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FINITE DELAY SOLUTIONS FOR SEQUENTIAL CONDITIONS

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We present an algorithm for deciding whether or not a condition C(X,Y) stated in sequential calculus admits a finite delay solution. This solves a problem stated in [3] concerning the existence of h-shift solutions for finite state conditions. We essentially apply methods developed in [3] to an argument used by Ever and Meyer [6] to solve the problem for sequential Boolean equations.

1. Synthesis Algorithms

Let C(X,Y) be a condition (i.e., a binary relation) on ω-sequences X = X(0) X(1)... and Y = Y(0) Y(1)... of members of finite sets I and J respecitvely. Let $\mathscr{A}: I^{\omega} \to J^{\omega}$ be an operator mapping ω-sequences over the set I into ω-sequences over the set J. Then $\mathscr{A}$ solves the condition C(X,Y) for Y, or $\mathscr{A}$ is a solution of C for Y, iff

$$(\forall X,Y)(Y = \mathscr{A}(X) \supset C(X,Y)).$$

If CL is a class of conditions denoting relations between ω-sequences, and OP is a class of operators, then a solvability algorithm for CL with respect to OP is an effective procedure which given any condition $C(X,Y) \in CL$, tells whether or not there is an operator $\mathscr{A} \in OP$ that solves C for Y. A solution algorithm, given a $C(X,Y) \in CL$ and an $\mathscr{A} \in OP$, decides whether or not $\mathscr{A}$ solves C for Y. If the members of OP are finitely presentable, then a <u>synthesis algorithm</u>, given an arbitrary $C(X,Y) \in CL$, 1) decides whether or not there is an $\mathscr{A} \in OP$ that solves C for Y, and 2) obtains a presentation of such an $\mathscr{A}$ if one exists. These types of algorithms are discussed, for example, in [2], [4], and [5]. The class of conditions that we are concerned with are those that can be stated in sequential calculus (SC), that is, the monadic second order theory of the natural numbers with the operation of successor. SC is the interpreted formalism containing: the first order predicate calculus where individual variables range over the set of natural numbers N; second order monadic predicate variables, which are interpreted as subsets of N; the unary function symbol ', interpreted as successor on N; quantification over both first and second order variables.

Note that we can easily establish a 1-1 correspondence between subsets of N and ω-sequences of members of $\{0,1\}$. If $\sigma \in \{0,1\}^{\omega}$, $\{n \mid \sigma(n) = 1\}$ is the set associated with σ. If $A \subseteq N$, then the sequence σ defined by $\sigma(n) = 1$ iff $n \in A$ is the sequence associated with A. In a similar fashion, n-tuples of subsets of N can be associated with members of $\{\{0,1\}^n\}^{\omega}$. Thus, $\langle A_1, \ldots, A_n \rangle$, $A_i \subseteq N$, is associated with $\sigma \in \{\{0,1\}^n\}^{\omega}$ just in case $\sigma(k) = \langle s_1, \ldots, s_n \rangle$ implies $s_i = 1$ iff $k \in A_i$. A formula

$C(X,Y)$ of SC with free predicate variables $X = \langle X_1, \ldots, X_n \rangle$ and $Y = \langle Y_1, \ldots, Y_m \rangle$ can then be seen as a condition on ω-sequences over $I = \{0,1\}^n$ and $J = \{0,1\}^m$.

A particularly interesting class of operators are those operators $Y = \mathscr{A}(X)$ mapping I^ω into J^ω that can be presented in the form

$$Y(t) = \Phi(\bar{X}(\phi t)),$$

where $Y(t)$ is the $(t+1)$ st element of the sequence Y, $\bar{X}(t)$ is the sequence $X(0)\ldots X(t)$, ϕ maps N into N, and if I^* denotes the set of all finite sequences (words) over I, Φ maps I^* into J. Operators that can be given in this form are continuous in the sense of the natural Cantor topology on the set of all ω-sequences over the alphabets I and J, and are hence called <u>continuous</u> operators. $\mathscr{A}$ is said to be <u>recursive</u> (RO) if Φ and ϕ are recursive, and <u>deterministic</u> (DO) if $\phi t \leq t$. A deterministic operator is <u>h-shift</u> if $\phi t = t-h$, where $\bar{X}(n)$ is Λ, the empty word, for $n < 0$. An operator is <u>h-delay</u> $(h > 0)$ if $\phi t = t+h-1$. An h-shift operator must produce $Y(t)$ based only on knowledge of $X(0)\ldots X(t-h)$, whereas an h-delay can look ahead to $X(t+h-1)$ before generating $Y(t)$. It is clear that $\{DO\} = \{0\text{-shift}\} \supseteq \{1\text{-shift}\} \supseteq \{2\text{-shift}\} \supseteq \ldots$, and that $\{DO\} = \{1\text{-delay}\} \subseteq \{2\text{-delay}\} \subseteq \ldots$.

An important class of recursive deterministic operators are the h-shift finite automata operators (FAO). Let $\langle S, s_0, \delta \rangle$ be a deterministic finite automaton system over alphabet I. That is, S is a finite set of states, $s_0 \in S$ is the initial state, and $\delta: S \times I \rightarrow S$ is the transition function. Let θ map I into the finite set J. Then the <u>h-shift FAO</u> $Y = \mathscr{A}(X)$ defined from $\mathfrak{A} = \langle S, s_0, \delta, \theta \rangle$ can be presented in the form

$$Z(t) = s_0 \quad \text{for} \quad t < h;$$

$$Z(t') = \delta(Z(t), X(t-h+1)) \quad \text{for} \quad t' \geq h;$$

$$Y(t) = \theta(Z(t));$$

where $Z(-1) = s_0$ and $\delta(s, \Lambda) = s_0$. Here, Z is an ω-sequence over the set of states S; X, the input sequence **X**, is a member of I^ω; and Y, the output sequence, is a member of J^ω. θ is called the output function, and J the output alphabet.

