustrated in Figure 6. Since the string is to be fixed at both ends throughout the motion,

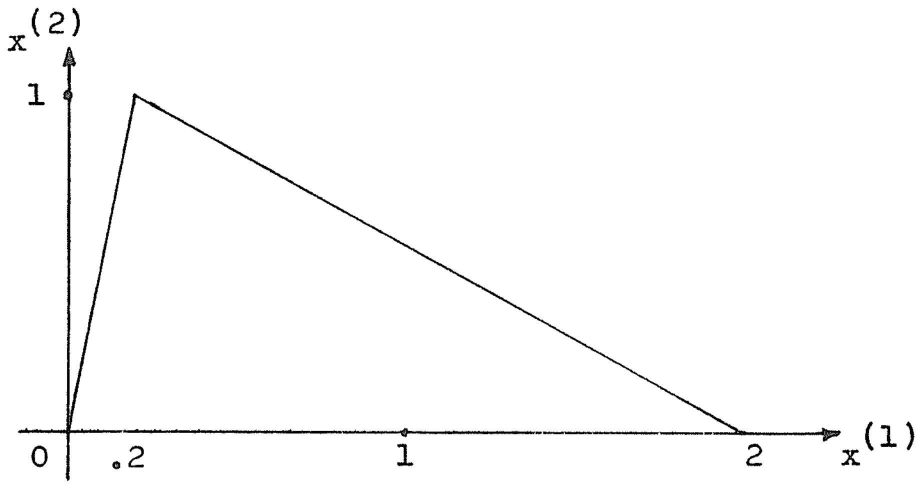


Figure 6

conditions (6.1) hold for all $k = 0,1,2,\ldots$. The initial conditions for the interior particles are given as follows:

$$x^{(1)}_{j,0} = (j-1)\Delta x, \quad j = 2,3,\ldots,n$$

$$x^{(2)}_{j,0} = 5x^{(1)}_{j,0}, \quad 0 < x^{(1)}_{j,0} \leq .2$$

$$x^{(2)}_{j,0} = -\frac{5}{9}x_{j,0} + \frac{10}{9}, \quad .2 < x^{(1)}_{j,0} < 2$$

$$v_{j,0} = (0,0), \quad j = 2,3,\ldots,n \,.$$

The parameter values chosen for this example are as follows:

$$(6.4) \qquad \begin{cases} n = 20 \quad (\Delta x = .1) \\ m = 1/n = .05 \\ T_0 = 12.5 \\ \Delta t = .001 \\ \alpha = .15 \\ \varepsilon = 0, .1, .5, 1. \end{cases}$$

Note from (6.4) that we have chosen a variety of values of ε, so we really have several examples in one.

Some of the results of these examples are illustrated in Figures 7, 8, and 9, each of which shows the pattern of motion of the string for one complete cycle, that is, one downward swing and one upward swing. Figure 7 shows clearly the effects of introducing non-linearity in the tension formula by comparing the linear case, $\varepsilon = 0$, with the case where $\varepsilon = .1$. Note in particular that in the case $\varepsilon = 0$ there is no motion in the horizontal direction (this fact is easily verified to be true exactly), whereas there is a significant amount of horizontal motion in the case $\varepsilon = .1$, as illustrated, for example, by the positions of the rightmost particles at $t = .42$. We see that there is a horizontal "bulging" effect in the non-linear case when the string reaches the extremity of its downward motion. This is not present in the linear case, but it is known from observation to be present in the real physical case. Thus this example illustrates the significant improvement of a nonlinear tension formula over a linear formula and of a model which includes horizontal motion over one which does not [4].

Another effect of the inclusion of non-linearity in the tension formula which is illustrated in Figure 7 is the change in the period of the string. Note the large difference in positions of the strings in the two cases during the upward half-cycle.

Figures 8 and 9 show the motion of the string with relatively large values of ε, .5 and 1, respectively. This means there is a relatively large degree of non-linearity in the tension formula. Note the more pronounced horizontal bulging effect which is present in these cases <u>during</u> the half-cycles as well as at their extremes. The greatest effect of this kind is observed in Figure 9 at $t = .64$ at the left side of the string.

Another item of some interest which was determined from the numerical printout is the amount of time required to reach steady state in each of these cases. The arbitrarily chosen criterion for when steady state occurred was the following:

$$\max_{1 \leq i \leq n+1} \| x_{i,k} - \bar{x}_i \|_\infty \leq .001 \,, \tag{6.5}$$

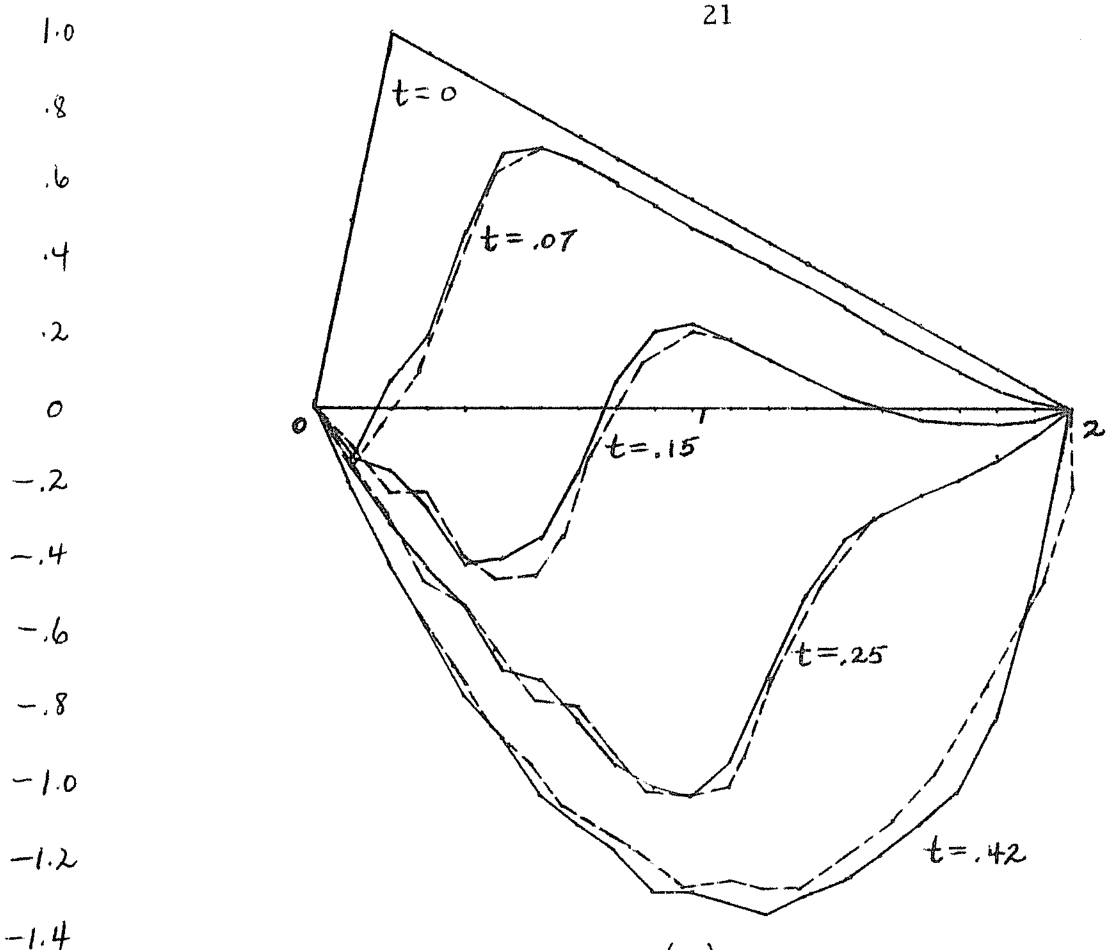
where $\bar{x}_i$ is the steady-state position of the i^{th} particle, and $x_{i,k}$ is the position of the i^{th} particle at time t_k. Table 1 shows time of steady state (t_{ss}) as a function of ε.

ε	t_{ss}
0	4.7
.1	4.7
.5	4.6
1	4.3

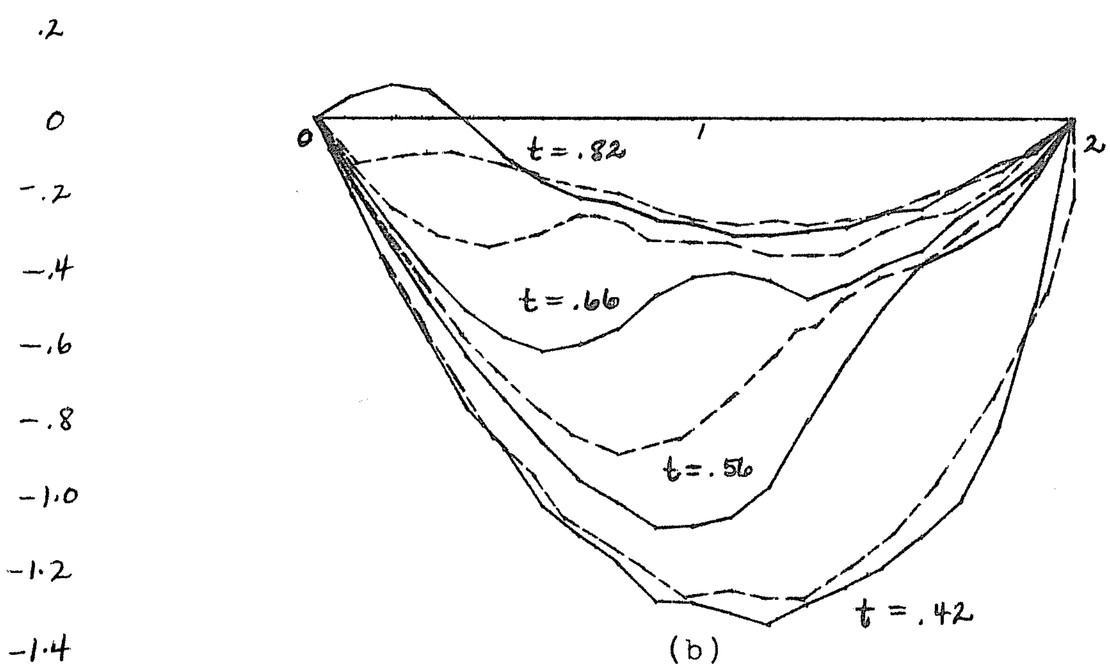
Table 1

Finally, Figures 10 and 11 show an enlarged view of the string motion in the initial moments for various values of ε. Once again the significance of the nonlinearity in the tension formula and of the motion in the horizontal direction is clearly illustrated. Also the flutter motion of consecutive particles, which was noted in example 6.1, is seen again in this example.

The average execution time for parameter cases run in this example was estimated to be approximately 265 time-steps per second of real Univac 1108 time.



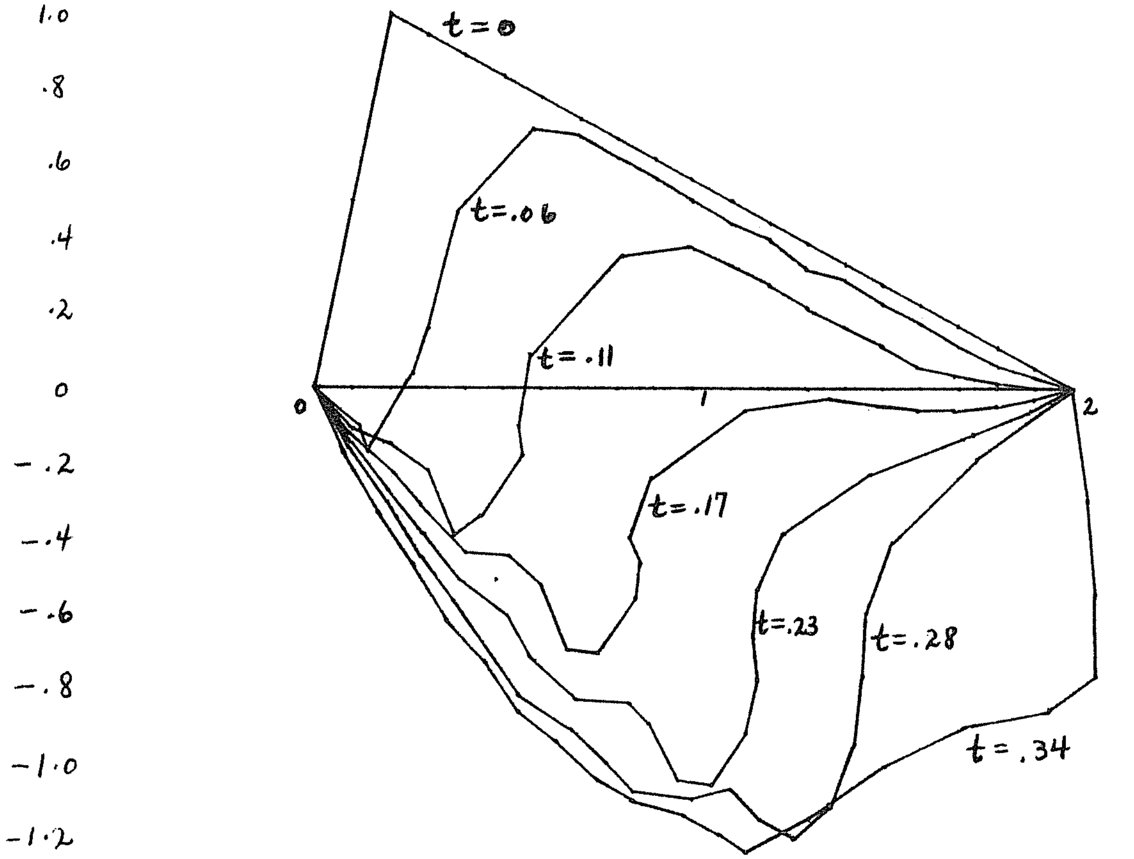
(a)



(b)

Figure 7

$\Delta t = .001$ $n = 20$ $T_o = 12.5$ $\alpha = .15$ $\varepsilon = 0$ (solid)
$\varepsilon = .1$ (dashed)



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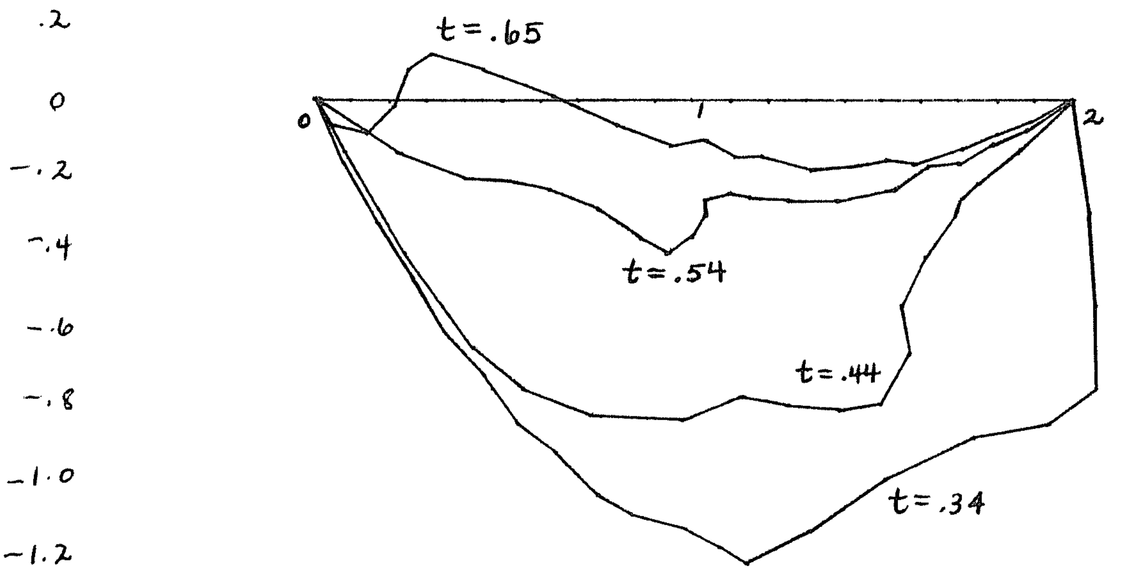
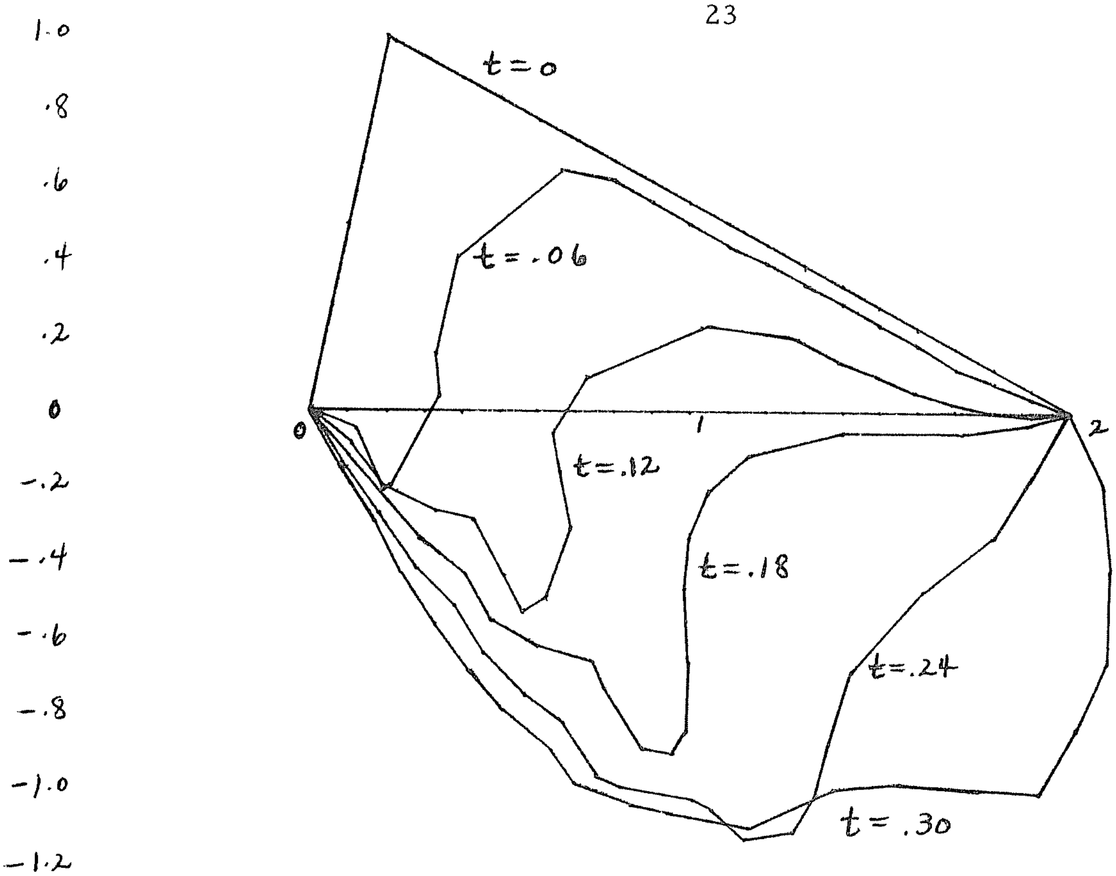


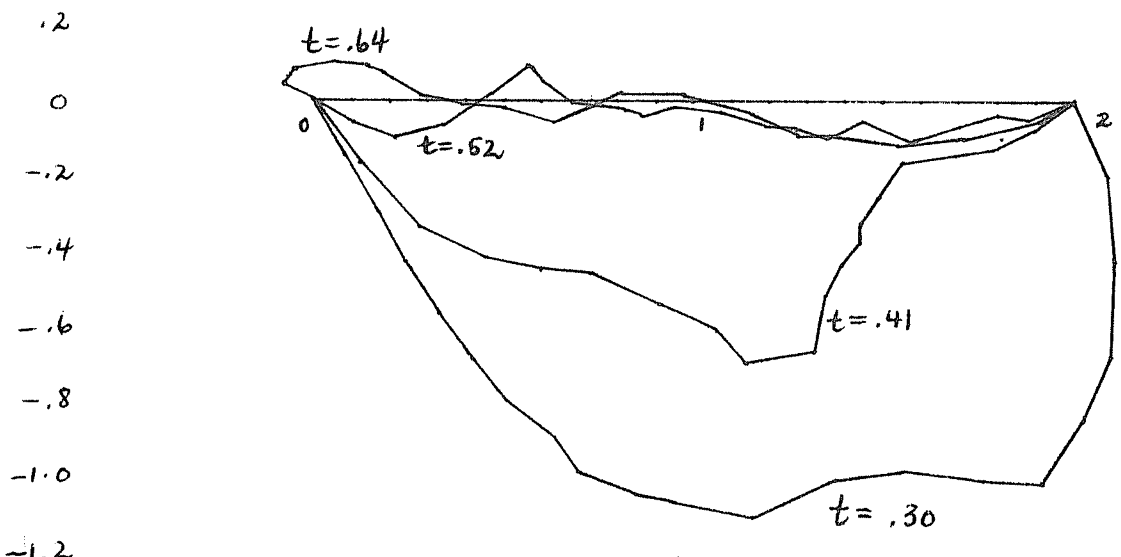
(b)

Figure 8

$\Delta t = .001 \quad n = 20 \quad T_o = 12.5 \quad \alpha = .15 \quad \varepsilon = .5$



(a)



(b)

Figure 9

$\Delta t = .001 \quad n = 20 \quad T_o = 12.5 \quad \alpha = .15 \quad \varepsilon = 1$

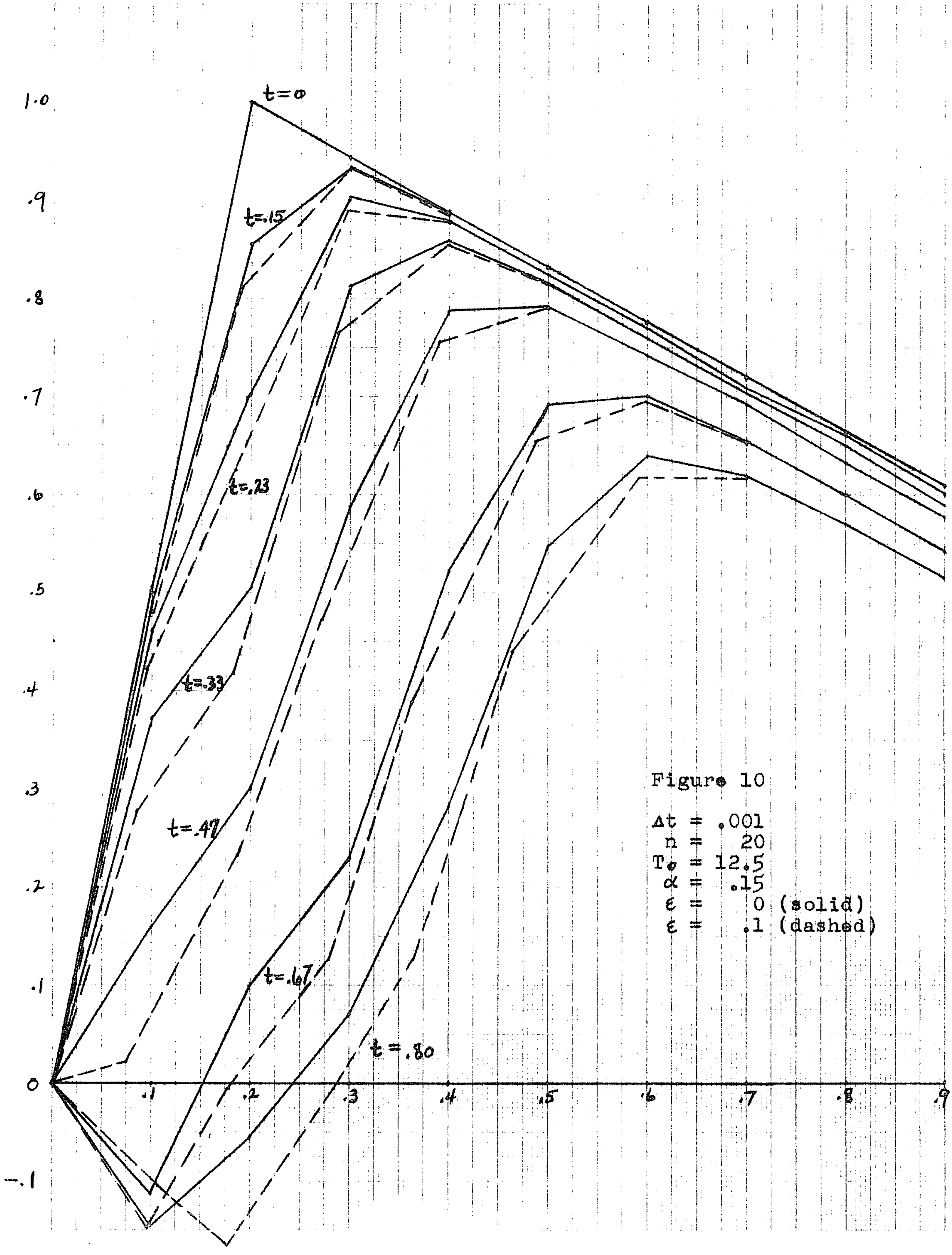


Figure 10

$\Delta t = .001$
$n = 20$
$T_0 = 12.5$
$\alpha = .15$
$\varepsilon = 0$ (solid)
$\varepsilon = .1$ (dashed)

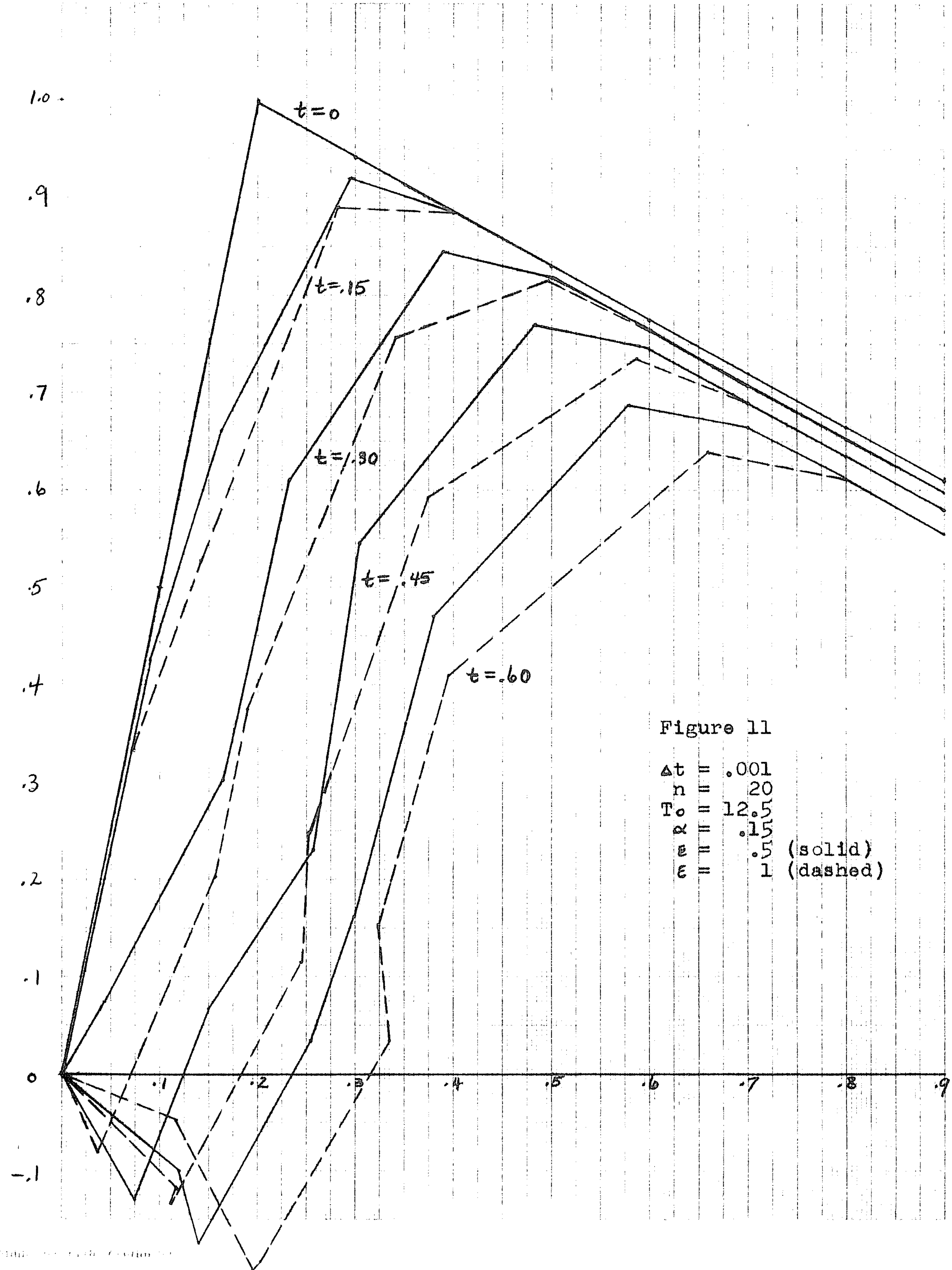


Figure 11

Δt = .001
n = 20
T_o = 12.5
α = .15
ε = .5 (solid)
ε = 1 (dashed)

Example 6.3

In this example an impulsive downward force is imparted to the center particle of a discrete string and the ensuing motion is then observed. The force of gravity will be taken to be zero in order to focus on the string motion effects due only to the striking of the center particle. Once again the end particles of the string will remain fixed throughout the motion, so conditions (6.1) hold. The initial conditions on the interior particles are given as follows:

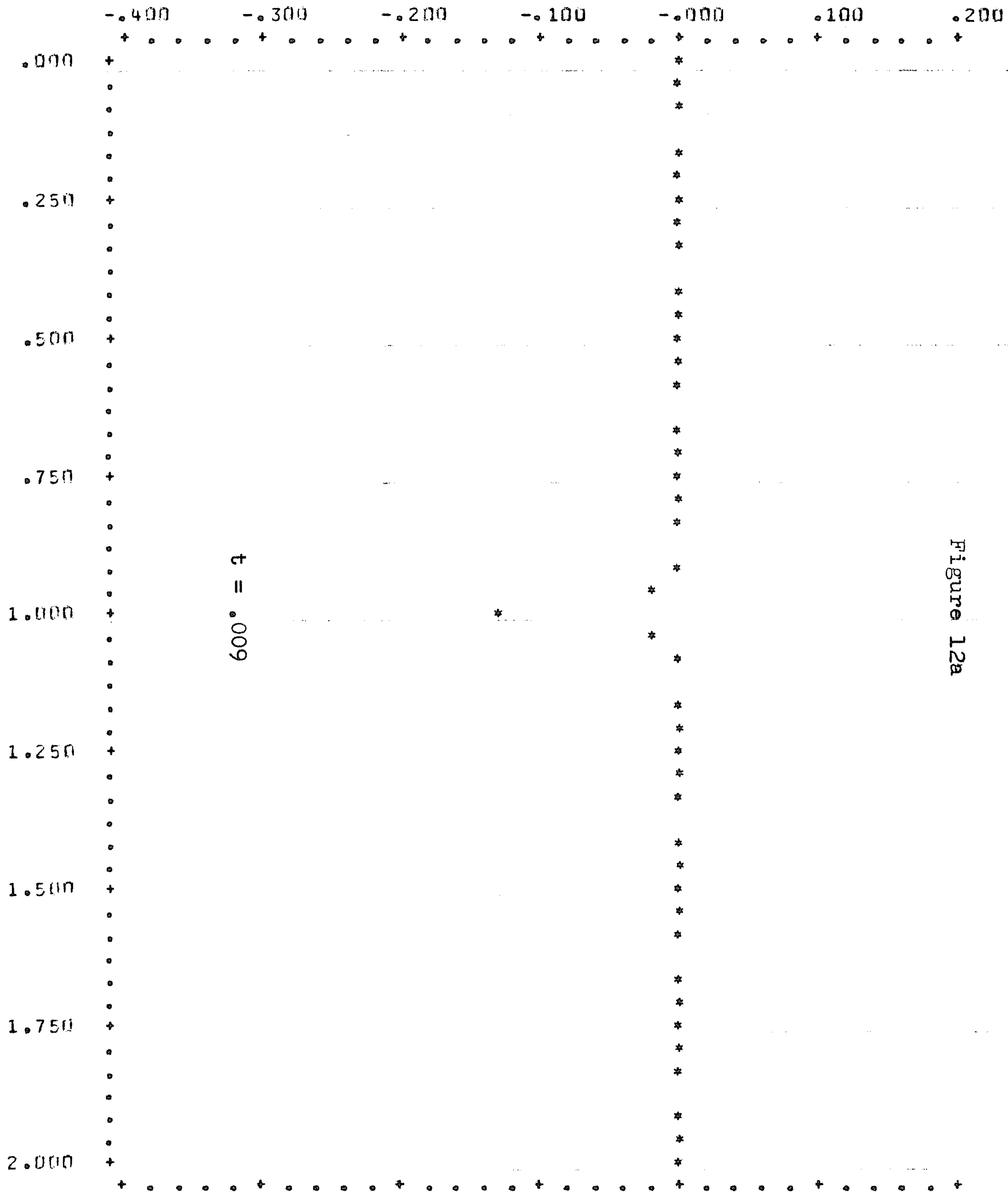
$$(6.6)\quad \begin{cases} x^{(1)}_{j,0} = (j-1)\Delta x, & j = 2,3,\ldots,n, \\ x^{(2)}_{j,0} = 0\ , & j = 2,3,\ldots,n, \\ v^{(1)}_{j,0} = 0\ , & j = 2,3,\ldots,n, \\ v^{(2)}_{j,0} = 0\ , & j = 2,3,\ldots,\frac{n}{2}, \frac{n}{2}+2,\ldots,n\ , \\ v^{(2)}_{\frac{n}{2}+1} = -20. & \end{cases}$$

For the parameters the following values have been chosen:

$$(6.7)\quad \begin{cases} n = 40 \quad (\Delta x = .05) \\ m = 1/n = .025 \\ T_0 = 12.5 \\ \Delta t = .0015 \\ \alpha = .15 \\ \varepsilon = 0, .5 \\ g = (0,0)\ . \end{cases}$$

Figures 12a - 12j show the first .525 time units of motion of the string for the case $\varepsilon = 0$, and Figures 13a - 13j do the same for the case $\varepsilon = .5$. Note in Figures 12a - 12f, which include the approximate period of time required for the wave generated by striking the center particle to reach the fixed ends, that all of the particles remain on or below the horizontal. Then Figures 12g - 12i show the waves reflected from the two ends traveling back toward and meeting at the center with much reduced amplitude. Finally, Figure 12j illustrates the string's shape early in the second full cycle as the wave travels toward the ends again with even further reduced amplitude. This type of motion continued until, at approximately $t = 1.7$, the maximum displacement of any particle was less than 10^{-3} units from the horizontal, so that our steady-state criterion was satisfied. Thus the motion of the string in this example is characterized by a relatively high amplitude first half-cycle with all particles remaining below the horizontal, followed by rapid attenuation of the wave in the succeeding half-cycles, leading to a quick approach to steady-state.

The same remarks made above for the case $\varepsilon = 0$ hold qualitatively for the case $\varepsilon = .5$ (see Figures 13a - 13j). The major differences due to a positive ε are the general reduction in amplitude of the wave throughout the motion and introduction of more "wiggles" in the shape of the string throughout its motion. Steady-state occurred at approximately the same time, $t = 1.7$.



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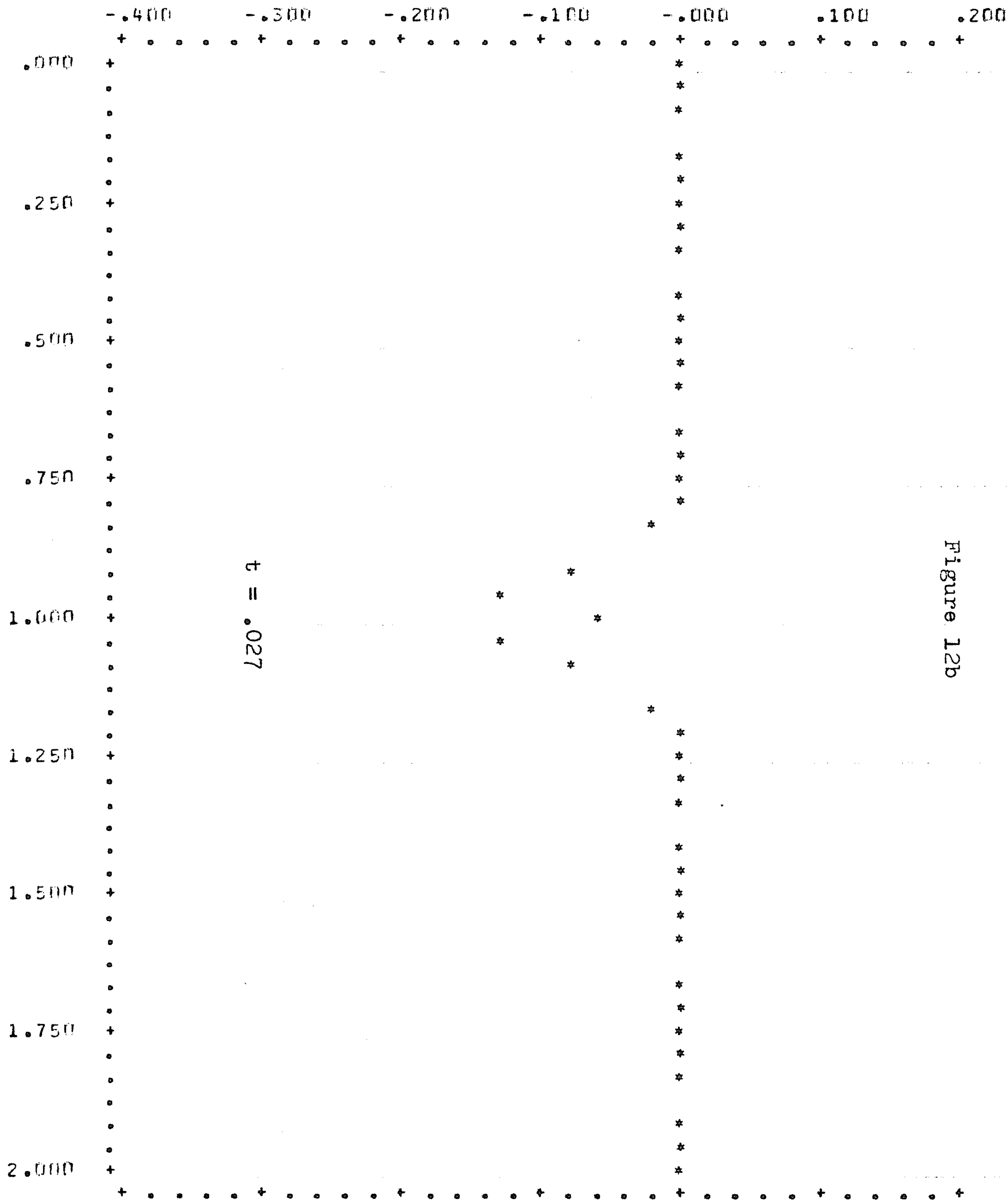
t = .009

Figure 12a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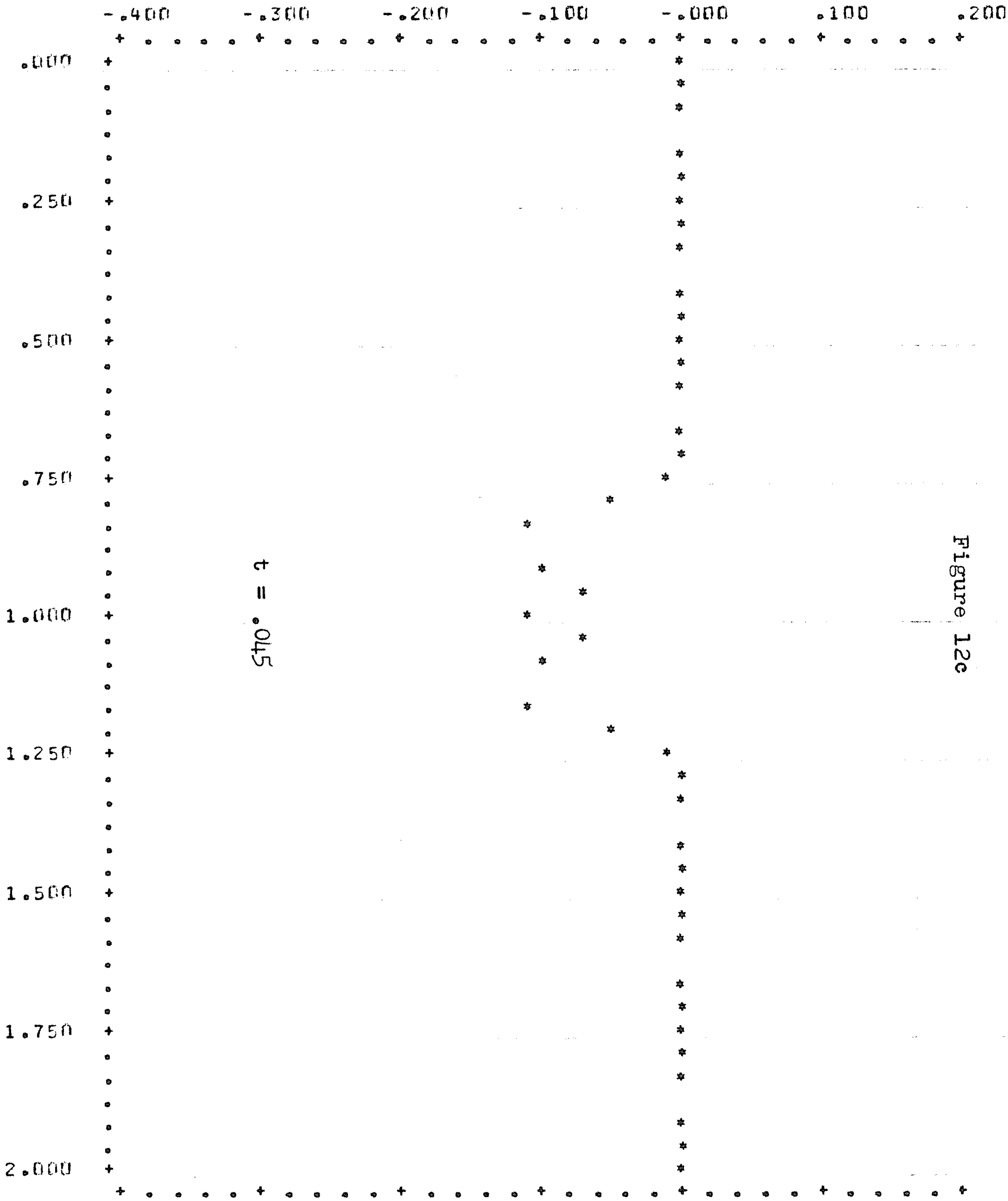
t = .027