Let $Y = \mathscr{A}(X)$ be a deterministic FAO mapping I^ω into J^ω. Then by an appropriate coding of $\mathscr{A}$ and of the finite sets I and J, a formula A(X,Y) of SC can be constructed such that A(X,Y) means $Y = \mathscr{A}(X)$. If C(X,Y) is a condition stated in sequential calculus, then the assertion "$\mathscr{A}$ solves C(X,Y) for Y" can be expressed as a sentence of SC. Büchi [2] gives a method for deciding the truth of sentences of SC, and thus there is a solution algorithm for SC with respect to FAO. Because all finite automata operators can be effectively enumerated, checking them one at a time to determine whether or not each solves some condition $C(X,Y)$ of SC provides a partial synthesis algorithm.

A condition $C(X,Y)$ is said to be <u>determined</u> if either there exists a 0-shift solution $Y = A(X)$ of $C(X,Y)$ for Y, or there exists a 1-shift solution $X = B(Y)$ of $\neg C(X,Y)$ for X. Suppose we consider the condition $C(X,Y)$ to be a game between two players I and J to be played as follows. At each instant of time $t = 0,1,2,\ldots,$ player I makes a move by selecting a member $X(t)$ of the set I, and then player J moves by selecting a member $Y(t)$ of the set J. At every time t, each player has complete information about all previous moves of his opponent; (i.e., before he moves at time t, player I can see $\bar{Y}(t-1)$, and player J can see $\bar{X}(t)$). The play $\langle X,Y\rangle$ of the game consists of all moves $X(0), X(1)\ldots, Y(0)\ Y(1)\ldots,$ of each player. Player J wins if the play $\langle X,Y\rangle$ satisfies C, otherwise player I wins. The condition C is determined if and only if one of the players has a winning strategy; i.e., player I has a 1-shift operator or player J has a 0-shift operator that beats all strategies of his opponent.

Büchi and Landweber [3] have given a synthesis algorithm for SC with respect to DO by showing that every condition $C(X,Y)$ of SC is determined, and in fact, one can either construct a 0-shift FAO that solves C for Y or a 1-shift FAO that solves $\neg$ C for X.

2. Delay Operators

We are concerned here with giving a synthesis algorithm for SC with respect to the class of finite delay operators. As we shall see, this is equivalent to the problem discussed by Büchi and Landweber [3] of finding an algorithm that determines for a given condition C(X,Y) of SC whether or not there exists an h such that C admits an h-shift but no (h+1)-shift solution for Y.

From the definition of an h-shift operator, it can be seen that the SC condition C(X,Y) has an h-shift solution for Y if and only if the formula

$$C_h(X,Y) \;.\!\equiv\!.\; (\exists Z).\; C(Z,Y) \;\wedge\; (\forall t)\,[Z(t) \longleftrightarrow X(t+h)]$$

has a 0-shift solution for Y. Hence, for any fixed h, we can use the Buchi-Landweber algorithm mentioned above to determine the existence of an h-shift solution. Since SC conditions are determined, $C_h(X,Y)$ does not have a 0-shift solution for Y if and only if $\neg\, C_h(X,Y)$ has a 1-shift solution for X. But

$$\neg\, C_h(X,Y) \;.\!\equiv\!.\; (\forall Z).[(\forall t)\; Z(t) = X(t+h)] \supset \neg\, C(Z,Y)$$

having a 1-shift solution for X is equivalent to $\neg\, C(X,Y)$ having an h-delay solution for X. Thus for every condition C(X,Y) defined in SC and for every fixed h, either C(X,Y) has an h-shift solution for Y or $\neg C(X,Y)$ has an h-delay solution for X.

Knowing if $\neg C(X,Y)$ has a finite delay solution for X tells us whether the Buchi-Landweber algorithm if applied to each C_h in succession, will ultimately encounter one that has no 0-shift solution for Y. On the other hand, if we can determine whether or not there is an h such that $C(X,Y)$ has no h-shift solution for Y, then we can tell if $\neg C(X,Y)$ has a finite delay solution. The Buchi-Landweber algorithm will produce a 1-shift solution of $\neg C_h(X,Y)$ for X if such an h exists, and this provides a finite delay solution of $\neg C$. Hence, the synthesis problem for finite delay operators and the h-shift problem of Buchi and Landweber are equivalent, and in fact if the existence of a finite delay solution for an SC condition can be determined, then the methods of [3] will yield a finite state presentation of a solution.

3. Finite Automata Graphs

Even and Meyer [6] solve the finite delay problem for a fragment of SC called "sequential Boolean equations." Let $F(X,Y,x)$ be a formula of SC having no constants or quantifiers, and containing only the second order variables $X = \langle X_1, \ldots, X_n \rangle$ and $Y = \langle Y_1, \ldots, Y_m \rangle$, and the first order variable x. Then the formula $G(X,Y) \mathrel{.\equiv.} \forall x\, F(X,Y,x)$ is a sequential Boolean equation. For example,

$$\forall x.\, (X_1(x) \wedge X_2(x')) \wedge (\neg X_1(x') \wedge Y_1(x'))$$

is a sequential Boolean equation. This equation would be more traditionally written as

$$x_1 dx_2 + d\bar{x}_1 \, dy_1 = 1.$$

To determine whether or not a sequential Boolean equation has an h-delay solution, they employ the concept of a finite automaton graph. A finite automaton graph G over a finite alphabet Σ is defined to be a system $\langle V, V_0, E\rangle$, where

> V is the finite set of <u>vertices</u> or <u>nodes</u>;
>
> $V_0 \subseteq V$ is the set of <u>initial vertices</u>;
>
> $E \subseteq V \times V \times \Sigma$ is the set of <u>labeled directed edges</u>.

Thus if $(v_1, v_2, \sigma) \in E$, we say that v_1 is connected to v_2 by an edge labeled σ, or simply v_1 is connected to v_2 by σ. We extend E to $V \times V \times \Sigma^*$ in the usual fashion:

$$E(v_1, v_2, \Lambda) \ .\equiv.\ v_1 = v_2;$$

$$E(v_1, v_2, \alpha\sigma) \ .\equiv.\ (\exists v_3)[E(v_1, v_3, \alpha) \wedge E(v_3, v_2, \sigma)];$$

where $v_1, v_2, v_3 \in V$, $\alpha \in \Sigma^*$, $\sigma \in \Sigma$, and Λ is the empty word. A sequence of edges of the form

$$(v_1, v_2, \sigma_1), (v_2, v_3, \sigma_2), \ldots, (v_{n-1}, v_n, \sigma_{n-1})$$

is called a path in G with label $\sigma_1\sigma_2 \ldots \sigma_{n-1}$. If $S \subseteq V$, $\alpha \in \Sigma^*$, define

$$\mathscr{E}(S,\alpha) \;.\!\equiv\!.\; \{v \in V \mid \exists u \in S \text{ and } E(u,v,\alpha)\}$$

to be the set of vertices accessible from S by a path with label α.