Figure 12b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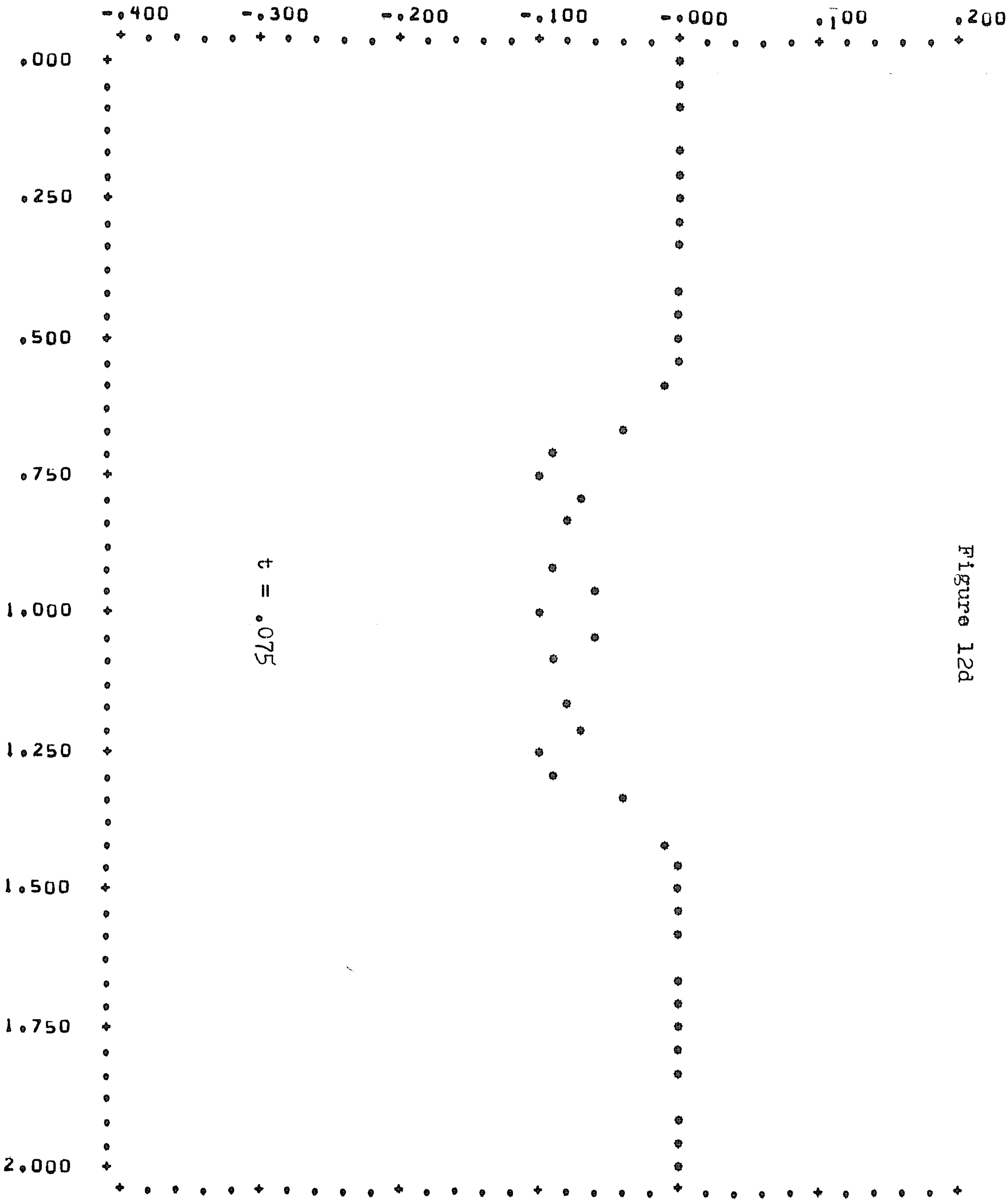
t = .045

Figure 12c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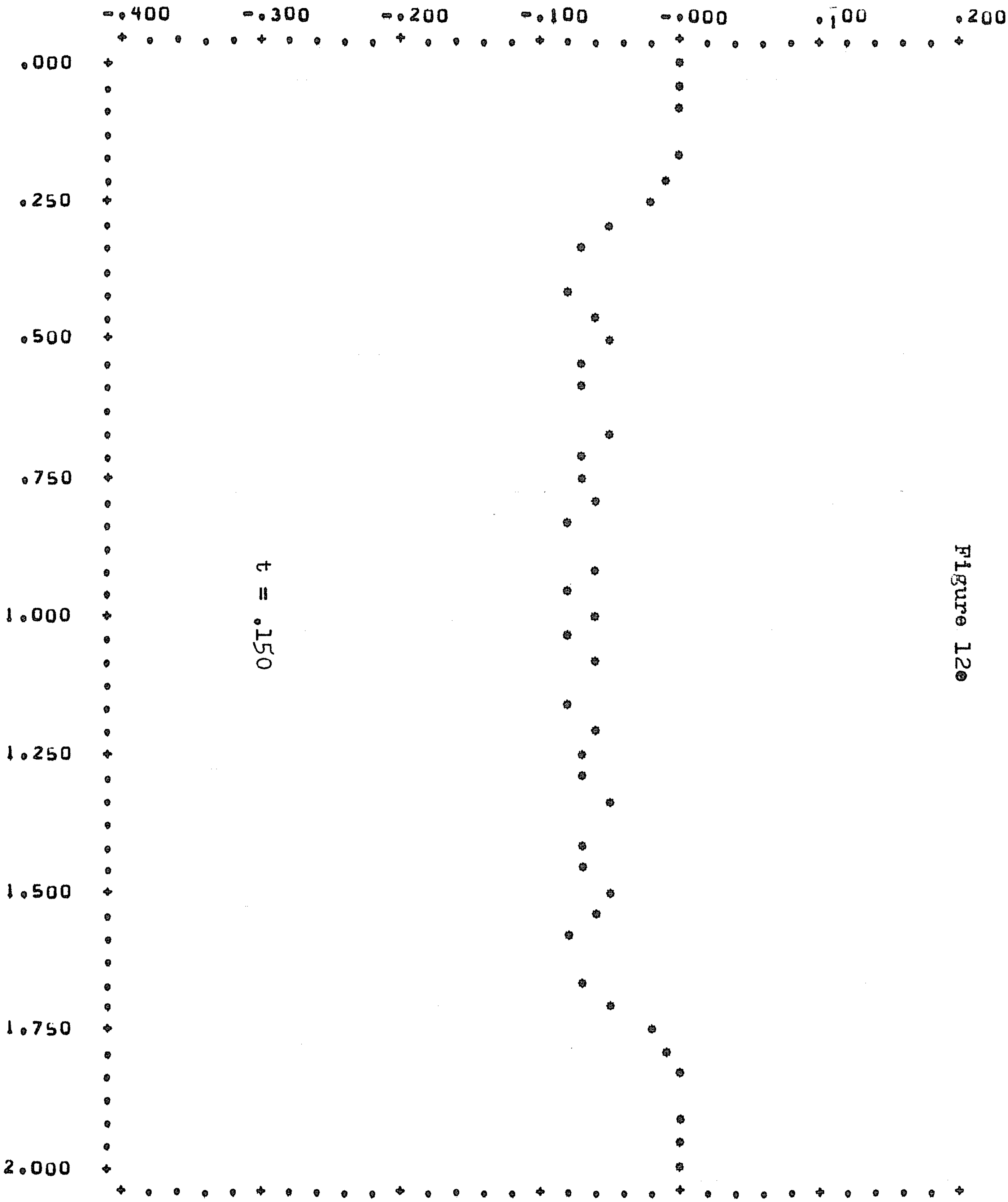
t = .075

Figure 12d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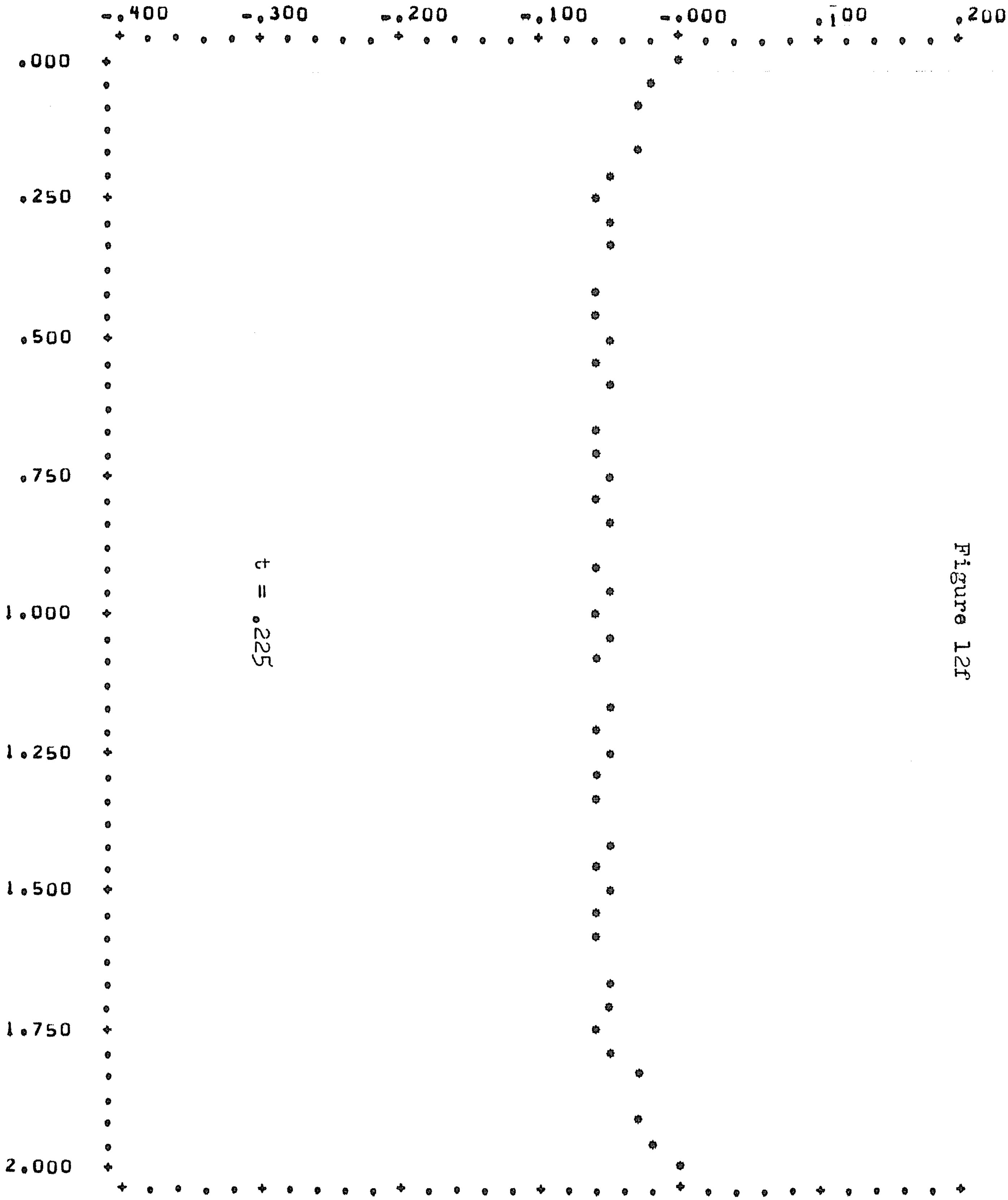
t = .150

Figure 12.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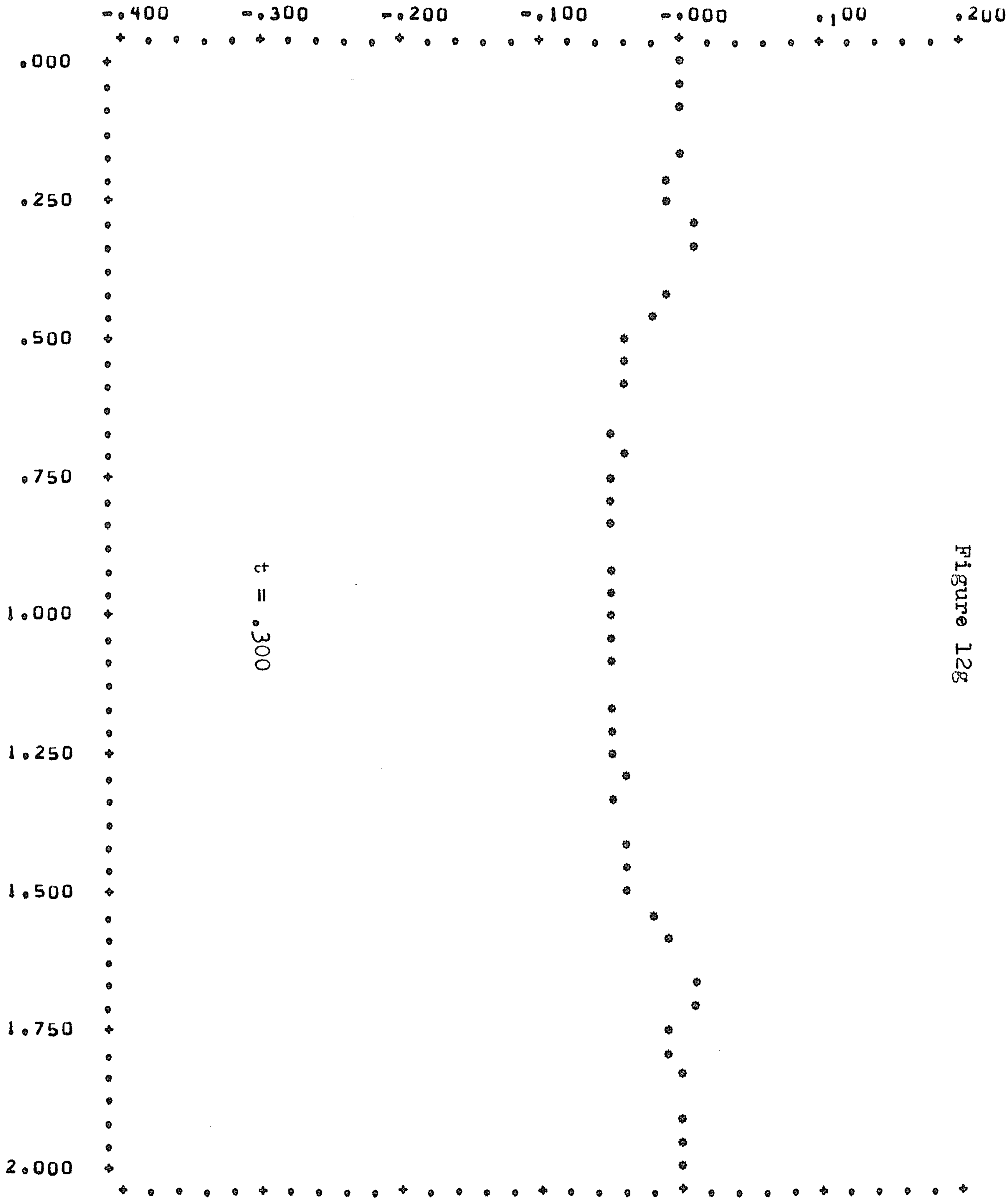
t = .225

Figure 12f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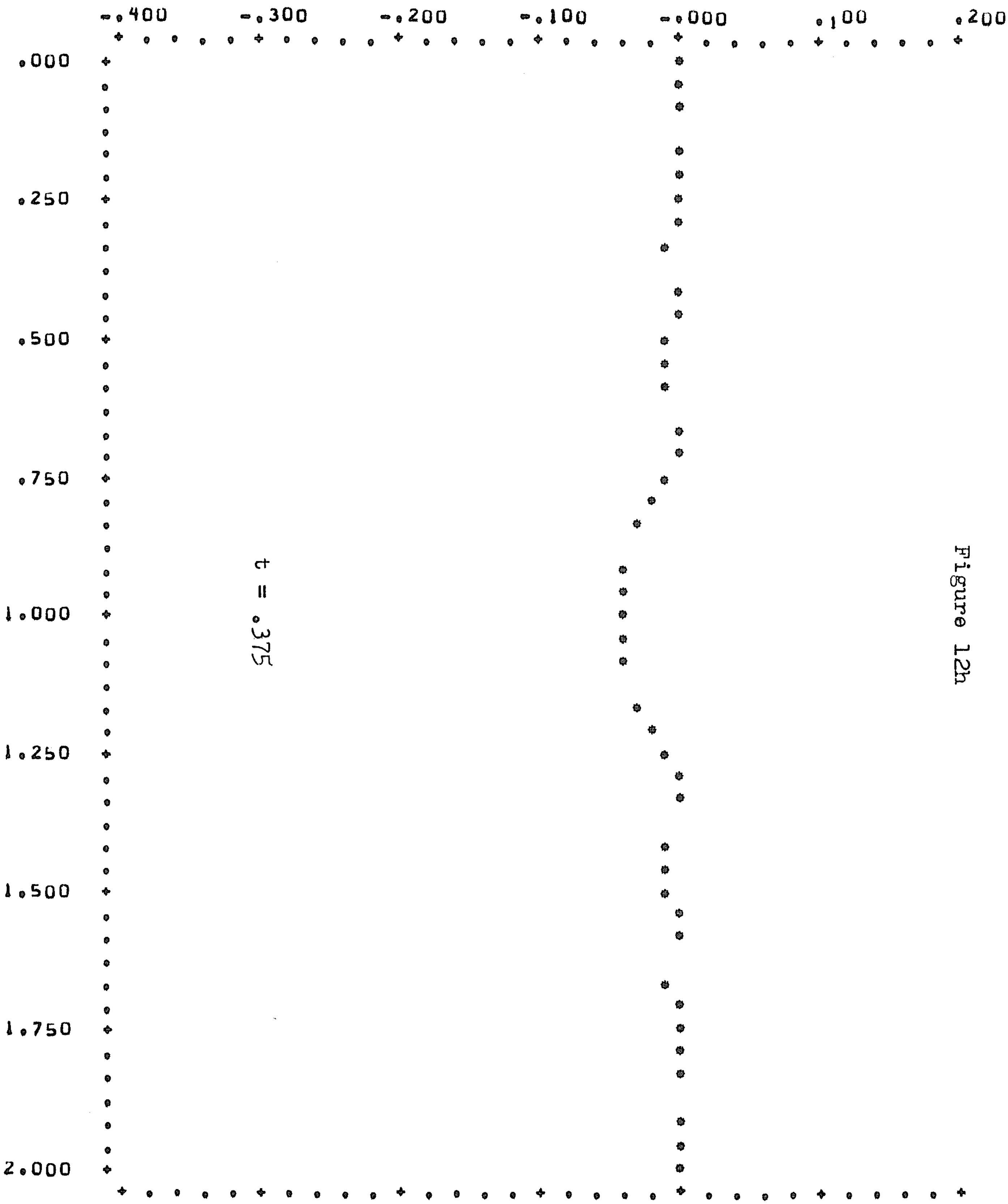
t = .300

Figure 12g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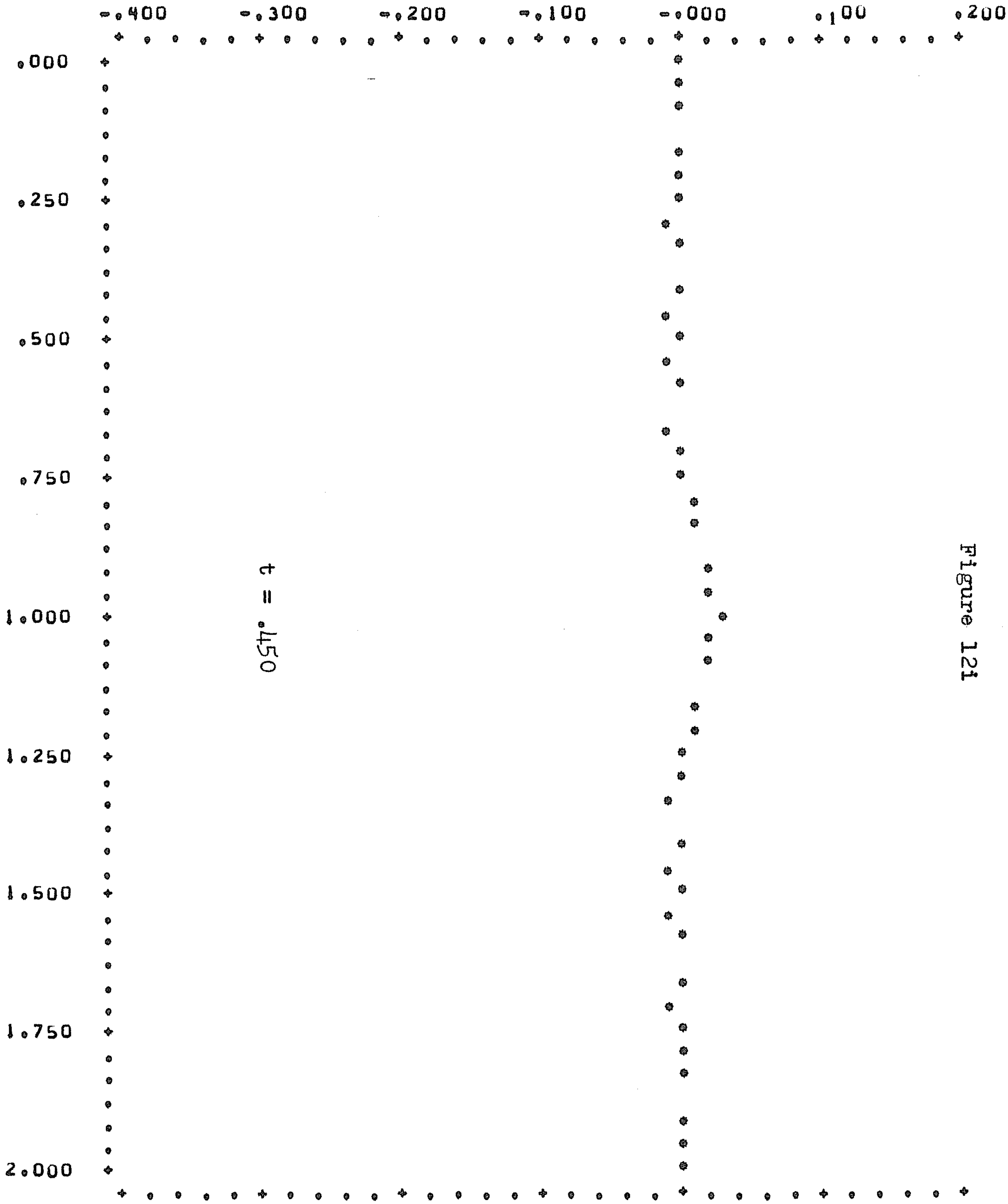
t = .375