The graph $G = \langle V, V_0, E \rangle$ is solvable with respect to Σ if for all $\alpha \in \Sigma^*$, $\mathscr{E}(V_0,\alpha) \neq \emptyset$. That is, G is solvable if for any word $\alpha \in \Sigma^*$, there is some path of G starting in V_0 labeled with α.

The concept of h-delay solvability is defined in a fashion similar to the corresponding concept for operators. The graph $G = \langle V, V_0, E \rangle$ is said to be solvable with delay h if there exist functions $\Phi_0 : \Sigma^h \to V_0$ and Φ: $V \times \Sigma^{h+1} \to V$, where Σ^h denotes words of length h on Σ, such that for any $\alpha \in \Sigma^\omega$, if

$$v(0) = \Phi_0(\alpha(0), \ldots, \alpha(h-1));$$

$$v(t+1) = \Phi(v(t), \alpha(t), \ldots, \alpha(t+h)),\ t \geq 0;$$

then $(v(t), v(t+1), \alpha(t))$ E holds for all $t \geq 0$. Thus the graph G is solvable with delay h if knowledge of the first h characters, $\alpha(0) \ldots \alpha(h-1)$, of the ω-sequence is sufficient to determine an initial vertex $v(0)$, and for every t, knowledge of $v(t)$, $\alpha(t) \ldots \alpha(t+h)$ allows the determination of a vertex $v(t+1)$ to which $v(t)$ is connected by $\alpha(t)$. Clearly if G is solvable with finite delay h, then G is solvable.

Even and Meyer show that for any sequential Boolean equation $F(X,Y)$, a graph G can be effectively constructed such that $F(X,Y)$ has an h-delay solution for Y if and only if G is solvable with delay h. The following theorem then completes their proof:

<u>Theorem 1.</u> (Even and Meyer) There is an effective procedure for deciding whether an arbitray finite automaton graph G is solvable with finite delay. Moreover, a solution (i.e., definition of Φ_0 and Φ) is effectively obtainable if one exsists.

Even and Meyer actually prove the theorem for $V_0 = V$, but the result for any finite automaton graph requires only trivial modification. Their proof gives a bound on h for any given graph G in the sense that they obtain a number N such that if G has an h-delay solution, then it has an h-delay solution for some $h < N$.

Meyer (private communication) suggested that this method be applied to the finite delay problem for the full sequential calculus. Given a condition C(X,Y) of SC, we show how to effectively construct a graph G such that C has an h-delay solution for Y if and only if G is solvable with delay h. Theorem 1 will then complete the solution.

4. Proving C has an h-delay Solution from G

We now attack the problem of describing the graph discussed above.

Let the FAO Z = C(X,Y), mapping ω-sequences on S, be given by

$$Z(0) = s_0;$$

$$Z(t') = \delta(Z(t), X(t), Y(t));$$

$$Z(t) = \theta(Z(t));$$

where $\langle S, s_0, \delta \rangle$ is a finite automaton system over $I \times J$, and $\theta: S \to S$ is the identity function. Let inf Z denote the set of states entered infinitely often by Z; i.e.

$$s \in \inf Z \;.\equiv.\; (\forall x)(\exists t)[x \le t \wedge Z(t) = s].$$

Then if $\mathcal{U} \subseteq 2^S$, we define the <u>ω-behavior of $\langle S, s_0, \delta, \mathcal{U} \rangle$</u> to be the relation C(X,Y) which holds for X and Y iff Z = C(X,Y) satisfies inf $Z \in \mathcal{U}$; i.e.,

(1) $$C(X,Y) \;.\equiv.\; (\exists Z)(Z(0) = s_0 \wedge (\forall t)(Z(t') = \delta(Z(t), X(t), Y(t)) \wedge \inf Z \in \mathcal{U}].$$

$\mathcal{U}$ is said to be the output condition of the FAO C. A <u>finite state condition</u> is one that is the ω-behavior of some FAO with output condition. By results of Buchi [1] and McNaughton [7], we know that finite state conditions are exactly those expressable in sequential calculus, and that given any formula C(X,Y) of SC, a finite automaton with output condition can effectively be constructed with ω behavior C.

We assume in the following that C(X,Y) is the ω-behavior of the automaton $\mathfrak{A} = \langle S, s_0, \delta, \mathcal{U} \rangle$. For each $A \in \mathcal{U}$, we choose some cyclic permutation of its elements, and denote the result of applying this permutation to $s \in A$ by A(s). In the notation $[A_1, s_1, \ldots, A_n, s_n]$, $A_1 \supsetneq \cdots \supsetneq A_n$ will range over strictly decreasing chains of members of $\mathcal{U}$, and $s_1, \ldots, s_n$ will range over members of $A_1, \ldots, A_n$, respectively.

We now define inductively the following subsets of S, whose use will be explained following the definition of G:

$$s \in \mathcal{R}_0[A_1, s_1, \ldots, A_n, s_n] \;.\equiv.\; \text{false} ;$$

$$s \in \mathcal{P}_k[A_1, s_1, \ldots, A_n, s_n] \;.\equiv.\; s \in \mathcal{R}_k[\;\;] \vee s \in A_1 \cap \mathcal{R}_k[A_1, s_1] \vee \ldots \vee s \in A_n \cap \mathcal{R}_k[A_1, s_1, \ldots, A_n, s_n];$$

(2)
$$s \in \mathcal{Q}_k[A_1, s_1, \ldots, A_n s_n] \;.\equiv.\; \bigvee_B \,.\, B \in \mathcal{U} \wedge s \in B \subsetneq A_n \wedge \bigwedge_{u \in B} u \in \mathcal{R}_k[A_1, s_1, \ldots, A_n, s_n, B, B(u)];$$

$$s \in \mathcal{R}_{k+1}[A_1, s_1, \ldots, A_n, s_n] \;.\equiv.\; \bigvee_{x \in I} \bigvee_{y \in J} \delta(x, y, s) \in (\{s_1, \ldots, s_n\} \cup \mathcal{P}_k[A_1, s_1, \ldots, A_n, s_n] \cup \mathcal{Q}_k[A_1, s_1, \ldots, A_n, s_n].$$

In this definition and for the remainder of the chapter, in the case $n = 0$, occurrences of A_n are to be suppressed and $\{s_1, \ldots, s_n\}$ is to be considered the empty set. For example,

$$s \in \mathcal{Q}_k[\;\;] \;.\equiv.\; \bigvee_B \,.\, B \in \mathcal{U} \wedge \bigwedge_{u \in B} u \in \mathcal{R}_k[B, B(u)],$$

where $B(u)$ is the successor of u in the chosen cyclic permutation of B.

Note that by induction on k, we have $\mathcal{R}_k[\alpha] \subseteq \mathcal{R}_{k+1}[\alpha]$, $\mathcal{P}_k[\alpha] \subseteq \mathcal{P}_{k+1}[\alpha]$, and $\mathcal{Q}_k[\alpha] \subseteq \mathcal{Q}_{k+1}[\alpha]$, where α is any $A_1, s_1, \ldots, A_n, s_n$. Let ℓ be the least number such that $\mathcal{R}_{\ell-1}[\alpha] = \mathcal{R}_{\ell}[\alpha]$, $\mathcal{P}_{\ell-1}[\alpha] = \mathcal{P}_{\ell}[\alpha]$, and $\mathcal{Q}_{\ell-1}[\alpha] = \mathcal{Q}_{\ell}[\alpha]$, for all α. Such an ℓ clearly exists, since all the $\mathcal{R},\mathcal{P},\mathcal{Q}$ are subsets of the finite set S, and there are only finitely many possible α. We now define the graph G that will be h-delay solvable iff C(X,Y) has an h-delay solution for Y.

Let $G = \langle N, N_0, E\rangle$ be defined as follows:

I. $N = \{[y,s,k,v]\}$, where:

1. $y \in J$, $s \in S$, $0 < k \leq \ell$;

2. v is of the form $[A_1,s_1,h_1,\ldots,A_n,s_n,h_n]$, with $n \geq 0$; $s_i \in A_i \in \mathcal{U}$, $1 \leq i \leq n$; $A_1 \not\supseteq A_2 \not\supseteq \cdots \not\supseteq A_n$; $0 < k \leq h_n < \ldots < h_1 \leq \ell$.

II. $N_0 = \{[y,s_0,k,v] \in N\}$.

III. If $v = [A_1,s_1,h_1,\ldots,A_n,s_n,h_n]$, $x \in I$, then $([y,s,k,v], [\hat{y},\hat{s},\hat{k},\hat{v}],x) \in E$ if and only if one of the following occurs:

(3)

(α) $\delta(s,x,y) = \hat{s} = s_i \in \{s_1,\ldots,s_n\}$;

$\hat{v} = [A_1,s_1,h_1,\ldots,A_i,A_i(s_i),\ h_i]$;

$\hat{k} = h_i$.

(β) $\delta(s,x,y) = \hat{s} \in \mathcal{P}_{k-1}[A_1,s_1,\ldots,A_n,s_n]$;

$\hat{\nu} = [A_1,s_1,h_1,\ldots,A_j,s_j,h_j]$,

where $\hat{s} \in A_j \cap \mathcal{R}_{k-1}[A_1,s_1,\ldots,A_j,s_j]$;

$\hat{k} = k - 1$.

(γ) $\delta(s,x,y) = \hat{s} \in \mathcal{Q}_{k-1}[A_1, s_1, \ldots, A_n, s_n]$;

$\hat{v} = [A_1, s_1, h_1, \ldots, A_n, s_n, h_n, B, B(\hat{s}), k-1]$,

where $B \in \mathcal{U}$, $\hat{s} \in B \subsetneq A_n$, and

$\bigwedge_{u \in B} u \in \mathcal{R}_{k-1}[A_1, s_1, \ldots, A_n, s_n, B, B(u)]$;

$\hat{k} = k-1$.

This graph can be effectively constructed, since membership in the $\mathcal{R}$, $\mathcal{P}$, and $\mathcal{Q}$ sets can be effectively determined, and ℓ can be effectively obtained.

G is constructed so that a solution of G will force the automaton $\mathfrak{A}$ to ultimately cycle through some accept set; i.e., some member of $\mathcal{U}$. The predicates $\mathcal{P}$, $\mathcal{Q}$, and $\mathcal{R}$ "control" the state sequence. Suppose, for example, that G is 1-delay solvable. If $X(0)\,X(1)\ldots \in I^{\omega}$, then a sequence $M(0)\,M(1)\ldots \in N^{\omega}$ with $M(0) \in N_0$ can be obtained such that for all t, $M(t)$ is connected to $M(t+1)$ by $X(t)$, and such that only $X(0)\ldots X(t)$ is needed to determine $M(t)$. If the vertices are of the form $M(t) = [Y(t), S(t), K(t), V(t)]$, we want $\langle X,Y \rangle = \langle X(0)\,X(1)\ldots, Y(0)\,Y(1)\ldots \rangle$ to satisfy $C(X,Y)$. Equivalently, we want the sequence $(X(0), Y(0))\,(X(1), Y(1))\ldots$ to force the automaton $\mathfrak{A}$ ultimately to cycle through some member of $\mathcal{U}$.

Suppose some $M(t) = [y, s, k, [A_1, s_1, h_1, \ldots, A_n, s_n, h_n]]$. The sets $A_1, \ldots, A_n$, forming a chain in $\mathcal{U}$, are the current candidates for a member of $\mathcal{U}$ through which $\mathfrak{A}$ will be forced to cycle. The $s_1, \ldots, s_n$ are "goals" in each A_i toward which $\mathfrak{A}$ is forced. k and $h_1, \ldots, h_n$ measure the "closeness" to a given goal. If the next vertex is reached by condition (α) of (3), then goal s_i has been reached, and in this case, all candidates below A_i in the chain are eliminated, the successor $A_i(s_i)$ of s_i is set up as a new goal, and the remembered "closeness", h_i, is returned. If case (β) is used, $\mathfrak{A}$ has gotten closer to goal s_j (i.e., k decreases), and all previous candidate sets lower than A_j are forgotten. If n = 0, this case amounts to the disregarding of all previous candidate sets, and starting anew. Since k is decreased, starting over in this fashion may happen only finitely often. In case (γ), a new smaller candidate set and goal to which $\mathfrak{A}$ is "closer", are added. In adding this new set, it is important to make sure that all of its members are in a proper $\mathcal{R}$. The controls imposed by the "closeness" index insures that $\mathfrak{A}$ will ultimately cycle constantly through some member of $\mathcal{U}$.