Figure 12h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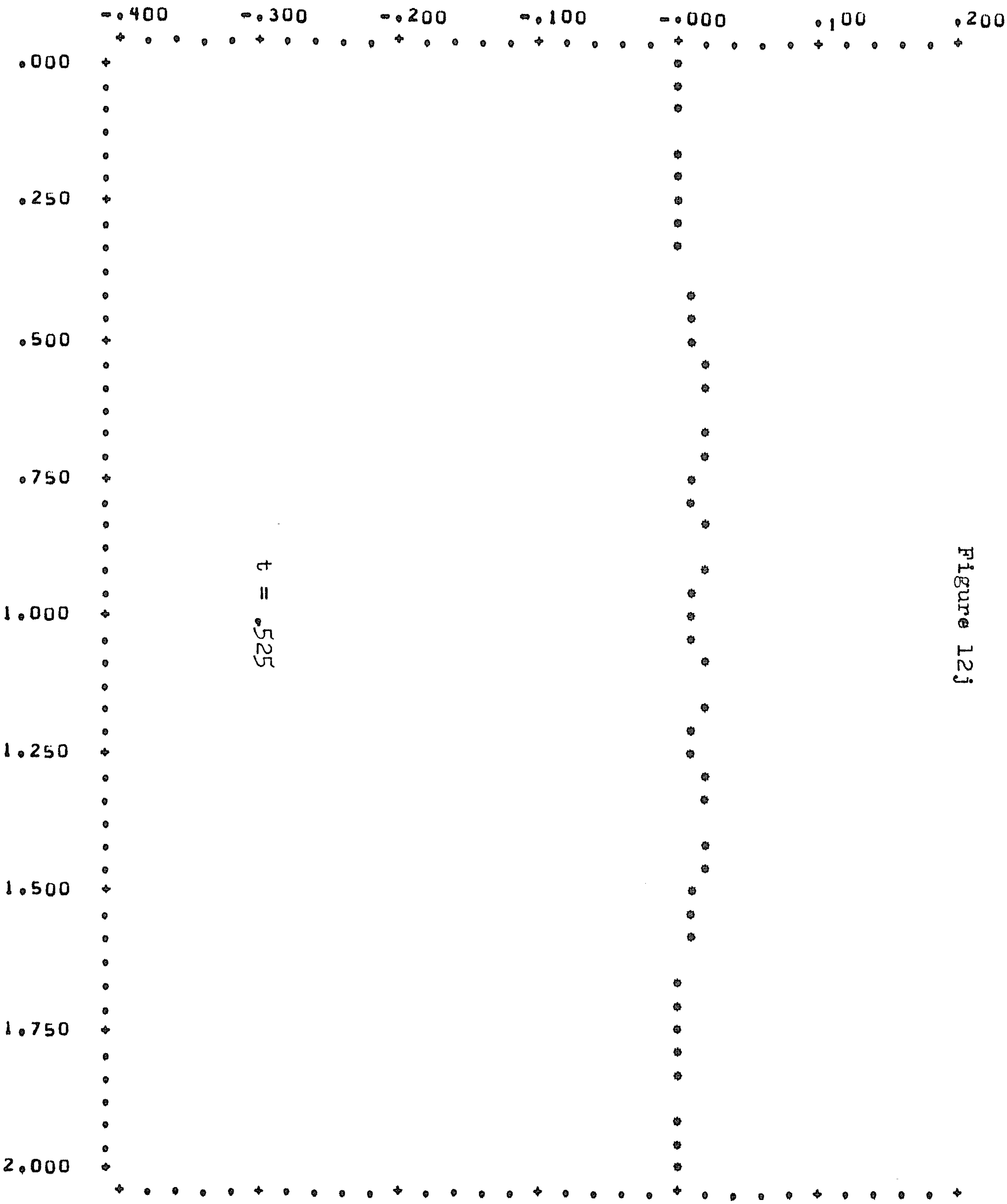
2.000
t = .450

Figure 121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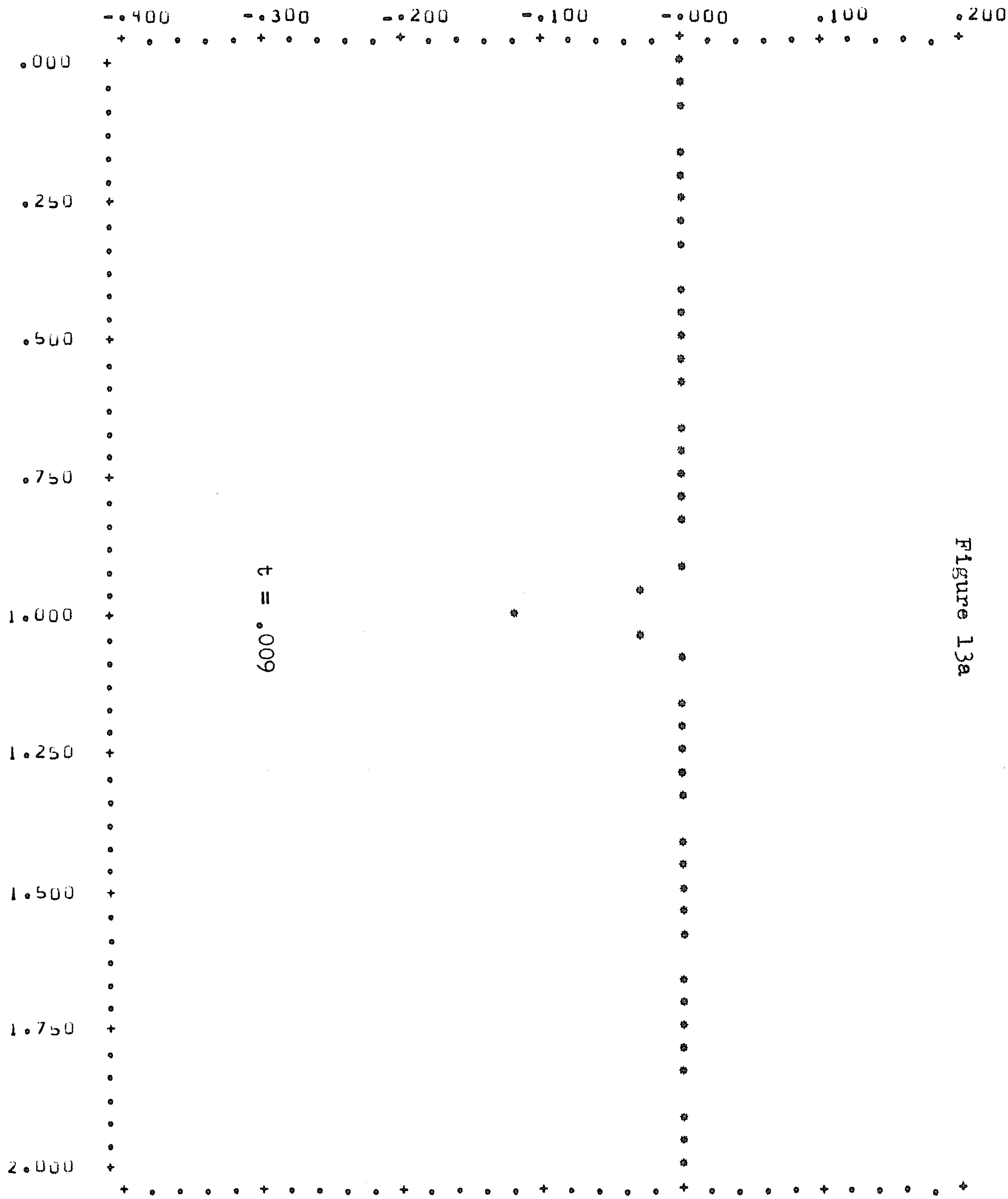
t = .525

Figure 12j



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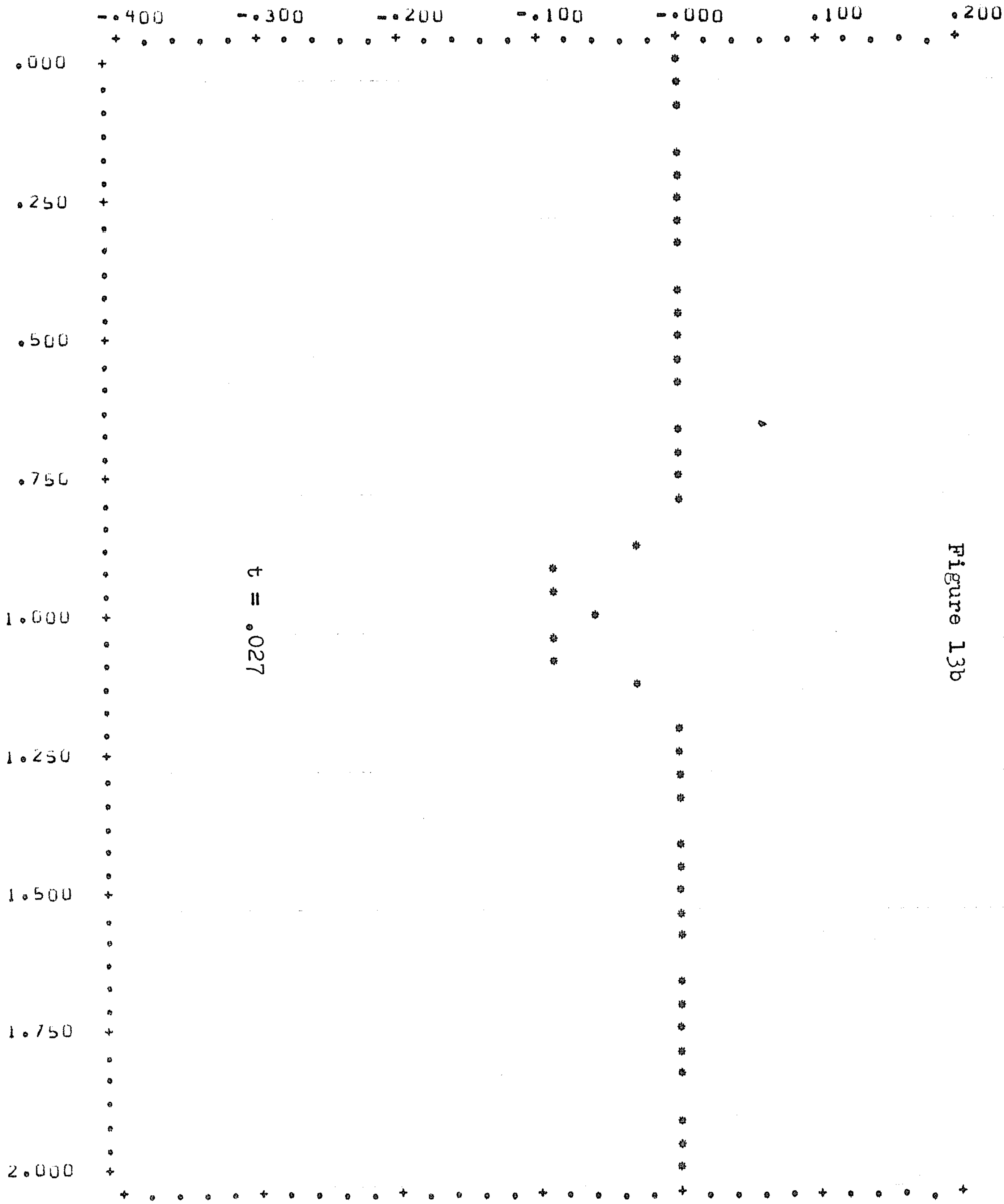
t = .009

Figure 13a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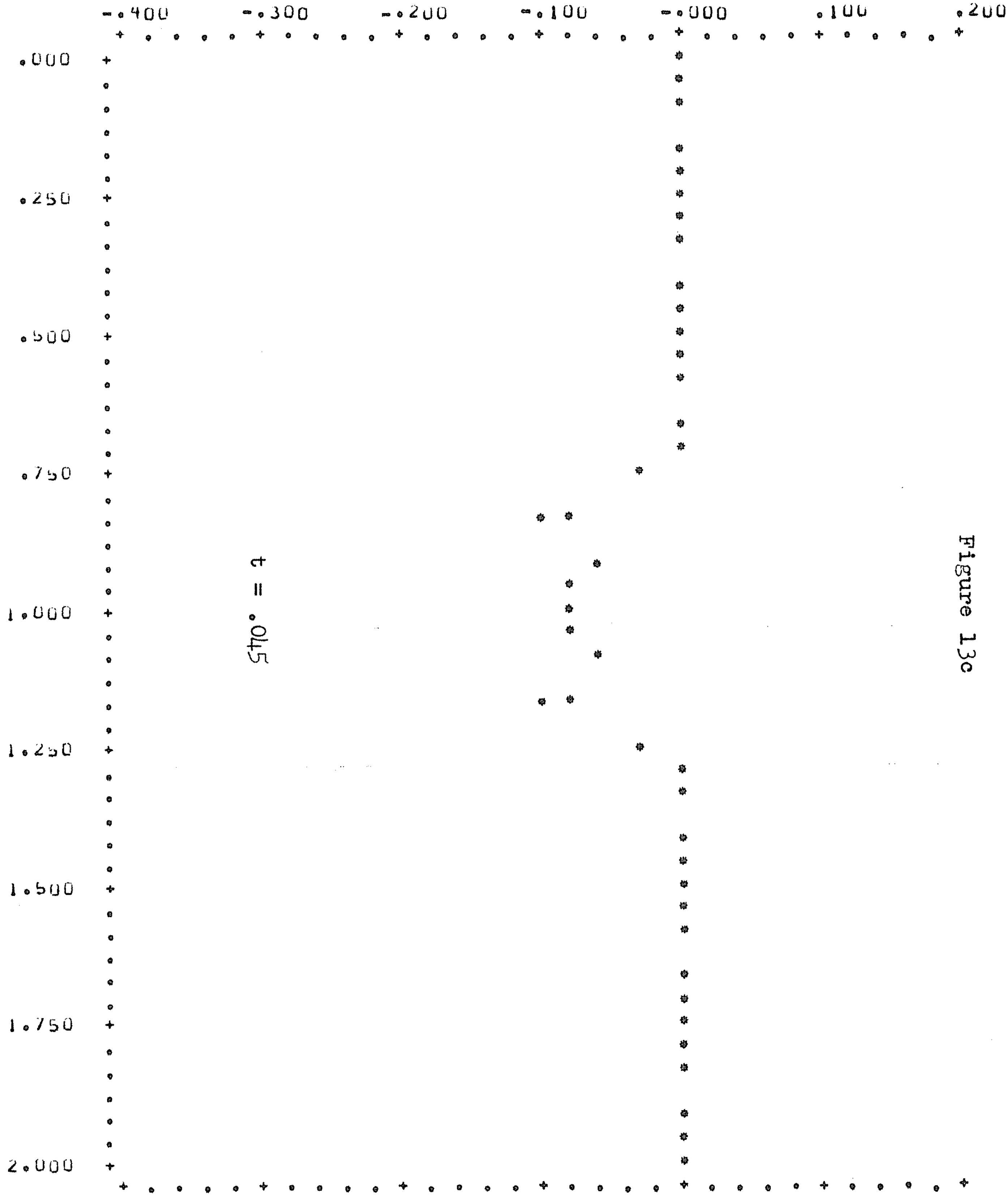
t = .027

Figure 13b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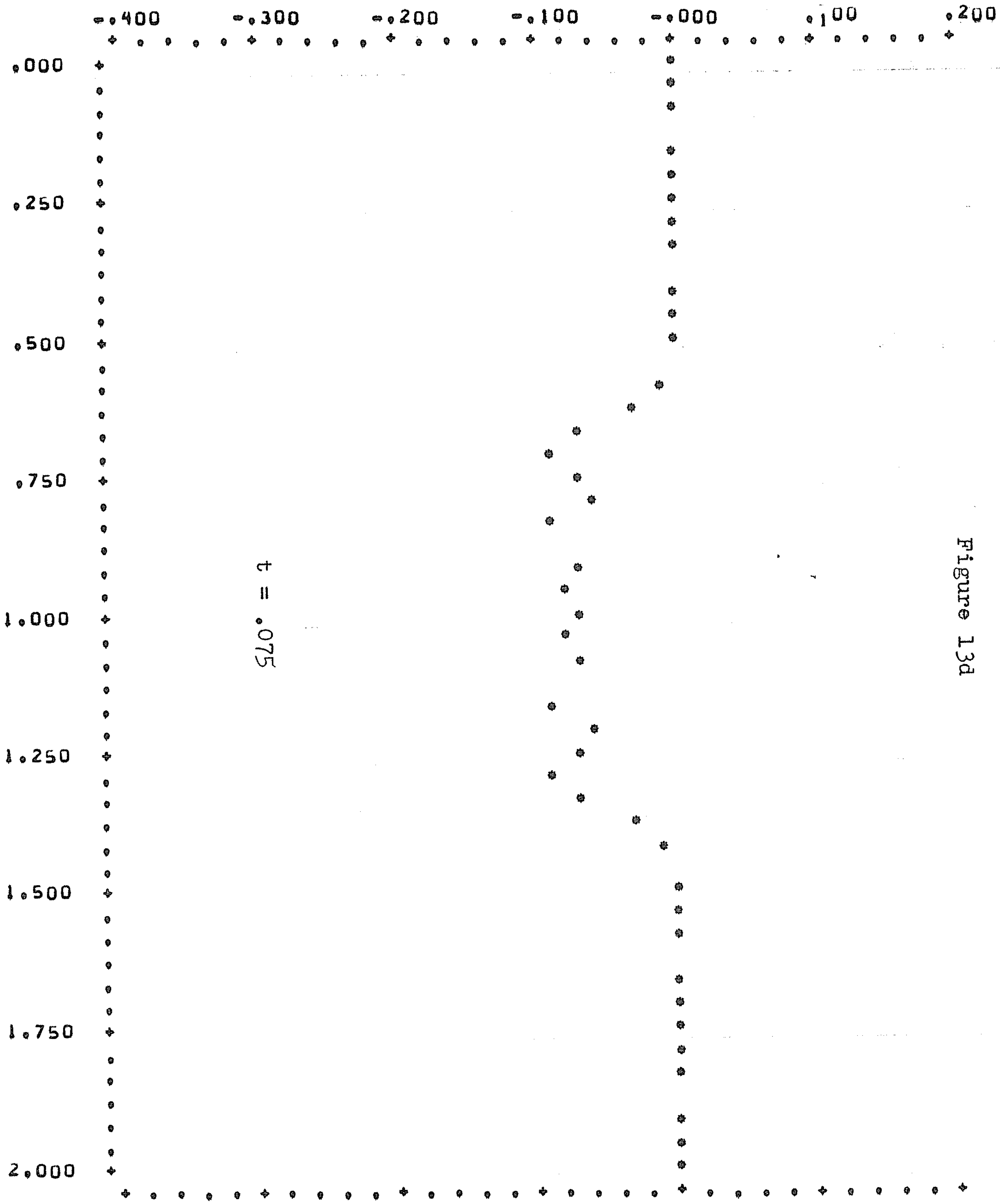
2.000
t = .045