The proof of the following theorem is essentially that of the main theorem of [3].

<u>Theorem 2</u>. If G is h-delay solvable, then C has an h-delay solution for Y.

Proof. If G is h-delay solvable, then ω-sequences X and Y can be obtained in the following way. Given $X(0)\ldots X(h-1) \in I^*$, a node $M0 = [Y(0), s0, K0, v0]$ of G can be found, with $s0 \in S_0$. For every t, given node $Mt = [Y(t), st, Kt, vt]$ and $X(t)\ldots X(t+h)$, a node $Mt' = [Y(t'), \delta(st, X(t), Y(t)) = st', Kt', vt']$ can be produced such that $(Mt, Mt', X(t)) \in E$. Thus, for every t, some $Y(t) \in J$ can be obtained given $\bar{X}(t+h) \in I^*$. We want to show that the resulting ω-sequences X and Y satisfy $C(X,Y)$.

To show this, we proceed as in [3]. Let $Mt = [Y(t), st, Kt, vt]$ represent the node chosen at time t, as indicated above. Suppose that for some time t_1, $vt_1 = [\ ]$. Then $t_2 > t_1$ implies (a) $Kt_2 < Kt_1$, and if $v_2t_2 = [A_1, s_1, h_1, \ldots, A_n, s_n, h_n]$, (b) each $h_i < Kt_1$. This is clearly true in case $t_2 = t_1 + 1$, since then only cases (β) or (γ) of (3) could be used. Assuming that it is true for some $t_2 > t_3$, one can observe that (α), (β), and (γ) of (3) preserve (a) and (b) for $t_2 + 1$.

Since $Kt > 0$, there is some t_1 such that $vt \neq [\ ]$ for all $t \geq t_1$. Thus for $t \geq t_1$, vt is of the form $[A_1, s_1, h_1, \ldots, A_n, s_n, h_n]$, with the level $n \geq 1$. Since n, the level of vt, is bounded by the lengths of chains in the finite set $\mathcal{U}$, some level must occur infinitely often. Let m be the smallest of these. Then $m \geq 1$, and there is a t_2 such that for $t \geq t_2$, the level of $vt \geq m$. Hence if $t \geq t_2$,

$$vt = [A_1, s_1, h_1, \ldots, A_m, s_m, h_m, \ldots, A_n, s_n, h_n], \text{ where } n \geq m \geq 1.$$

Furthermore, $n = m$ occurs infinitely often. From this, we can see that for $t > t_2$, only cases (α) with $i \geq m$, (β) with $j \geq m$, and (γ) with $n \geq m$ of (3) can occur, and therefore A_m in vt must remain constant after time t_2. By (3), the definition of G, $st \in A_m$ for $t \geq t_2$, and so $\inf \{st\} \subseteq A_m \in \mathcal{U}$. It remains only to show that $A_m \subseteq \inf \{st\}$.

Suppose that case (α) of (3) with $i = m$ occurred only finitely many times. Then for some $t_3 > t_1$, $t \geq t_3$ would imply that only cases (α) with $i > m$, (β) with $j \geq m$, and (γ) with $n \geq m$ would be used. Inspection of (3) shows that each application of case (β) with $i = m$ produces a lower value of K than the previous application. This is true since we are assuming that (α) can occur only with $i > m$, and any h_j, $j > m$, added to v after an application of (β) with $i = m$ must be less than the value of K at that application. Thus, cases (α) with $i = m$ and (β) with $j = m$ are used only finitely often. But this contradicts the fact that the level of vt is m for infinitely many t. Hence case (α) with $i = m$ must occur infinitely often. Let $t_3 < t_4 < t_5 < \ldots$ be the infinitely many consecutive times $t > t_2$ where case (α) with $i = m$ occurs. Then clearly $st_{k+1} = A_m(st_k)$, $k = 3, 4, \ldots$. But $A_m(s)$ is a cyclic permutation of A_m, and hence $A_m \subseteq \inf \{st\}$. Thus, $\inf \{st\} = A_m \in \mathcal{U}$. ∎

4. Proving G is h-delay Solvable from C

To prove the converse of theorem 2, intermediate steps are required. First define C^h by

$$C^h(X,Y) \;.\!\equiv.\; (\exists Z).C(X,Z) \wedge (\forall t)[Z(t) \longleftrightarrow Y(t+h-1)].$$

C has an h-delay solution for Y iff C^h is 1-delay solvable for Y. We will construct a graph G^h from C^h so that

$$\begin{array}{l} C \text{ h-delay solvable for } Y \\ \quad\Downarrow \text{(a)} \\ C^h \text{ 1-delay solvable for } Y \\ \quad\Downarrow \text{(b)} \\ G^h \text{ 1-delay solvable} \\ \quad\Downarrow \text{(c)} \\ G \text{ h-delay solvable.} \end{array}$$