Figure 13c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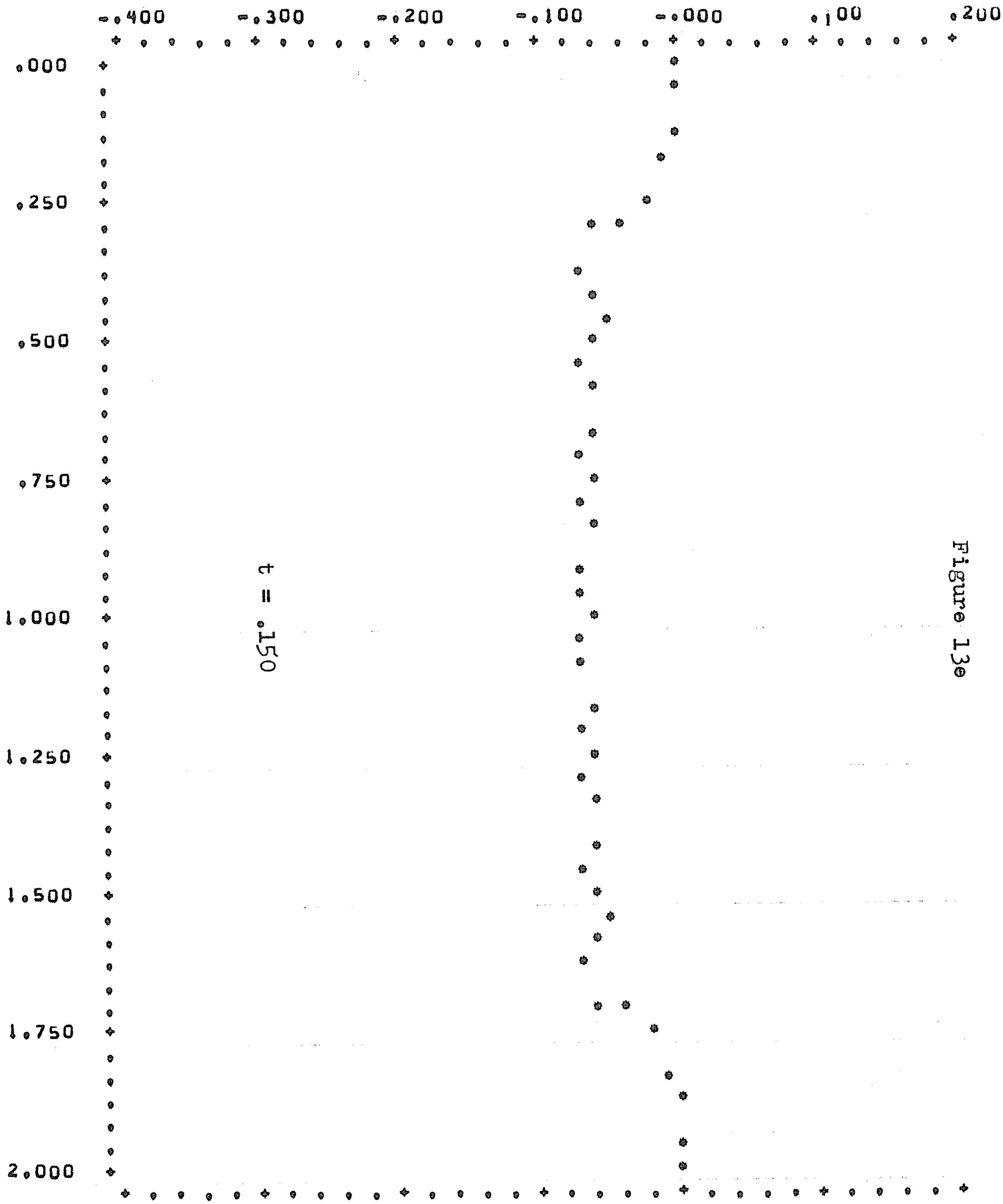
t = .075

Figure 13d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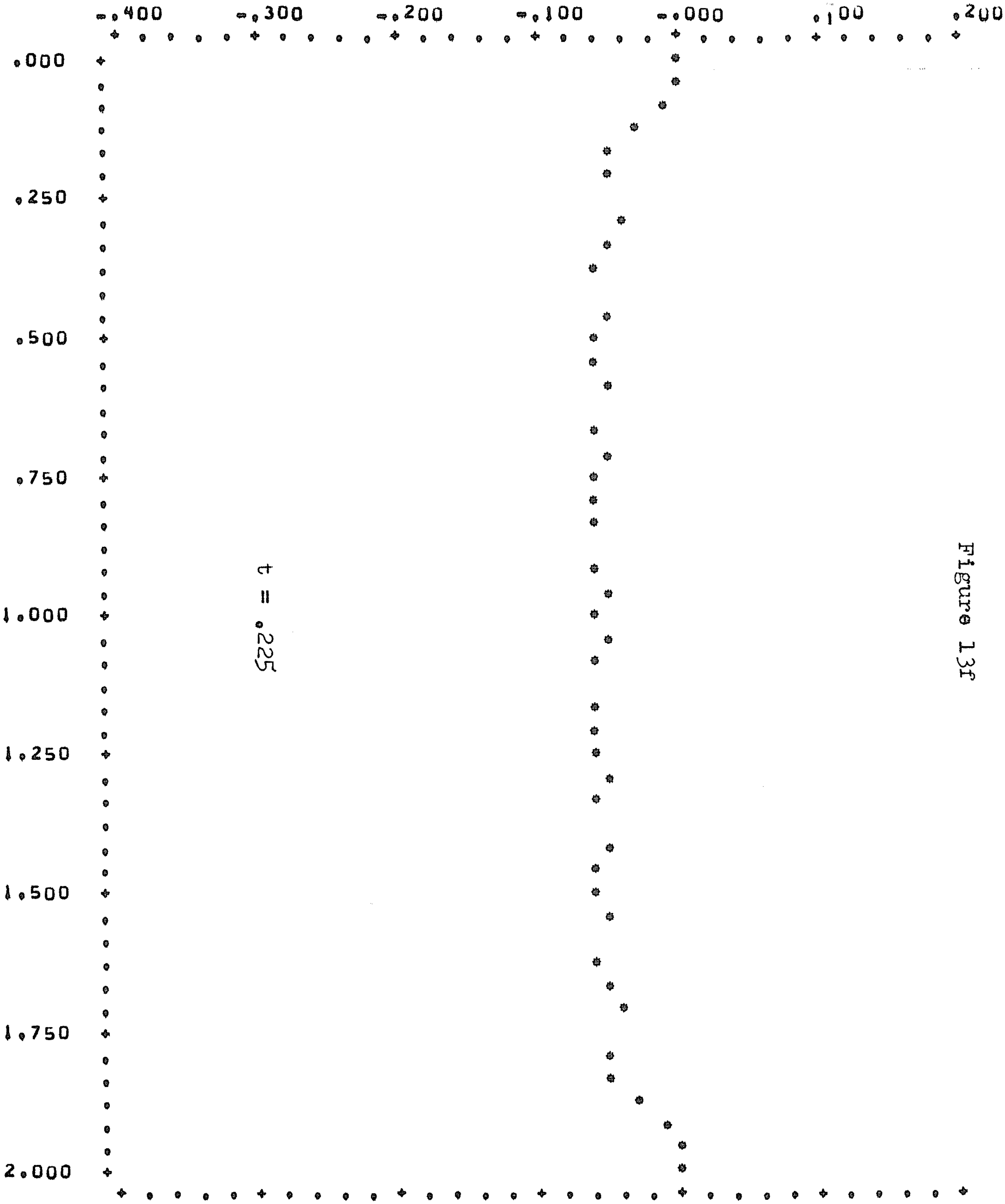
t = .150

Figure 13e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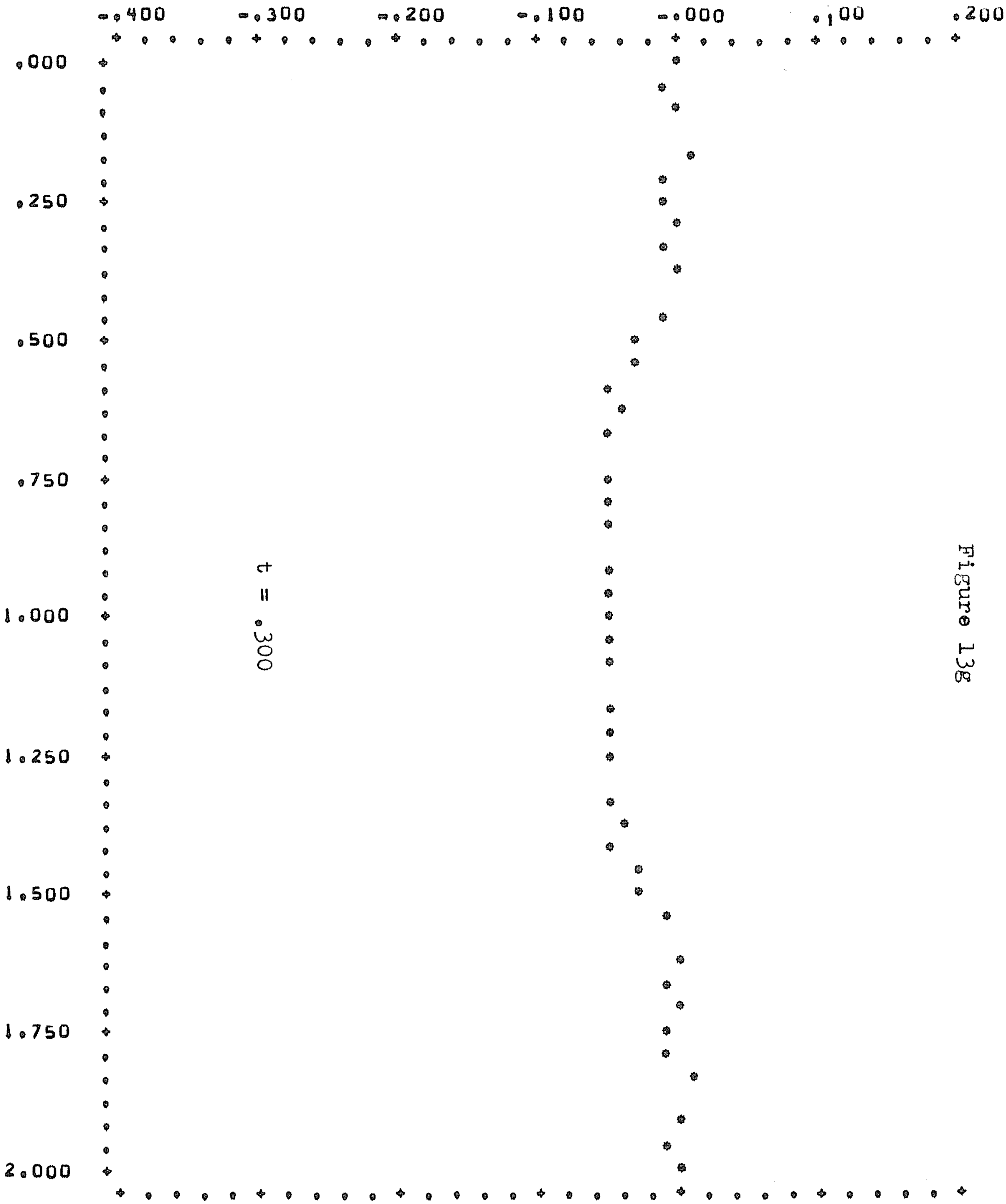
t = .225

Figure 13f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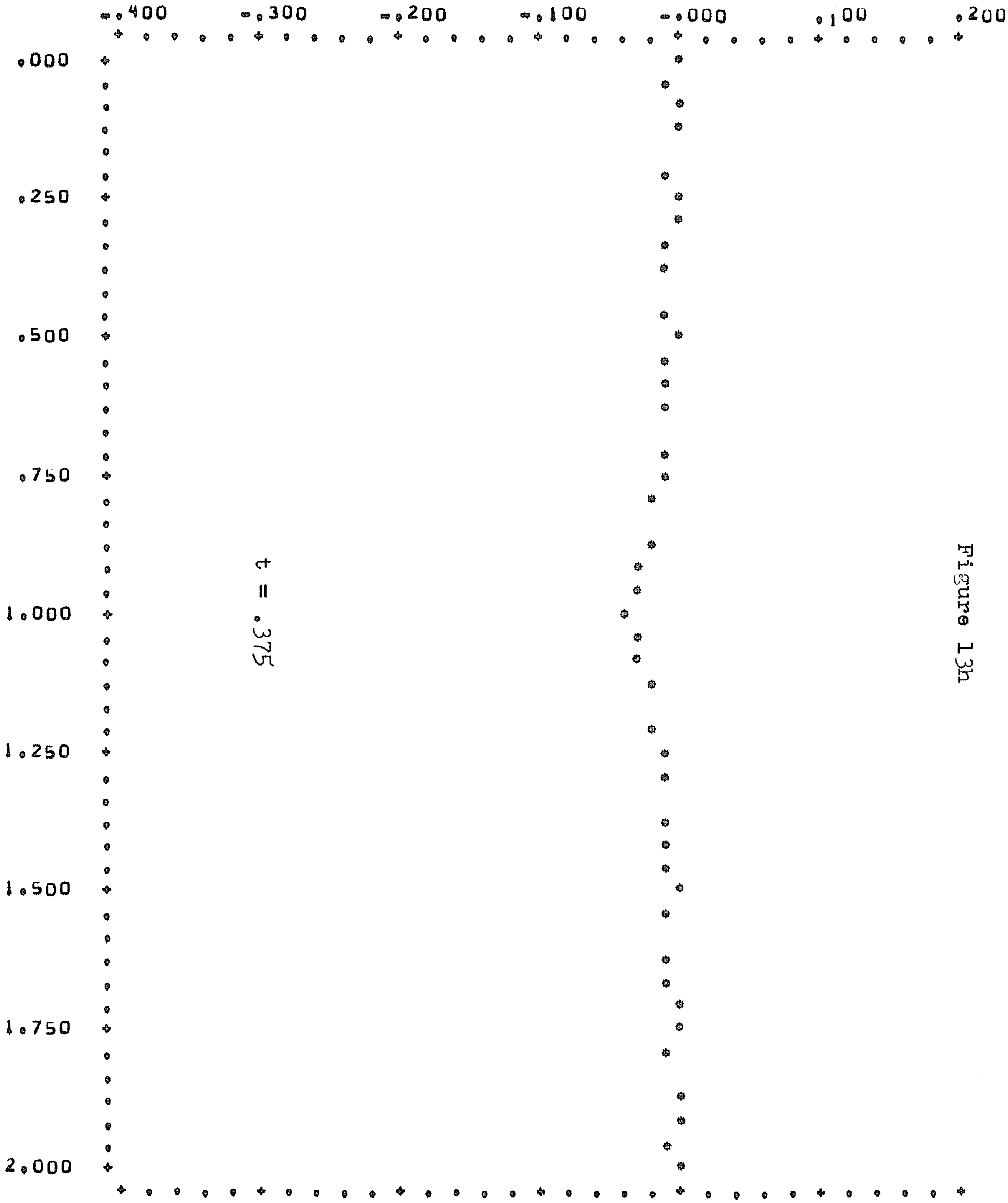
t = .300

Figure 13g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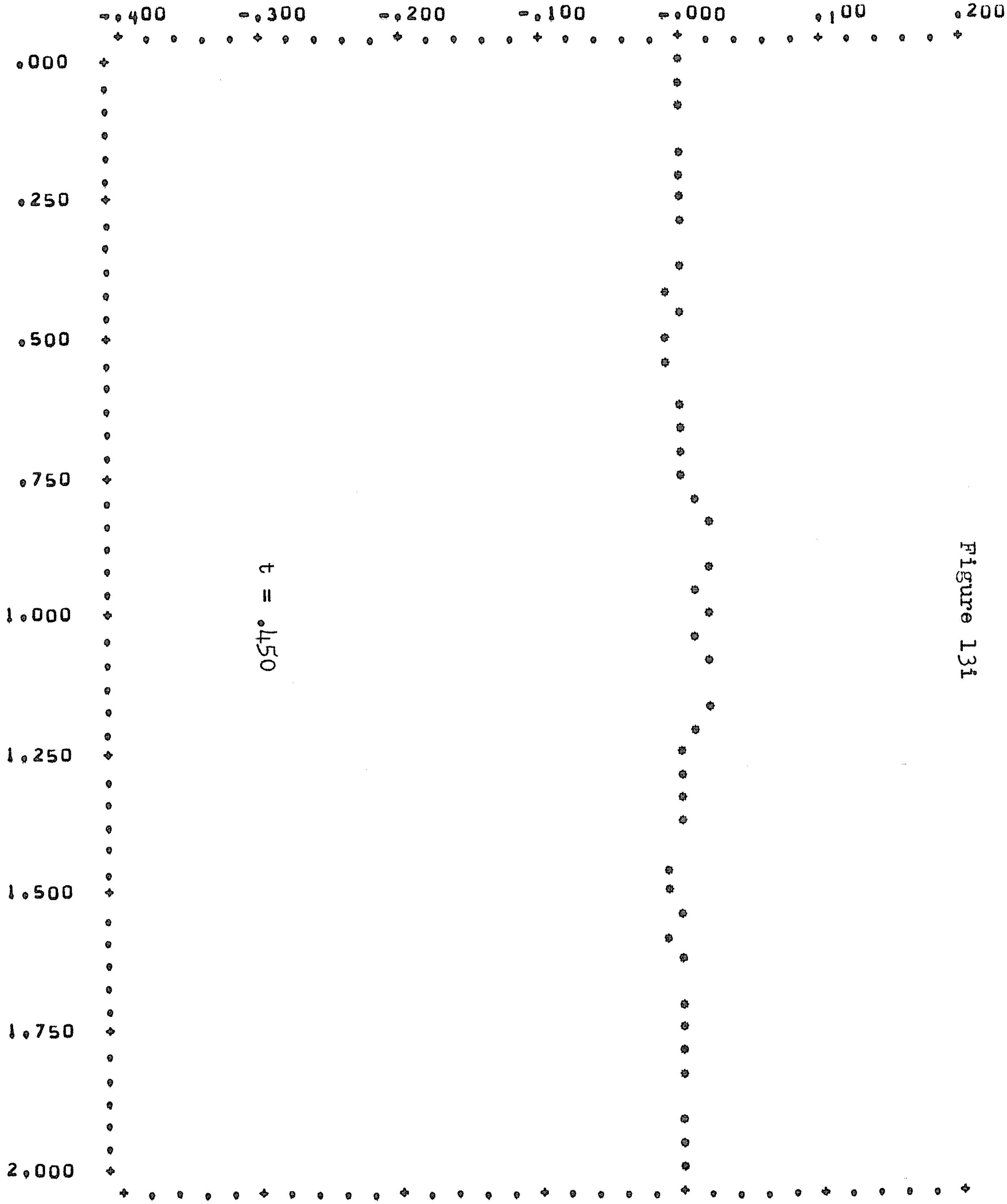
t = .375

Figure 13h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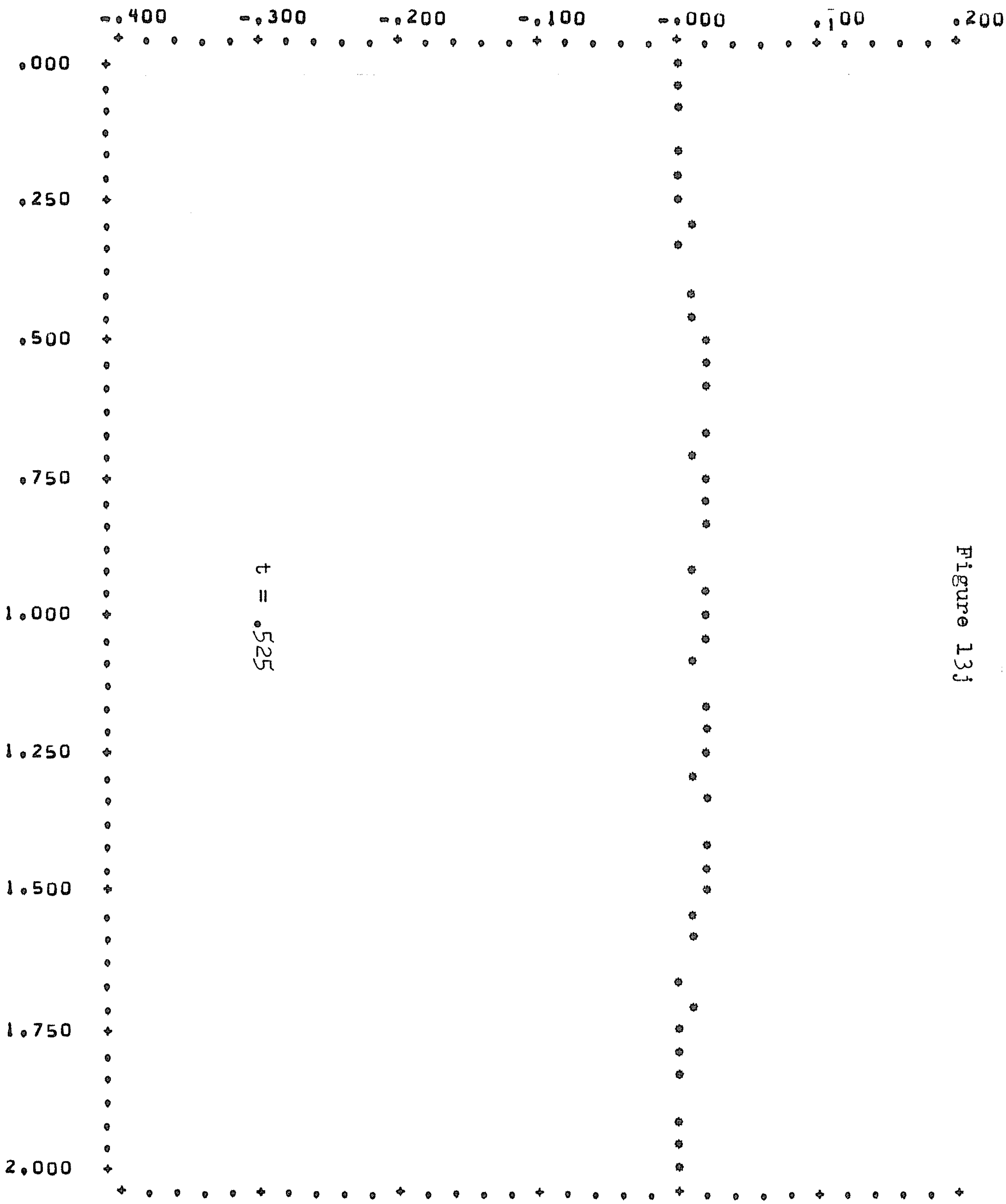
t = .450

Figure 13i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t = .525

Figure 13j

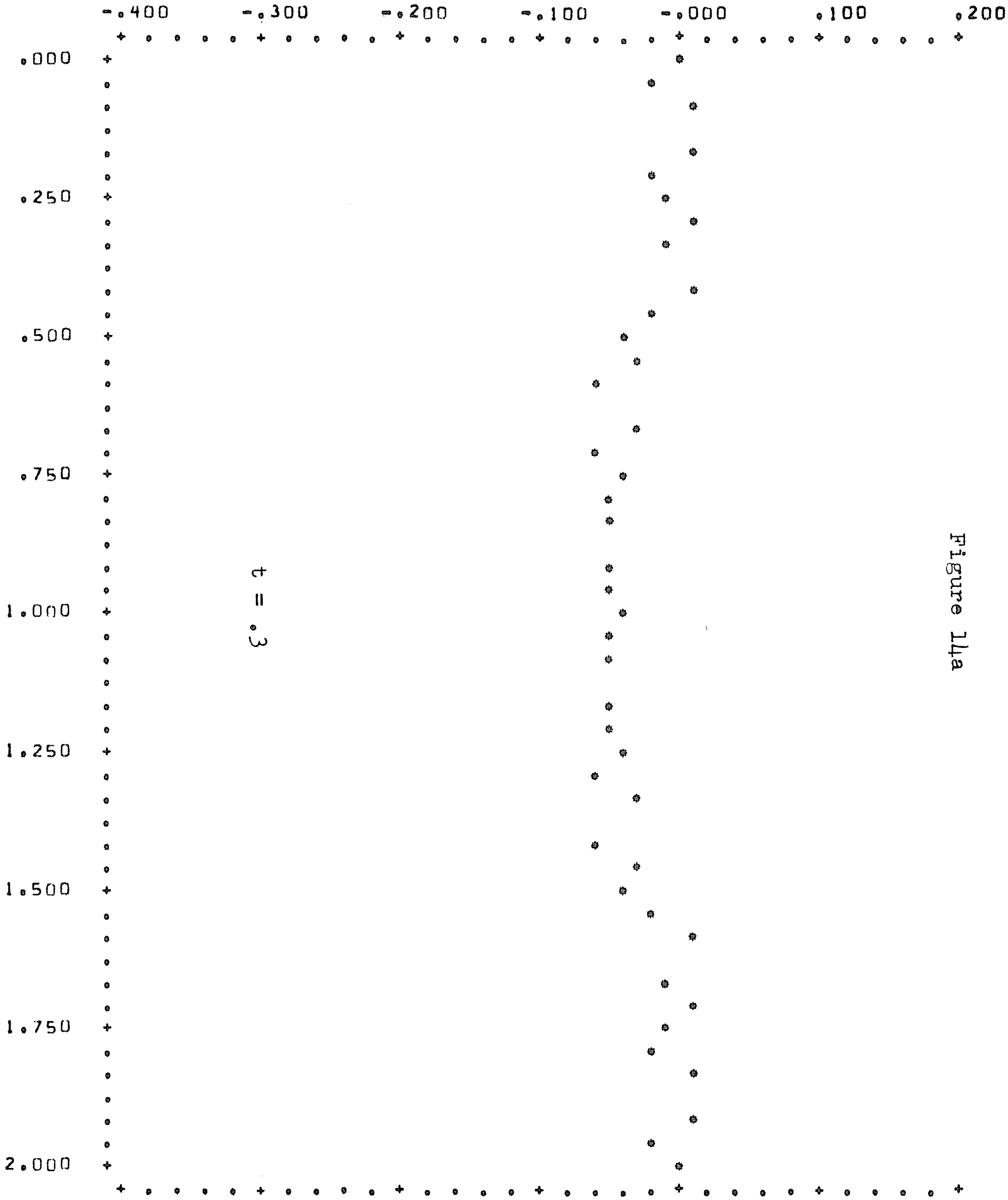
Example 6.4

This example is intended to demonstrate instability in a numerical computation. The initial conditions are the same as in (6.6) and the parameter values are as given in (6.7) with $\varepsilon = 0$ except for the value of Δt, which is here taken to be

$$\Delta t = .003 . \tag{6.8}$$

Figures 14a - 14e show the first 1.5 time units of motion of the string during which it is clear that the computation is already becoming unbounded. Thus an increase in Δt of a mere .0015 units has caused a change from a stable computation to one which is unstable.

Finally, we point out that the average Univac 1108 execution time for parameter cases considered in examples 6.3 and 6.4 was estimated to be approximately 240 time-steps per second of real time.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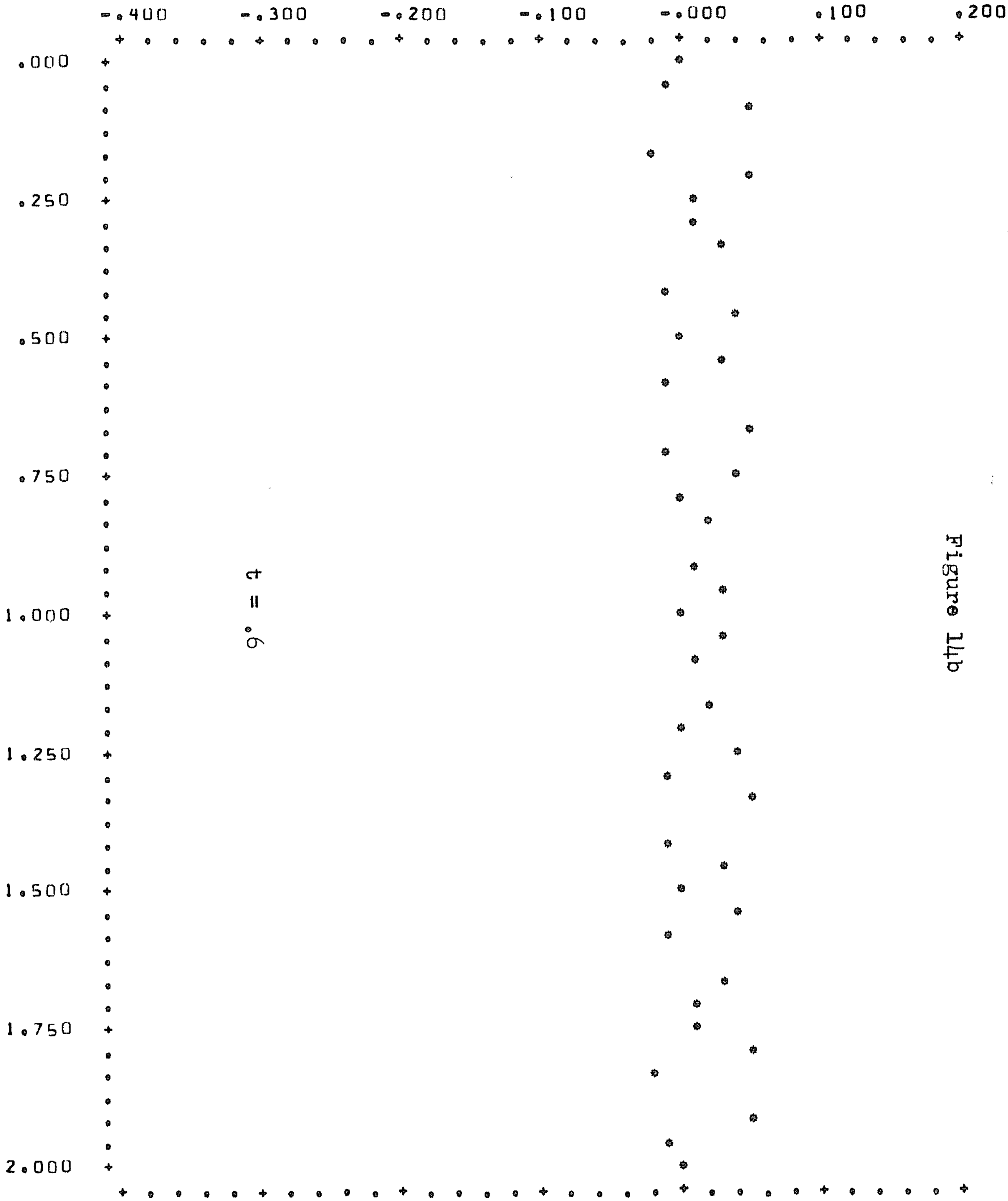
t = .3

Figure 14a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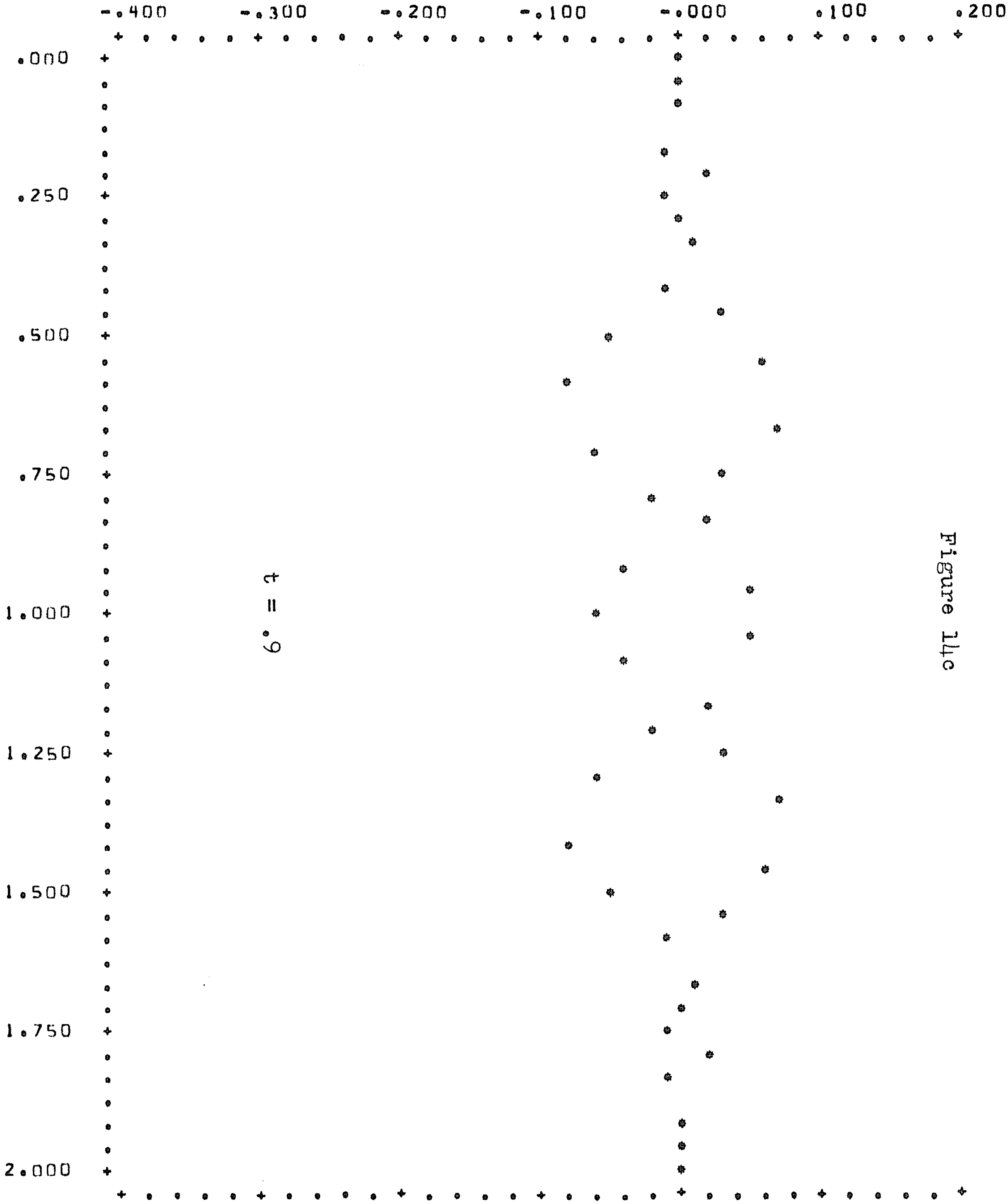
t = .6

Figure 14b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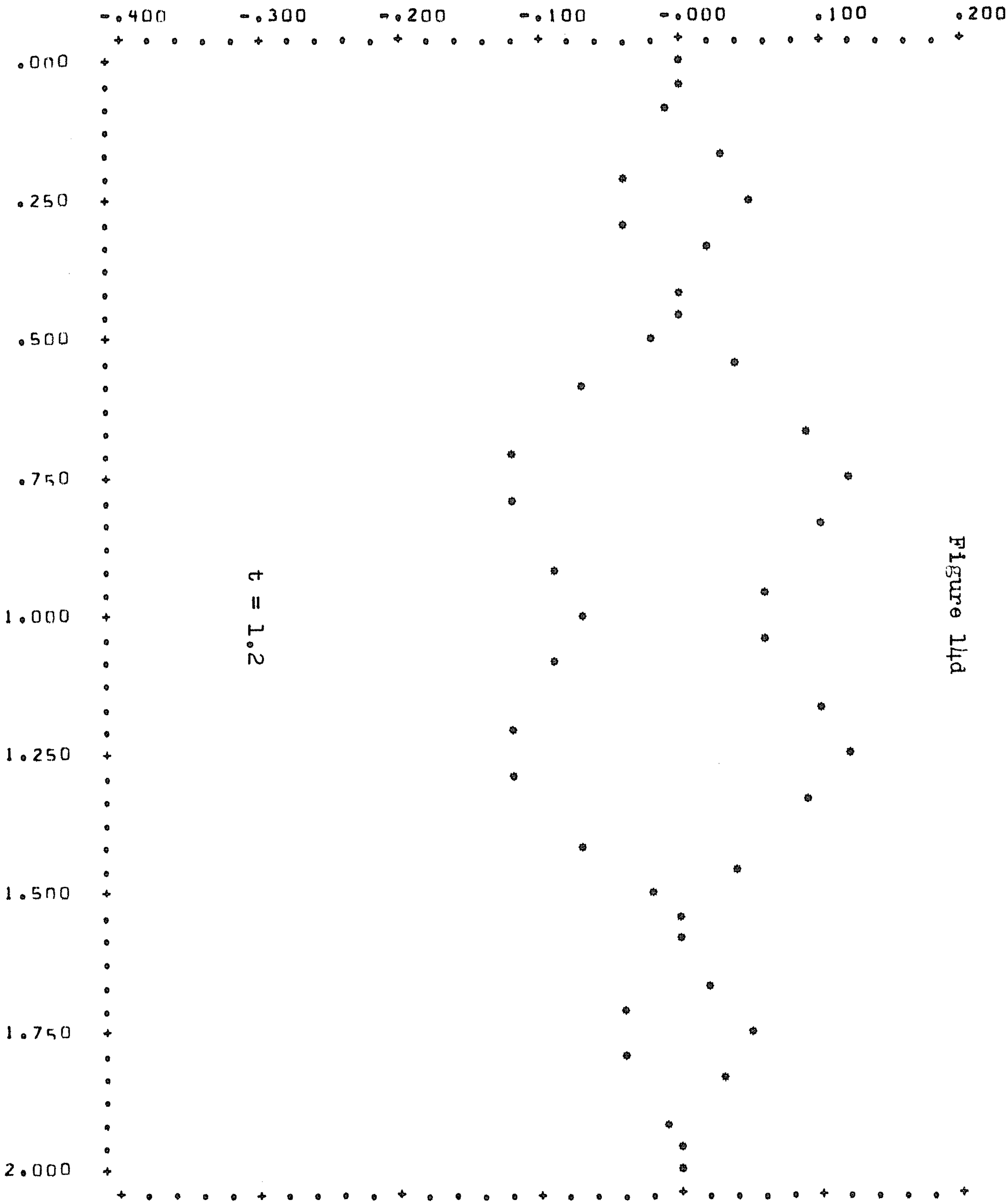
2.000
t = .9

Figure 14c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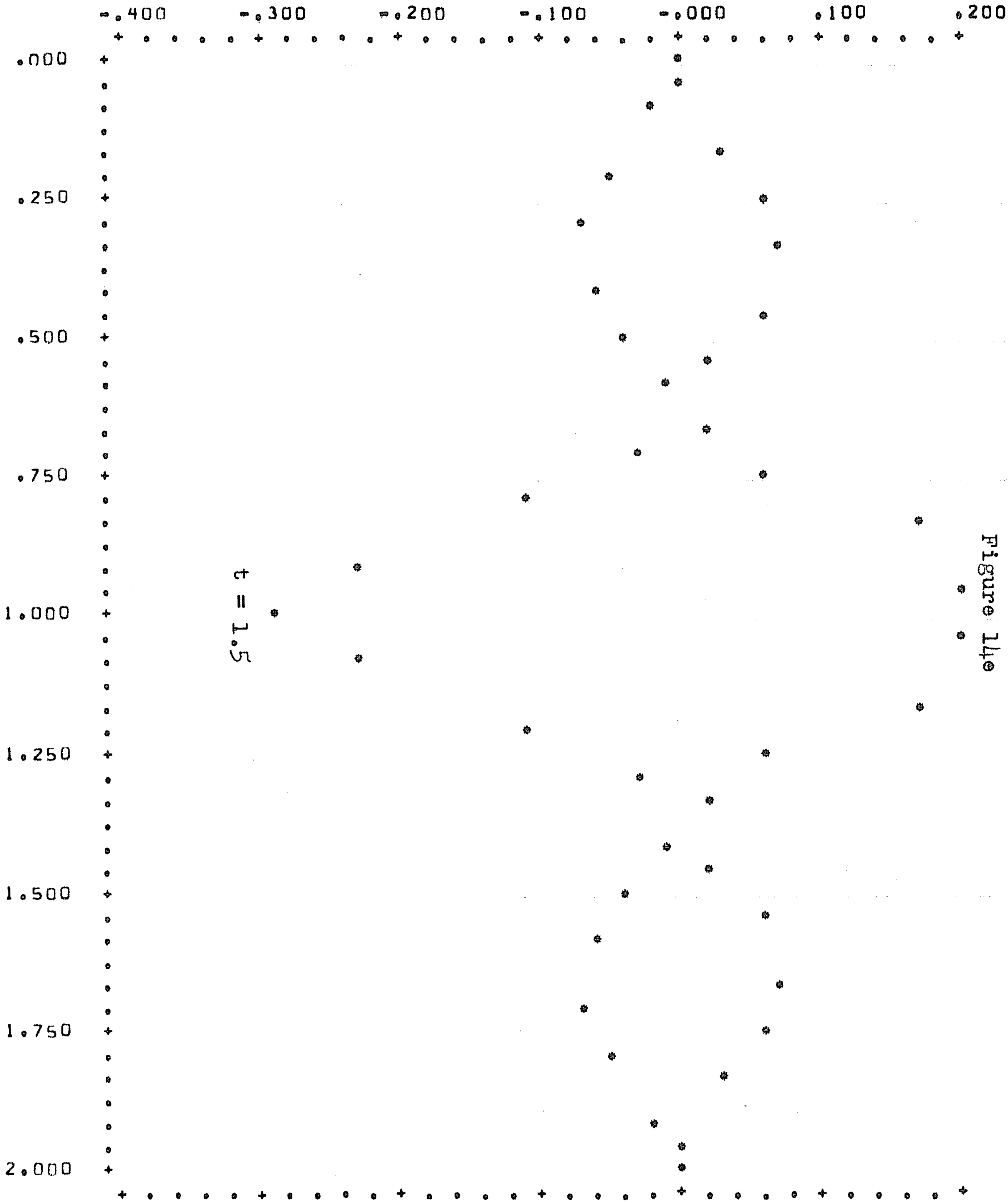
2.000
t = 1.2

Figure 14d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1.750
2.000
t = 1.5

Figure 14e

7. Studies of a Free-end String

Now we proceed to the study of a string fixed only at one end. Referring to the string description in Section 2 and to Figure 1, the problem may be described as follows. Initially the string is assumed to be hanging in its equilibrium position with particle P_1 at the point $(0,0)$ and particle P_{n+1} at $(2,0)$. For $t > 0$ particle P_1 is no longer fixed at the origin but is free to move according to the governing law of motion. Thus we may use the same definitions, law of motion, and computational procedure as in the case of the string fixed at both ends, except that we must include the motion of P_1. The necessary modifications may be made as follows. In step 3 of Section 5 include the computation of $a_{1,k}$ which is given by

$$(7.1) \qquad m a_{1,k} = \tau_{1,k} + \mu_{1,k} + G, \quad k = 0,1,2,\ldots ,$$

where $\mu_{1,k}$ and G are defined in (4.2) and (4.3), respectively, and

$$(7.2) \qquad \tau_{1,k} = |T_{1,2,k}| \frac{x_{2,k} - x_{1,k}}{r_{1,2,k}} ,$$

where $|T_{1,2,k}|$ is defined by equation (4.5).