Then by Theorem 2, C(X,Y) has an h-delay solution for Y if and only if G is h-delay solvable. Let $\mathfrak{A} = \langle S, s_0, \delta, \mathcal{U} \rangle$ be the automaton that determines C, as above. Define an automaton $\mathfrak{A}^h = \langle S^h, s_0{}^h, \delta^h, \mathcal{U}^h \rangle$ with behavior $C^h(X,Y)$ by:

$$S^h = \{\langle s, x_0, \ldots, x_{h-2}\rangle \mid s \in S,\ x_i \in I \cup \{\Lambda\}\};$$

$$s_0^h = \langle s_0, \Lambda, \ldots, \Lambda\rangle \in S^h;$$

(4)
$$\delta^h(\langle s, x_0, \ldots, x_{h-2}\rangle, x, y) = \begin{cases} \langle \delta(s, x_0, y),\ x_1, \ldots, x_{h-2}, x\rangle & \text{if } x_0 \in I; \\ \langle s, x_1, \ldots, x_{h-2}, x\rangle & \text{if } x_0 = \Lambda; \end{cases}$$

$$\mathcal{U}^h = \{\bar{A} \subseteq S^h \mid \pi_1(\bar{A}) \in \mathcal{U}\};$$

where $\pi_1(\langle a_1, \ldots, a_n\rangle) = a_1$, $\pi_1\{\langle a_1^1, \ldots, a_n^1\rangle, \ldots, \langle a_1^k, \ldots, a_n^k\rangle\} = \{a_1^1, \ldots, a_1^k\}$. That is, π_1 is the first component projection function. $\mathfrak{A}^h$ with input (X(0), Y(0)), (X(1), Y(1)),..., simulates the action of $\mathfrak{A}$ on the input string (X(0), Y(h-1)), (X(1), Y(h)),... . It is clear that $C^h(X,Y)$ is the ω-behavior of $\mathfrak{A}^h$.

For $\bar{A} \subseteq S^h$ and $\bar{s} \in S^h$, let $\pi_1(\bar{A}) = A \subseteq S$, and $\pi_1(\bar{s}) = s \in S$. Recall that for every $A \in \mathcal{U}$, A(s) denotes some cyclic permutation of A. We would like to define a "permutation" for each $\bar{A} \in \mathcal{U}^h$ whose action on the first component mirrors the cyclic permutation of $\pi_1(\bar{A})$. For each $\bar{A} \in \mathcal{U}^h$, choose some sequence $\bar{s}_1 \ldots \bar{s}_n \in \bar{A}^*$ that satisfies

$$(5) \quad \begin{array}{ll} \text{i)} & (\forall \bar{s} \in \bar{A})(\exists i,\ 1 \le i \le n)\ \bar{s}_i = \bar{s}; \\ \text{ii)} & A(\pi_1(\bar{s}_j)) = \pi_1(\bar{s}_{j+1}),\ 1 \le j < n; \\ \text{iii)} & A(\pi_1(\bar{s}_n)) = \pi_1(\bar{s}_1). \end{array}$$

This sequence is called the permuting sequence for $\bar{A}$. Note that repetitions can occur in $\bar{s}_1 \ldots \bar{s}_n$. Let $\text{In}(\bar{A})$ be the set of all initial segments of the permuting sequences of $\bar{A}$, i.e , $\text{In}(\bar{A}) = \{\bar{s}_1 \ldots \bar{s}_j \mid 1 \le j \le n\}$. Define the "permutation" $\bar{\bar{A}}: \text{In}(\bar{A}) \to \text{In}(\bar{A})$ and the projection $P: \text{In}(\bar{A}) \to \bar{A}$ by

$$\bar{\bar{A}}(\bar{s}_1 \ldots \bar{s}_j) = \bar{s}_1 \ldots \bar{s}_{j+1},\ 1 \le j < n;$$

$$\bar{\bar{A}}(\bar{s}_1 \ldots \bar{s}_n) = \bar{s}_1\ ;$$

$$P(\bar{s}_1 \ldots \bar{s}_j) = \bar{s}_j,\ 1 \le j \le n.$$

If $\bar{s} \in \bar{A}$, we will let $\bar{\bar{A}}(\bar{s})$ denote $\bar{\bar{A}}(\bar{s}_1 \ldots \bar{s}_j)$, where j is the least number such that $\bar{s} = \bar{s}_j$. Now by (5),

$$(6) \quad \pi_1(P(\bar{\bar{A}}(\bar{s}_1 \ldots \bar{s}_j))) = A(\pi_1(P(\bar{s}_1 \ldots \bar{s}_j))),\ 1 \le j \le n, \quad \text{and}$$

$$\pi_1(P(\bar{\bar{A}}(\bar{s}))) = A(\pi_1(\bar{s})),$$

so the permuting sequence for $\bar{A}$ gives a "permutation" of $\bar{A}$ that parallels the fixed permutation of $A = \pi_1(\bar{A})$ when projected.

For the remainder of the paper, the notation $[\bar{A}_1, \sigma_1, \ldots, \bar{A}_m, \sigma_m]$ will imply that $\bar{A}_1 \supsetneq \bar{A}_2 \supsetneq \cdots \supsetneq \bar{A}_m$ is a strictly decreasing chain in $\mathcal{U}^h$, $A_1 \supsetneq A_2 \supsetneq \cdots \supsetneq A_m$ is a strictly decreasing chain in $\mathcal{U}$ (where $\pi_1(\bar{A}_i) = A_i$), and $\sigma_i \in \text{In}(\bar{A}_i)$. Then if $\bar{s} = \langle s, x_0, \ldots, x_{h-2}\rangle \in S^h$, define the following subsets of S^h:

$$\bar{s} \in \mathscr{R}_0^h[\bar{A},\sigma_1,\ldots,\bar{A}_n,\sigma_n] \;.\!\equiv\!.\; \text{false};$$

$$\bar{s} \in \mathscr{P}_k^h[\bar{A},\sigma_1,\ldots,\bar{A}_n,\sigma_n] \;.\!\equiv\!.\; \bar{s} \in \mathscr{R}_k^h[\;] \vee$$

$$\bar{s} \in \bar{A}_1 \cap \mathscr{R}_k^h[\bar{A}_1,\sigma_1] \vee \ldots \vee$$

$$\bar{s} \in \bar{A}_n \cap \mathscr{R}_k^h[\bar{A}_1,\sigma_1,\ldots,\bar{A}_n,\sigma_n];$$

$$\bar{s} \in \mathscr{Q}_k^h[\bar{A}_1,\sigma_1,\ldots,\bar{A}_n,\sigma_n] \;.\!\equiv\!.\; \bigvee_{\bar{B}}.\; \bar{B} \in \mathscr{U}^k$$