The boundary conditions for this problem are

$$(7.3) \qquad \left\{ \begin{array}{l} x_{n+1,k} = (2,0) \\ v_{n+1,k} = (0,0) \end{array} \right\} \quad k = 0,1,2,\ldots ,$$

whereas, for initial conditions, we have

$$(7.4) \qquad \left\{ \begin{aligned} v_{j,0} &= (0,0) \\ x_{j,0} &= \hat{x}_j \end{aligned} \right\} , \quad j = 1,2,\ldots,n;$$

where $\hat{x}_j$ is the equilibrium (steady-state) position of the j^{th} particle of a string hanging from both ends. $\hat{x}_j$, $j = 2,3,\ldots,n$, may be computed by setting all time-dependent variables in equation (4.11) to zero and solving the resulting system of nonlinear equations. Thus one would solve the following system by the generalized Newton's method (see [5]):

$$(7.5) \qquad \frac{T_0}{m\Delta x} \{(1-\varepsilon)(x_{j+1} - 2x_j + x_{j-1}) + \frac{\varepsilon}{\Delta x} [r_{j,j+1} x_{j+1} - (r_{j,j+1} + r_{j-1,j}) x_j + r_{j-1,j} x_{j-1}]\} - g = 0, \quad j = 2,3,\ldots,n.$$

Call any solution so obtained $\hat{x}_j$. (Note that $\hat{x}_1 = (0,0)$ and $\hat{x}_{n+1} = (2,0)$.)

In case $\varepsilon = 0$ an alternative method, which is faster, may be used to solve for the initial position, for this corresponds to the linear case of uniform tension T_0 in the string. Indeed, one may take the classical continuous wave equation

$$(7.6) \qquad u_{tt} = \frac{T_0}{\rho} u_{xx} - g_0 ,$$

where u is the displacement in the vertical direction, ρ is the linear density of the string, and g_0 is the acceleration of gravity, set $u_{tt} = 0$; $\rho = nm/2$, and solve the resulting ordinary differential equation

$$(7.7) \qquad \frac{d^2u}{dx^2} = \frac{nm\,g_0}{2\,T_0}$$

subject to the conditions that u vanish at $x = 0$ and $x = 2$. The solution is

$$(7.8) \qquad u(x) = \frac{nm\,g_0}{4\,T_0}\,x(x-2).$$

Then, recalling that the horizontal components of particle positions in the linear case $\varepsilon = 0$ are just the positions equally spaced by Δx units on the interval [0,2], we have

$$(7.9) \qquad \left\{ \begin{aligned} \hat{x}_j^{(1)} &= (j-1)\Delta x \\ \hat{x}_j^{(2)} &= u(\hat{x}_j^{(1)}) \end{aligned} \right\} , \; j = 2,3,\ldots,n ,$$

and initial conditions (7.4) are thus determined.

A comparison between the two methods of computing these initial conditions when $\varepsilon = 0$ was actually made for T_0 in the range 12.5 - 50 to justify the use of the second, faster, method as a good approximation to the first. Table 2 shows the results of this comparison for the case $n = 20$. The quantity δ is the maximum absolute discrepancy between positions computed by the two methods.

T_0	δ
12.5	.003
20	.001
25	.001
35	.001
50	.001

Table 2

Example 7.1

The purpose of this example is to contrast the cases of a linear tension formula and a nonlinear formula. The parameter values are given as follows:

	Case 1	Case 2
n	20	20
T_0	12.5	12.5
m	.05	.05
Δt	.002	.003
α	.15	.15
ε	.2	.0

Note that there are different values of Δt for the two cases. The smaller value of Δt in the nonlinear case ($\varepsilon = .2$) is chosen to ensure numerical stability in the computation, since it is known from empirical evidence that there is a greater restriction on the size of Δt in a nonlinear case than in a linear one.

Figure 15 is a side-by-side plot of the first downward swing for these two cases at time intervals of .18 units. The solid curves correspond to the case $\varepsilon = .2$, and the dashed curves correspond to the case $\varepsilon = 0$. One can see in the figure that the linear case tends to lag slightly behind the nonlinear case in the downward motion.

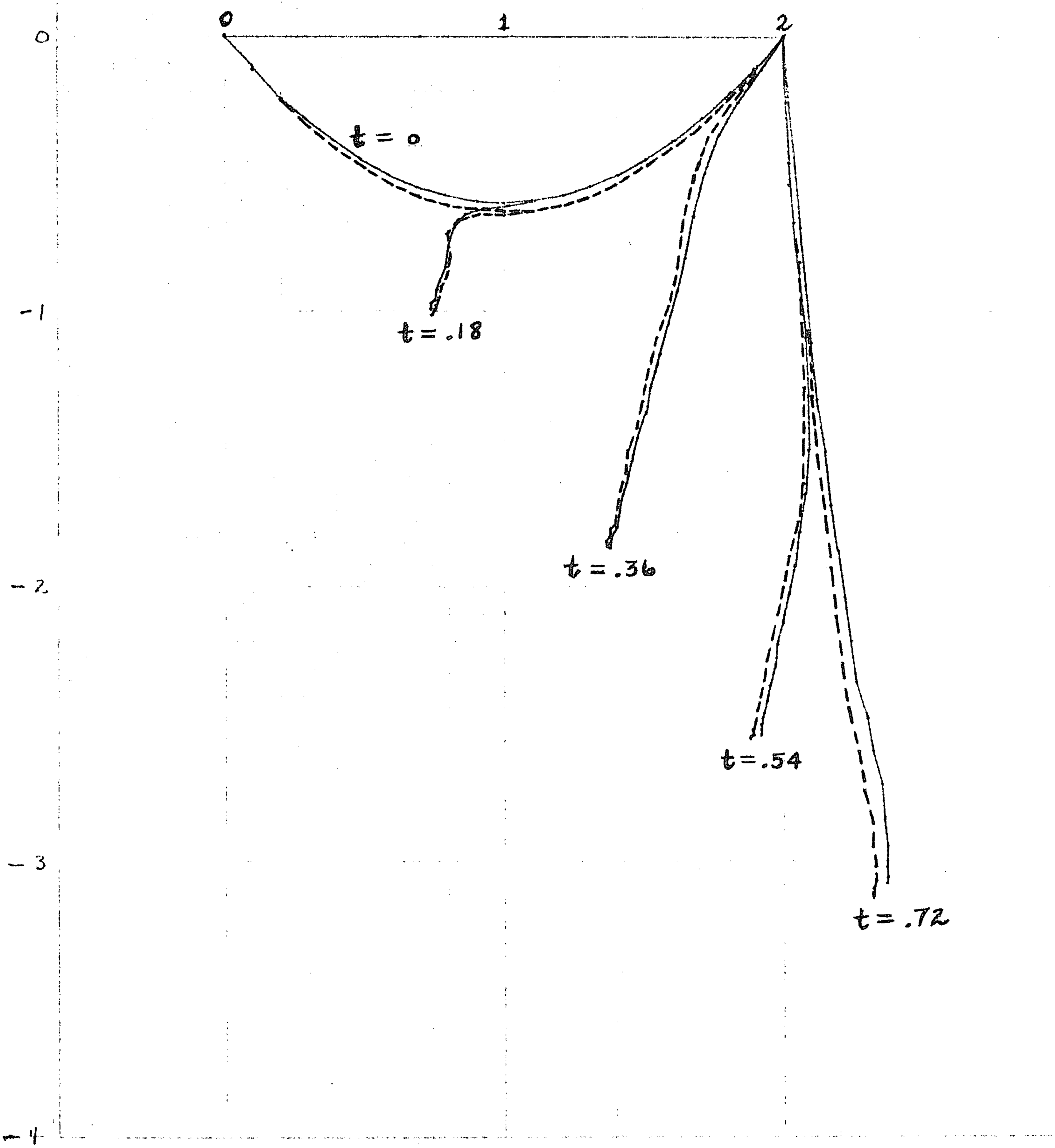
Example 7.2

This example contrasts cases with different values of the viscosity parameter α. The tension law is taken to be non-linear for greater generality. The parameter values chosen are as follows:

	Case 1	Case 2
n	20	20
T_0	12.5	12.5
m	.05	.05
Δt	.002	.002
α	.15	.05
ε	.2	.2

Figure 15

$T_0 = 12.5 \quad \alpha = .15 \quad \varepsilon = .2$ (solid) $\quad \Delta t = .002$
$\varepsilon = 0$ (dashed) $\quad \Delta t = .003$



Note that all parameters except α are the same in both cases. The value of α in Case 1 is a reference value which has been used in all examples described previously. Figure 16 shows the first downswing of the motion of the string in each case at uniform time intervals of .18 units. The solid curves correspond to Case 1 ($\alpha = .15$), and the dashed curves correspond to Case 2 ($\alpha = .05$). As expected, the string's motion is seen to be less restricted for the less viscous case, as indicated by the greater stretching of the string at the extremity of its first downswing.

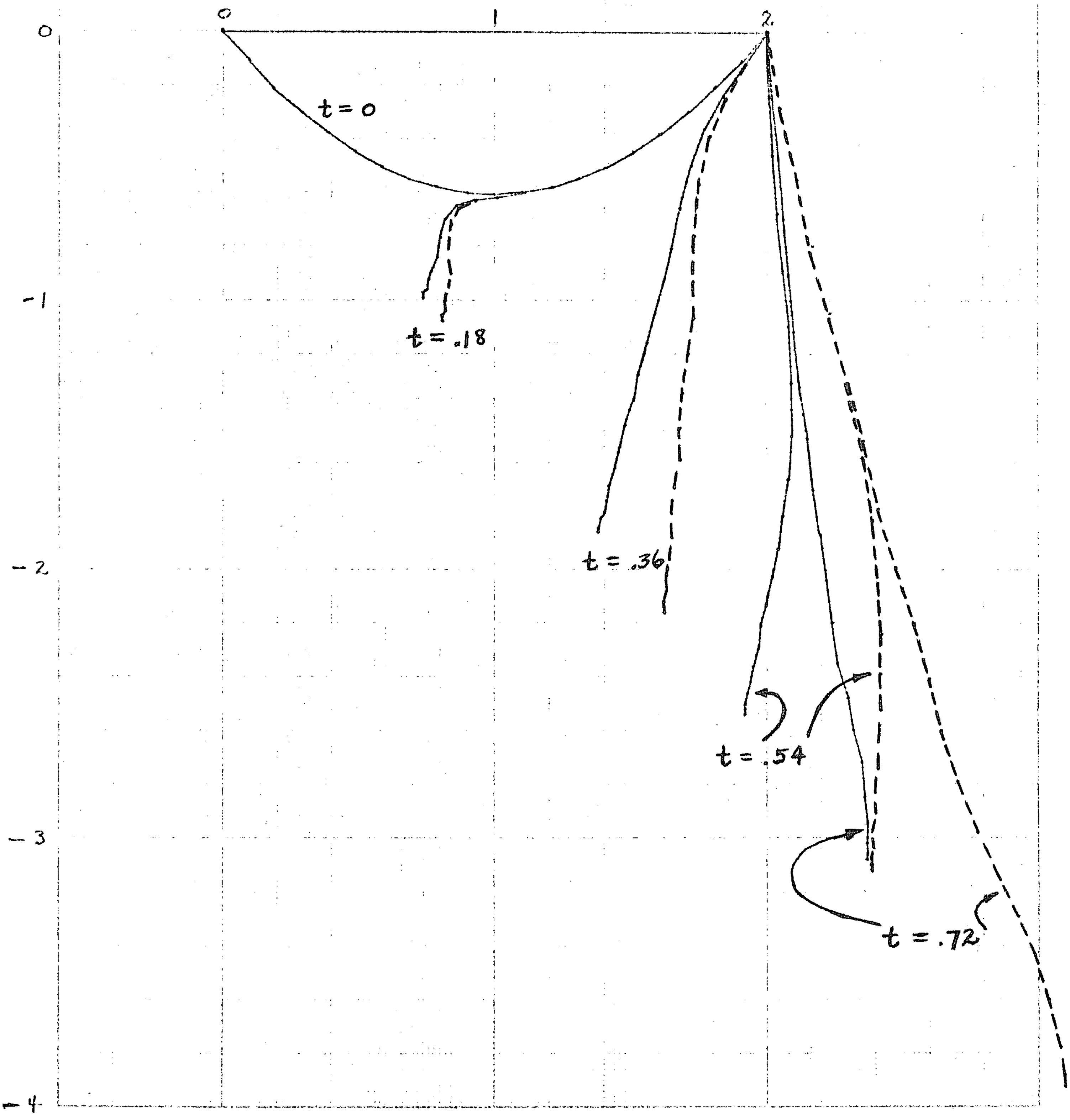
Example 7.3

For a final example of the motion of a free-end string, we choose to contrast cases of different reference tensions T_0. The parameter choices are as follows:

	Case 1	Case 2
n	20	20
T_0	12.5	25
m	.05	.05
Δt	.002	.002
α	.15	.15
ε	.2	.2

Figure 16

$T_0 = 12.5$ $\alpha = .15$ (solid) $\varepsilon = .2$ $\Delta t = .002$
$\alpha = .05$ (dashed)



Once again all parameters except T_0 are held constant in the two cases so that all differences are due to the choices of T_0. Also ε is taken to be non-zero so that the cases are nonlinear.

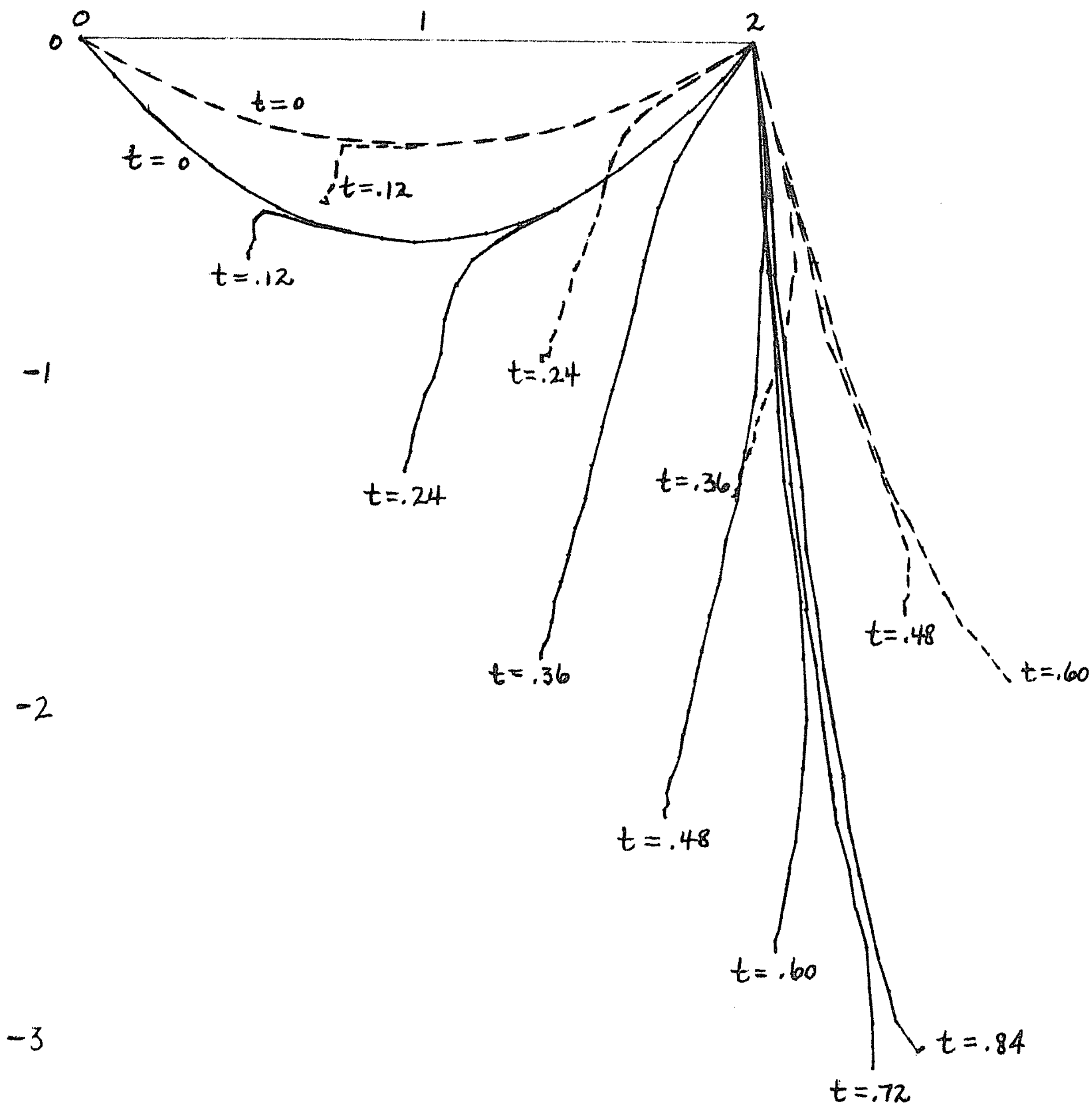
Figure 17 illustrates the differences between these two cases for the first downward swing of the string. Note that there is a great difference in initial positions of the string in the two cases and that there is a large difference in the nature of the downward motion in the two cases. The greater tension in Case 2 results in a shallower down-swing than in Case 1 but one which swings farther outward in the horizontal direction. In short, in Case 2 the horizontal forces, due primarily to tension, are greater with respect to the vertical forces, due primarily to gravity, than in Case 1.

In Table 3 is shown the amount of time required to reach steady-state in each of the cases considered for a free-end string. The steady state, or equilibrium position for a string fixed at one end only is, of course, such that the horizontal component is identical for each particle and the vertical component is constant for each particle. In our case this implies $x_j = (2, x_j^{(2)})$, $j = 1,2,\ldots,n+1$. The criterion chosen for the occurrence of steady-state was the same as in section 6 (see (6.5)), i.e., when the following condition is satisfied:

$$\max_{1 \leq i \leq n+1} \{ \| x_{i,k} - \bar{x}_i \|_\infty \} \leq .001 \quad ,$$

Figure 17

To = 12.5 (solid) α = .15 ε = .2 Δt = .002
To = 25.0 (dashed)



where $\bar{x}_i$ is the steady-state position of the i-th particle, and $x_{i,k}$ is the position of the i-th particle at time t_k.

m	n	T_0	α	ε	t_{ss}
.05	20	12.5	.15	.2	2.7
.05	20	12.5	.05	.2	14.7
.05	20	12.5	.15	0	5.0
.05	20	25	.15	.2	2.6

Table 3

For these parameter cases the average execution speed (including the time required to compute the initial position by the Generalized Newton's Method) was approximately 200 time-steps per second of real Univac 1108 time. This compares closely with the figure of approximately 265 for the 21-particle asymmetric string, fixed at both ends, which was discussed in section 6. The difference in these two figures may be attributed to the time required to compute the initial position for the free-end cases. The number of Generalized Newton iterations required was approximately 100 for the specified tolerance of $.5 \times 10^{-4}$.

We conclude our discussion of the free-end string by giving Table 4,which indicates a possible relationship between the final (steady-state) length ℓ of the string and the reference tension T_0 for the case $\varepsilon = 0$. (Recall that T_0 represents the tension between two consecutive particles spaced by a distance $\Delta x = 2/n$.)

T_0	ℓ	$T_0 \times \ell$	
		n = 20	n = 100
12.5	2.70	33.8	32.0
15	2.25	33.8	
17.5	1.94	34.0	
20	1.70	34.0	32.4
22.5	1.50	33.8	
25	1.35	33.8	32.5

Table 4

The table suggests that the product of T_0 and ℓ is constant, at least for the range of T_0 indicated, and that this constant approaches the magnitude of the acceleration due to gravity as the number of particles n gets large.

Finally, we remark that the computational method described in this section for the free-end string problem may be compared with a related problem which has been solved using classical mathematics only. That is the problem of the suspended rope which was solved (see [1],[7]) using Bessel functions. However the suspended rope problem is a much simpler problem since it assumes motion in one direction only. In particular it does not allow for the free end of the rope to move above its equilibrium height. So thus far there is no known classical mathematical solution for the free-end string problem which we have solved by our discrete method.

The appendix contains a listing of the Fortran program used in the computation for the free-end string problem.

8. Remarks

We conclude this report with a few remarks about extensions which can be made. First, the problem can be extended to three dimensions. Further, one could take into account rotary motion of the particles. One could also refine the "single-strand" model for the arrangement of the string's component particles to allow for multiple strands, or rows, of particles. This last refinement has associated with it the difficulty of determining an appropriate description of the forces acting between particles in different rows.

In Section 4 the formula chosen for the magnitude of the tension between consecutive particles was quadratic in distance between the particles. We have already noted that polynomials of higher degree could be chosen, but this has not yet been implemented. One could also consider using different polynomials in different intervals of distance between particles, e.g., a piecewise linear polynomial (see [3]).

Finally, there is current investigation into the stability of the computational method described in section 5. The first efforts have been empirical in nature, and some stability conditions have been hypothesized, but thus far no mathematical proofs of these conditions have been obtained.