(7)
$$\wedge\; \bar{B} \subsetneq \bar{A}_n \wedge \pi_1(\bar{B}) \subsetneq \pi_1(\bar{A}_n)$$

$$\wedge \bigwedge_{\bar{u}\in\bar{B}} \bar{u} \in \mathscr{R}_k^h[\bar{A}_1,\sigma_1,\ldots,\bar{A}_n,\sigma_n,\bar{B},\bar{B}(\bar{u})];$$

$$\bar{s} \in \mathscr{R}_{k+1}^h[\bar{A}_1,\sigma_1,\ldots,\bar{A}_n,\sigma_n] \;.\!\equiv\!.\; \bigvee_{x\in I}\;\bigvee_{y\in J}$$

$$[\delta^h(\bar{s},x,y) \in \mathscr{P}_k^h[\bar{A}_1,\sigma_1,\ldots,\bar{A}_n,\sigma_n]$$

$$\cup\;\; \mathscr{Q}_k^h[\bar{A}_1,\sigma_1,\ldots,\bar{A}_n,\sigma_n] \vee$$

$$\bigvee_{1\le i\le n} (\delta^h(\bar{s},x,y) \in \bar{A}_i \wedge \delta(s,x_0,y) = \pi_1 P(\sigma_i))].$$

We further define R_k^h, P_k^h, and Q_k^h, in the same manner as $\mathscr{R}_k^h, \mathscr{P}_k^h$, and $\mathscr{Q}_k^h$, except that "$\bigwedge_{x\in I}$" replaces "$\bigvee_{x\in I}$" in the definition of $\mathscr{R}_{k+1}^h$. Let $\bar{s} = \langle s,x_0,\ldots,x_{h-2}\rangle \in S^h$. Then:

$$\bar{s} \in R_0^h[\bar{A}_1,\sigma_1,\ldots,\bar{A}_n,\sigma_n] \;.\!\equiv\!.\; \text{false};$$

$$\bar{s} \in P_k^h[\bar{A}_1,\sigma_1,\ldots,\bar{A}_n,\sigma_n] \;.\!\equiv\!.\; \bar{s} \in R_k^h[\;] \;\vee$$

$$\bar{s} \in \bar{A}_1 \cap R_k^h[\bar{A}_1,\sigma_1] \vee \ldots \vee$$

$$\bar{s} \in A_n \cap R_k^h[\bar{A}_1,\sigma_1,\ldots,\bar{A}_n,\sigma_n];$$

$$\bar{s} \in Q_k^h[\bar{A}_1,\sigma_1,\ldots,\bar{A}_n,\sigma_n] \;.\!\equiv\!.\; \bigvee_{\bar{B}\in\mathcal{U}^h}$$

(8)
$$\bar{s} \in \bar{B} \subsetneq \bar{A}_n \wedge B \subsetneq A_n$$

$$\wedge \bigwedge_{\bar{u}\in\bar{B}} \bar{u} \in R_k^h[\bar{A}_1,\sigma_1,\ldots,\bar{A}_n,\sigma_n,\bar{B},\bar{B}(\bar{u})];$$

$$\bar{s} \in R_{k+1}^h[\bar{A}_1,\sigma_1,\ldots,\bar{A}_n,\sigma_n] \;.\!\equiv\!.\; \bigwedge_{x\in I}\;\bigvee_{y\in J}$$

$$[\delta^h(\langle s,x_0,\ldots,x_{h-2}\rangle,x,y)$$

$$\in P_k^h[\bar{A}_1,\sigma_1,\ldots,\bar{A}_n,\sigma_n] \cup Q_k^h[\bar{A}_1,\sigma_1,\ldots,\bar{A}_n,\sigma_n] \wedge$$

$$\bigvee_{1\le i\le n} [\delta^h(\bar{s},x,y) \in \bar{A}_i \wedge \delta(s,x_0,y) = \pi_1 P(\sigma_i)].$$

For $s \in S$, $j \in J$, $\delta(s,\Lambda,y) = s$. As before, if $n = 0$ surpress all occurrences of $\bar{A}_n$ and A_n. We can see that these definitions are effective and that $\mathcal{R}_k[\alpha] \subseteq \mathcal{R}_{k+1}[\alpha]$, $\mathcal{P}_k[\alpha] \subseteq \mathcal{P}_{k+1}[\alpha]$, $\mathcal{Q}_k[\alpha] \subseteq \mathcal{Q}_{k+1}[\alpha]$, $R_k[\alpha] \subseteq R_{k+1}[\alpha]$, $P_k[\alpha] \subseteq P_{k+1}[\alpha]$, $Q_k[\alpha] \subseteq Q_{k+1}[\alpha]$, and that integers $\bar{\ell}$ and $\hat{\ell}$ can be effectively found such that $\mathcal{R}_{\bar{\ell}}[\alpha] = \mathcal{R}_{\bar{\ell}-1}[\alpha]$, $\mathcal{P}_{\bar{\ell}}[\alpha] = \mathcal{P}_{\bar{\ell}-1}[\alpha]$, $\mathcal{Q}_{\bar{\ell}}[\alpha] = \mathcal{Q}_{\bar{\ell}-1}[\alpha]$, $R_{\hat{\ell}}[\alpha] = R_{\hat{\ell}-1}[\alpha]$, $P_{\hat{\ell}}[\alpha] = P_{\hat{\ell}-1}[\alpha]$, $Q_{\hat{\ell}}[\alpha] = Q_{\hat{\ell}-1}[\alpha]$, for all α.

With $\bar{s} = \langle s, x_0, \ldots, x_{h-2} \rangle \in S^h$, we have the following:

$$\bar{s} \notin P^h_{\hat{\ell}}[\bar{A}_1, \sigma_1, \ldots, \bar{A}_n, \sigma_n] \;.\!\equiv\!.\; \bar{s} \notin R^h_{\hat{\ell}}[\quad] \wedge$$

$$(\bar{s} \notin \bar{A}_1 \vee \bar{s} \notin R^h_{\hat{\ell}}[\bar{A}_1, \sigma_1]) \wedge \ldots \wedge$$

$$(\bar{s} \notin \bar{A}_1 \vee \bar{s} \notin R^h_{\hat{\ell}}[A_1, \sigma_1, \ldots, \bar{A}_n, \sigma_n]);$$

$$\bar{s} \notin Q^h_{\hat{\ell}}[\bar{A}_1, \sigma_1, \ldots, \bar{A}_n, \sigma_n] \;.\!\equiv\!.\; \bigwedge_{\bar{B} \in \mathcal{U}^h}$$

$$[\bar{s} \in \bar{B} \wedge \bar{B} \subsetneq \bar{A}_n \wedge B \subsetneq A_n] \supset$$