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Appendix

```
      THIS PROGRAM COMPUTES THE TWO-DIMENSIONAL POSITIONS AND
      VELOCITIES OF THE PARTICLES COMPOSING A DISCRETE STRING OF
      UNIT TOTAL MASS WHICH IS FIXED AT ONE END ONLY.  THE COMPUTA-
      TIONS ARE DONE AT A SPECIFIED TIME INCREMENT STARTING FROM THE
      STEADY-STATE POSITION OF A STRING HANGING WITH BOTH ENDS FIXED.
      THE FORCES CONSIDERED TO BE ACTING ON THE PARTICLES ARE NON-
      LINEAR TENSION, VISCOSITY, AND GRAVITY.


@FOR,IS   ,ENDFREE

      PARAMETER NPART=  21,N=NPART-1,NH=N/2+1

C          NPART = TOTAL NUMBER OF PARTICLES IN STRING.

      REAL MASS
      DIMENSION X(NPART,2),V(NPART,2,2),A(NPART,2,2),X0(NPART,2)
     * ,V0(NPART,2),ALPHAM(2),XS(NPART,2)

C          DEFINITIONS OF DIMENSIONED VARIABLES.

C         X(I,J) = J-TH COMPONENT OF POSITION OF I-TH PARTICLE.
C       V(I,J,1) = J-TH COMPONENT OF VELOCITY OF I-TH PARTICLE AT PREVIOUS
C                  TIME-STEP.
C       V(I,J,2) = SAME AS V(I,J,1) EXCEPT AT CURRENT TIME-STEP.
C       A(I,J,1) = J-TH COMPONENT OF ACCELERATION OF I-TH PARTICLE AT
C                  PREVIOUS TIME-STEP.
C       A(I,J,2) = SAME AS A(I,J,1) EXCEPT AT CURRENT TIME-STEP.
C        X0(I,J) = STARTING VALUE USED IN ITERATIVE METHOD FOR COMPUTING
C                  J-TH COMPONENT OF INITIAL POSITION OF I-TH PARTICLE.
C        V0(I,J) = J-TH COMPONENT OF INITIAL VELOCITY OF I-TH PARTICLE.
C        XS(I,J) = J-TH COMPONENT OF INITIAL POSITION OF I-TH PARTICLE.

      DATA G,ERR,ISTDY,RHO/32.2,1.E-7,1,.5/

C          DEFINITIONS OF DATA VARIABLES.

C              G = ACCELERATION DUE TO GRAVITY.
C            ERR = SMALL POSITIVE NUMBER USED TO RESOLVE FLOATING POINT
C                  ZERO.
C          ISTDY = VARIABLE CONTROLLING METHOD OF COMPUTING INITIAL POS-
C                  ITIONS OF PARTICLES.
C                  ISTDY.EQ.0 IMPLIES (FOR THE CASE EPS = 0) USE CONTIN-
C                             UOUS APPROXIMATION FOR INITIAL POSITIONS.
C                             (IF EPS.GT.0 THIS IS AN ERRONEOUS CONTROL
C                             VARIABLE VALUE.)
C                  ISTDY.NE.0 IMPLIES USE THE GENERALIZED NEWTON'S METHOD
C                             FOR COMPUTING INITIAL POSITIONS.
C            RHO = LINEAR MASS DENSITY OF STRING.


C     USE A VARIABLE OTHER THAN THE PARAMETER VARIABLE NPART TO HOLD
```

```
C       THE TOTAL NUMBER OF PARTICLES.

        NP=NPART

C       COMPUTE INDIVIDUAL PARTICLE MASS, REFERENCE HORIZONTAL PARTICLE
C       SPACING, AND CONSTANTS TO BE USED IN INITIAL POSITION COMPUTATION.

        MASS=1./N
        DX=2./N
        RHOG2=RHO*G*.5
        C=G/N*DX

C       SET INITIAL VELOCITIES TO ZERO AND STARTING VALUES FOR POSITIONS
C       TO ZERO IN THE VERTICAL DIRECTION AND TO EQUALLY-SPACED VALUES
C       IN THE HORIZONTAL DIRECTION.

        DO 2 I=1,NPART
        X0(I,1)=(I-1)*DX
        X0(I,2)=0.
        V0(I,1)=0.
      2 V0(I,2)=0.

C       WRITE HEADING LINE DEFINING TYPE OF PROBLEM THIS PROGRAM SOLVES.

        WRITE(6,95) NP    ,DX,MASS
        WRITE(6,96)

C       PRINT OUT INITIAL PARTICLE VELOCITIES.

        CALL PRINT(0,V0,1,0,NP)

C       INITIALIZE CURRENT VALUES OF PROBLEM PARAMETERS.

        EPS=-10.
        TEN0=-10.
        DT=-10.
        ALPHA=-10.

C       UPDATE PREVIOUS VALUES OF PROBLEM PARAMETERS.

      3 EOLD=EPS
        TOLD=TEN0
        DTOLD=DT
        AOLD=ALPHA

C       READ PARAMETERS FOR A DATA CASE.

        READ(5,98,END=30) LMAX,INCPR,INCPL,DT,TEN0,EPS,ALPHA,OMEGA

C          DEFINITIONS OF DATA PARAMETERS.

C           LMAX = MAXIMUM NUMBER OF TIME-STEPS THIS CASE IS TO BE RUN.
C          INCPR = TIME-STEP INCREMENT FOR PRINTING OF PARTICLE POSITIONS.
C          INCPL = TIME-STEP INCREMENT FOR PLOTTING OF PARTICLE POSITIONS.
```

```
C            DT = TIME-STEP SIZE.
C          TENO = REFERENCE TENSION BETWEEN CONSECUTIVE PARTICLES.
C           EPS = PARAMETER OF NONLINEARITY IN TENSION FORMULA.
C         ALPHA = COEFFICIENT OF VISCOSITY.
C         OMEGA = OVER-RELAXATION FACTOR FOR GENERALIZED NEWTON'S METHOD.

C     TEST TO SEE IF TENO OR EPS HAS CHANGED FROM PREVIOUS DATA CASE.

      IF(ABS(EOLD-EPS)+ABS(TOLD-TENO).LT.ERR) GO TO 1003

C     IF SO, COMPUTE VALUES DEPENDENT UPON THESE PARAMETERS.

      EDX=EPS/DX
      EPS1=1.-EPS
      CON=C/TENO
      TMDX=TENO/MASS/DX

C     TEST FOR METHOD OF COMPUTING INITIAL POSITIONS OF PARTICLES.

      IF(ISTDY.NE.0) GO TO 203

C     TEST WHETHER EPS IS ZERO IN THIS CASE.  IF NOT, THE CONDITION IS
C     ERRONEOUS, SO IGNORE CURRENT CASE AND READ THE NEXT ONE.

      IF(ABS(EPS).GT.ERR) GO TO 104

C     USE CONTINUOUS APPROXIMATION FOR INITIAL POSITIONS.

      RGT2=RHOG2/TENO
      DO 103 I=1,NPART
      XS(I,1)=X0(I,1)
  103 XS(I,2)=RGT2*X0(I,1)*(X0(I,1)-2.)
      GO TO 5

C     TEST FOR WHETHER OR NOT STEADY-STATE CALCULATION FAILED FOR THE
C     CURRENT VALUES OF TENO AND EPS IN A PREVIOUS CASE WITH A DIFFERENT
C     VALUE OF OMEGA.

 1003 IF(IFLAG.NE.0) GO TO 203

C     IF STEADY-STATE POSITION HAS ALREADY BEEN OBTAINED, HENCE THE CASE
C     ALREADY RUN, FOR THE CURRENT PARAMETER VALUES, IGNORE THE CURRENT
C     DATA CASE AND READ THE NEXT ONE.  OTHERWISE PROCEED TO RUN THIS
C     CASE.

      IF(ABS(DTOLD-DT)+ABS(AOLD-ALPHA).LT.ERR) GO TO 3
      GO TO 5

C     COMPUTE INITIAL POSITIONS BY GENERALIZED NEWTON'S METHOD.

  203 CALL STEADY(EPS1,EDX,OMEGA,CON,X0,XS,IFLAG)

C     TEST FOR WHETHER OR NOT INITIAL POSITION COMPUTATION SUCCEEDED.
```

```
C      IF SO, PROCEED TO RUN CURRENT CASE.  IF NOT, WRITE MESSAGE INDIC-
C      ATING THIS AND READ NEXT DATA CASE.

       IF(IFLAG.EQ.0) GO TO 5
  104  WRITE(6,90) TENU,EPS,ALPHA,DT,OMEGA
       GO TO 3

C      PRINT OUT PARAMETER VALUES FOR CURRENT DATA CASE.

    5  WRITE(6,97) TENU,DT,ALPHA,EPS
       WRITE(6,99)

C      PRINT OUT INITIAL PARTICLE POSITIONS.

       CALL PRINT(0,XS,1,1,NP)

C      TEST FOR WHETHER OR NOT INITIAL POSITIONS WERE THE ONLY COMPUT-
C      ATIONS DESIRED FOR THIS CASE.

       IF(LMAX.EQ.0) GO TO 3

C      COMPUTE HALF THE TIME-STEP SIZE AND VISCOSITY COEFFICIENT DIVIDED
C      BY PARTICLE MASS.

       DT2=DT/2.
       ALPHAM=ALPHA/MASS

C      WRITE MESSAGE INDICATING THE TIME-STEP INCREMENT FOR PLOTTING OF
C      PARTICLE POSITIONS.

       WRITE(6,94) INCPL

C      STORE INITIAL VALUES INTO WORKING ARRAYS.

       DO 6 I=1,NPART
       X(I,1)=XS(I,1)
       X(I,2)=XS(I,2)
       V(I,1,1)=V0(I,1)
    6  V(I,2,1)=V0(I,2)

C      COMPUTE INITIAL ACCELERATIONS.

       CALL ACCEL(TMDX,EPS,FDX,X,V,ALPHAM,DX,$28,A(1,1,2))

C      COMPUTE VELOCITIES AND POSITIONS AT FIRST TIME-STEP.

       DO 10 I=1,N
       V(I,1,2)=V(I,1,1)+DT*A(I,1,2)
       V(I,2,2)=V(I,2,1)+DT*A(I,2,2)
       X(I,1)=X(I,1)+DT2*(V(I,1,2)+V(I,1,1))
   10  X(I,2)=X(I,2)+DT2*(V(I,2,2)+V(I,2,1))

C      TEST FOR PRINTING OF POSITIONS AT FIRST TIME-STEP.
```

```
      IF(INCPR.EQ.1) CALL PRINT(1,X,INCPR,INCPL,NP)
C     TEST FOR MAXIMUM OF 1 TIME-STEP FOR THIS CASE.
      IF(LMAX.LT.2) GO TO 3
C     PROCEED TO COMPUTE FOR THE REMAINING TIME-STEPS FOR THIS CASE.
      DO 25 L=2,LMAX
C     UPDATE PREVIOUS VALUES OF VELOCITIES AND ACCELERATIONS.
      DO 13 I=1,N
      V(I,1,1)=V(I,1,2)
      V(I,2,1)=V(I,2,2)
      A(I,1,1)=A(I,1,2)
   13 A(I,2,1)=A(I,2,2)
C     COMPUTE ACCELERATIONS AT CURRENT TIME-STEP.
      CALL ACCEL(TMDX,EPS,EDX,X,V,ALPHAM,DX,$28,A(1,1,2))
C     COMPUTE VELOCITIES AND POSITIONS AT CURRENT TIME-STEP.
      DO 15 I=1,N
      V(I,1,2)=V(I,1,1)+DT*(1.5*A(I,1,2)-.5*A(I,1,1))
      V(I,2,2)=V(I,2,1)+DT*(1.5*A(I,2,2)-.5*A(I,2,1))
      X(I,1)=X(I,1)+DT2*(V(I,1,2)+V(I,1,1))
   15 X(I,2)=X(I,2)+DT2*(V(I,2,2)+V(I,2,1))
C     TEST FOR PRINTING OR PLOTTING OF POSITIONS AT CURRENT TIME-STEP.
      IF(MOD(L,INCPR).EQ.0.OR.MOD(L,INCPL).EQ.0) CALL PRINT(L,X,INCPR,
     * INCPL,NP)
   25 CONTINUE
C     TEST FOR WHETHER OR NOT THE PRINTING INCREMENT EXCEEDS THE
C     MAXIMUM NUMBER OF TIME-STEPS.  IF NOT, READ NEXT DATA CASE.
      IF(INCPR.LE.LMAX) GO TO 3
C     IF SO, PRINT OUT THE POSITIONS AT THE LAST TIME-STEP, AND THEN
C     READ NEXT DATA CASE.
      WRITE(6,93)
      CALL PRINT(LMAX,X,1,0,NP)
      GO TO 3
C     ERROR RETURN POINT FROM ACCELERATION COMPUTATION, INDICATING THAT
C     COMPUTATION HAS BECOME UNSTABLE.  PRINT MESSAGE AND PROCEED TO
C     NEXT DATA CASE.
   28 WRITE(6,91)
      GO TO 3
```

```
C     TERMINATION POINT FOR PROGRAM.  CONTROL REACHES HERE AFTER
C     ATTEMPTING TO READ PAST LAST DATA RECORD.

   30 STOP

C     FORMAT STATEMENTS FOR MAIN PROGRAM.

   99 FORMAT('0INITIAL POSITION')
   98 FORMAT(3I5,12F5.0)
   97 FORMAT(//1H0 'T0 =' E11.3,3X 'DT =' E11.3,3X 'ALPHA =' E11.3,3X
     * 'EPS =' F11.3//)
   96 FORMAT('0INITIAL VELOCITY')
   95 FORMAT(1H0 I6,2X'PARTICLE  FREE END  STRING ON (0,2). PARTICLE SPA
     *CING ='1PE11.3,3X 'PARTICLE MASS =' E11.3/1X 'FORCES CONSIDERED--
     * TENSION (NONLINEAR),VISCOSITY,GRAVITY.')
   94 FORMAT(' PLOTTING FREQUENCY =' I5,2X 'TIME STEPS'/)
   93 FORMAT('0FINAL POSITION'//)
   91 FORMAT('0OVERFLOW IN ACCELERATION CALCULATION. ABANDON CURRENT DAT
     *A CASE.')
   90 FORMAT('0INITIAL CONDITION CALCULATION FAILED FOR THIS CASE.'
     * 'T0 =' F5.1,5X 'EPS =' F6.3,5X'ALPHA ='F6.3,5X'DT ='F6.4,5X
     * 'OMEGA =' F6.2)

      END


@FOR,IS   ,ACCEL

C     THIS ROUTINE COMPUTES THE ACCELERATIONS OF ALL THE STRING
C     PARTICLES AT A SINGLE TIME-STEP.

      SUBROUTINE ACCEL(TMDX,EPS,EDX,X,V,ALPHAM,DELX,$,A)
      PARAMETER NPART=  21,N=NPART-1,NH=N/2+1
      DIMENSION X(NPART,2),A(NPART,2),V(NPART,2)
      DATA G/32.2/
      TY2=0.
      TX2=0.
      DO 10 I=1,N
      TY1=TY2
      TX1=TX2
      DX=X(I+1,1)-X(I,1)
      DY=X(I+1,2)-X(I,2)
      DDX=ABS(DX/DELX)
      DDY=ABS(DY/DELX)
      DD=DX**2+DY**2
      IF(DD.LT.0. .OR. DD.GT. 10.  ) RETURN 8
      T=TMDX*(1.-EPS+EDX*SQRT(DD))
      TY2=T*DY
      TX2=T*DX
      A(I,1)=TX2-TX1-ALPHAM*V(I,1)
   10 A(I,2)=TY2-TY1-ALPHAM*V(I,2)-G
      RETURN
      END
```

```
@FOR,IS   ,PRINT

C       THIS ROUTINE PRINTS AND/OR PLOTS (BY PRINTER) THE PARTICLE
C       POSITIONS (X) AT A SPECIFIED TIME-STEP (L) IF THIS TIME-STEP
C       IS A MULTIPLE OF SPECIFIED INCREMENTS FOR PRINTING AND PLOTTING
C       (INCPR,INCPL).

        SUBROUTINE PRINT(L,X,INCPR,INCPL,NP)
        PARAMETER NPART= 21
        DIMENSION X (NPART,2)
        DIMENSION TITLE(8),XTITLE(8),YTITLE(8)
        DATA (TITLE(I),I=1,8),(XTITLE(I),I=1,8),(YTITLE(I),I=1,8)/24*
       * 6H       /
    99 FORMAT(1H0I5,21F6.3)
    98 FORMAT(6X 21F6.3)
    97 FORMAT(1H0 5X 21F6.3)
        IF(MOD(L,INCPR).NE.0) GO TO 7
        NN=MIN0(NP,21)
        WRITE(6,99) L,(X (I,2),I=1,NN)
        WRITE(6,98) (X (I,1),I=1,NN)
     5 IF(NN.GE.NP) GO TO 7
        N1=NN+1
        NN=NN+MIN0(NP-NN,21)
        WRITE(6,97) (X(I,2),I=N1,NN)
        WRITE(6,98) (X(I,1),I=N1,NN)
        GO TO 5
     7 IF(INCPL.EQ.0) RETURN
        IF(MOD(L,INCPL).NE.0) RETURN
        CALL GRPH2N(X (1,2),'R',X (1,1),'R',-NP,'SMALL',-4.,.5,0.,.5,
       * TITLE,XTITLE,YTITLE,'*')
        CALL GRPHND
        RETURN
        END


@FOR,IS   ,STEADY

C       THIS ROUTINE COMPUTES, BY THE GENERALIZED NEWTON'S METHOD, THE
C       STEADY-STATE, OR EQUILIBRIUM, POSITIONS OF THE PARTICLES OF A
C       STRING HANGING WITH BOTH ENDS FIXED.

        SUBROUTINE STEADY(EPS1,EDX,OMEGA,CON,X0,X,IFLAG)
        PARAMETER NPART= 21,NHM1=NPART/2,NH=NHM1+1,NHP1=NH+1
        DIMENSION X0(NPART,2),X(NPART,2),EX(NH),EY(NH)
        DATA TOL/5.E-5/,MAXIT/250/
    98 FORMAT('0CONVERGENCE ATTAINED IN STEADY-STATE CALCULATION AFTER'
       * I4,2X 'ITERATIONS'/1X 'MAX. ERROR =' E11.4,3X 'TOLERANCE='E11.4,
       * 5X 'OMEGA =' F6.2/)
    97 FORMAT('0ITERATION FAILED TO CONVERGE IN STEADY-STATE CALCULATION
       *AFTER' I4,2X 'ITERATIONS'/1X 'MAX. ERROR ='E11.4,3X'TOLERANCE ='
       * E11.4/)
        IFLAG=0
```

```
      DO 5 I=1,NHP1
      X(I,1)=XO(I,1)
    5 X(I,2)=-XO(I,2)
      DO 20 IT=1,MAXIT
      DX2=X(2,1)-X(1,1)
      DX2SQ=DX2**2
      DO 10 I=2,NH
      DX1=DX2
      DX2=X(I+1,1)-X(I,1)
      DY2=X(I+1,2)-X(I,2)
      DY1=X(I,2)-X(I-1,2)
      DX1SQ=DX2SQ
      DX2SQ=DX2**2
      DY1SQ=DY1**2
      DY2SQ=DY2**2
      R1=SQRT(DX1SQ+DY1SQ)
      R2=SQRT(DX2SQ+DY2SQ)
      ANUM=EPS1*(DX2-DX1)+EDX*(R2*DX2-R1*DX1)
      DNOM=2.*EPS1+EDX*(DX2SQ/R2+R2+DX1SQ/R1+R1)
      EX(I)=OMEGA*ANUM/DNOM
      X(I,1)=X(I,1)+EX(I)
      DX2=X(I+1,1)-X(I,1)
      DX1=X(I,1)-X(I-1,1)
      DX1SQ=DX1**2
      DX2SQ=DX2**2
      R2=SQRT(DX2SQ+DY2SQ)
      R1=SQRT(DX1SQ+DY1SQ)
      ANUM=EPS1*(DY2-DY1)+EDX*(R2*DY2-R1*DY1)-CON
      DNOM=2.*EPS1+EDX*(DY2SQ/R2+R2+DY1/R1+R1)
      EY(I)=OMEGA*ANUM/DNOM
   10 X(I,2)=X(I,2)+EY(I)
      X(NHP1,1)=2.-X(NHM1,1)
      X(NHP1,2)=X(NHM1,2)
      IF(IT.LT.50)GO TO 20
C     TEST FOR CONVERGENCE
      EMAX=0.
      DO 15 I=2,NH
   15 EMAX=DMAX1(EMAX,ABS(EX(I)),ABS(EY(I)))
      IF(EMAX.GT.TOL) GO TO 20
      WRITE(6,98) IT,EMAX,TOL,OMEGA
      NS=NH+2
      DO 17 I=NS,NPART
      X(I,2)=X(NPART-I+1,2)
   17 X(I,1)=2.-X(NPART-I+1,1)
      RETURN
   20 CONTINUE
      WRITE(6,97) MAXIT,EMAX,TOL
      IFLAG=-1
      RETURN
      END
```

BIBLIOGRAPHIC DATA SHEET

1. Report No. WIS-CS-72-158

2.

3. Recipient's Accession No.

4. Title and Subtitle

Numerical Studies of Discrete Vibrating Strings

Appendix: Typical FORTRAN Program for a Vibrating String

5. Report Date: November 1972

6.

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10. Project/Task/Work Unit No.

11. Contract/Grant No.

12. Sponsoring Organization Name and Address

13. Type of Report & Period Covered

14.

15. Supplementary Notes

16. Abstracts

Discrete string vibrations are studied by means of a new two dimensional model. Linear and nonlinear wave motions are constructed with equal ease. Conditions and problems in which horizontal motion is particularly noticeable are emphasized. A large variety of computer examples are given.

17. Key Words and Document Analysis. **17a.** Descriptors

numerical analysis, wave motion, vibrating strings.

17b. Identifiers/Open-Ended Terms

17c. COSATI Field/Group

18. Availability Statement

Available to public.

19. Security Class (This Report) UNCLASSIFIED

20. Security Class (This Page) UNCLASSIFIED

21. No. of Pages: 77

22. Price