(9) $$\bigvee_{\bar{u} \in \bar{B}} \bar{u} \notin R^h_{\hat{\ell}}[\bar{A}_1, \sigma_1, \ldots, \bar{A}_n, \sigma_n, \bar{B}, \bar{B}(\bar{u})];$$

$$\bar{s} \notin R^h_{\hat{\ell}}[\bar{A}_1, \sigma_1, \ldots, \bar{A}_n, \sigma_n] \;.\!\equiv\!.\; \bigvee_{x \in I} \bigwedge_{y \in J}$$

$$[\delta^h(\bar{s}, x, y) \notin P^h_{\hat{\ell}}[\bar{A}_1, \sigma_1, \ldots, \bar{A}_n, \sigma_n]$$

$$\cup\, Q^h_{\hat{\ell}}[\bar{A}_1, \sigma_1, \ldots, \bar{A}_n, \sigma_n] \wedge (\delta^h(\bar{s}, x, y) \in \bar{A}_i$$

$$\supset \delta(s, x_0, y) \neq \pi_1 P(\sigma_1))].$$

Lemma 1. For all $k \geq 0$, all α: $R^h_k[\alpha] \subseteq \mathcal{R}^h_k[\alpha]$; $P^h_k[\alpha] \subseteq \mathcal{P}^h_k[\alpha]$; $Q^h_k[\alpha] \subseteq \mathcal{Q}^h_k[\alpha]$.

Proof. Induction on k. ∎

Lemma 2. $\langle s_0, \Lambda, \ldots, \Lambda \rangle = s_0^h \notin R^h_{\hat{\ell}}$ implies $\neg C^h(X,Y)$ has a 1-shift solution for X.

Proof. The proof of this lemma follows a similar proof by Buchi and Landweber [3]. If $\mathfrak{A}^h$ on input $\langle X, Y \rangle = (X(0), Y(0))\,(X(1), Y(1))\ldots$ traverses the ω-sequence $Z = Z(0)Z(1)\ldots \in (S^h)^\omega$, then $C^h(X,Y)$ if and only if $\inf\{Z(t)\} \in \mathcal{U}^h$, or equivalently, $\inf\{\pi_1(Z(t))\} \in \mathcal{U}$. Consider the following formulas, where $\pi_2(\langle a_1, a_2, \ldots, a_n \rangle) = a_2$, and $(\mu x)\, E(x)$ denotes the least x in I such that $E(x)$ holds, if such an x exists.

$$Z(0) = \langle s_0, \Lambda, \ldots, \Lambda \rangle = s_0^h \in S^h; \quad W0 = \{\{s_0\}\}; \; V0 = [\ ].$$

Assume $Vt = [\bar{A}_1, \sigma_1, \ldots, \bar{A}_n, \sigma_n]$. Then

$$X(t) = (\mu x) \bigwedge_{y \in J} [\delta^h(Z(t), x, y) \notin P^h_{\hat{\ell}}[\bar{A}_1, \sigma_1, \ldots, \bar{A}_n, \sigma_n]$$

$$\cup\, Q^h_{\hat{\ell}}[\bar{A}_1, \sigma_1, \ldots, \bar{A}_n, \sigma_n] \wedge (\delta^h(Z(t), x, y)$$

$$\in \bar{A}_i \supset \delta(\pi_1(Z(t)), \pi_2(Z(t)), y) = \pi_1(\delta^h(Z(t), x, y)) \neq \pi_1 P(\sigma_i))];$$

$$Z(t') = \delta^h(Z(t), X(t), Y(t));$$

(10) $$Wt' = \{B \cup \{\pi_1(Z(t'))\} \mid B \in Wt \vee B \text{ empty}\};$$

$$(\alpha) \text{ If } \bigvee_{\bar{B} \in \mathcal{U}^h} B \in \mathcal{U} \cap Wt' \wedge Z(t') \in \bar{B} \wedge$$

$$\bigvee_{0 \le i \le n} [(0 < i < n \wedge \bar{A}_i \supsetneqq \bar{B} \supsetneqq \bar{A}_{i+1} \wedge A_i \supsetneqq B \supsetneqq A_{i+1})$$

$$\vee (i = n \wedge \bar{B} \subsetneqq \bar{A}_n \wedge B \subsetneqq A_n) \vee (i = 0 \wedge \bar{B} \supsetneqq \bar{A}_1$$

$B \supseteq A_1)] \wedge Z(t') \notin R^h_\ell [\bar{A}_1, \sigma_1, \ldots, \bar{A}_i, \sigma_i, \bar{B}, \bar{B}(Z(t'))]$,

let $\bar{B}$ be the largest such. Then $Vt' =$

$[\bar{A}_1, \sigma_1, \ldots, \bar{A}_n, \sigma_n, \bar{B}, \bar{B}(Z(t'))]$.

(β) If not (α), let i be such that $Z(t') \in \bar{A}_i$,

$Z(t') \notin \bar{A}_{i+1}$, (where $\bar{A}_{n+1}$ is considered empty).

Then $Vt' = \bar{A}_1, \sigma_1, \ldots, \bar{A}_i, \sigma_i]$.

Note also the formulas:

If $Vt = [\bar{A}_1, \sigma_1, \ldots, \bar{A}_n, \sigma_n]$, then

$Z(t) \notin R^h_\ell [\bar{A}_1, \sigma_1, \ldots, \bar{A}_n, \sigma_n]$;

$\bar{A}_1 \supsetneq \cdots \supsetneq \bar{A}_n$;

(11) $A_1 \supsetneq \cdots \supsetneq A_n$;

$A_i \in \mathcal{U} \cap Wt$, $1 \le i \le n$;

$Z(t) \in \bar{A}_n$, if $n \neq 0$;

σ_i an initial segment of the permuting sequence for $\bar{A}_i$;

Wt is a chain of subsets of S.

Since $\langle s_0, \Lambda, \ldots, \Lambda\rangle \notin R^h_\ell[\ ]$ by assumption, $Z(0)$, $W0$, and $V0$ given by (10) satisfy the conditions (11) with $t = 0$. Assume that (11) holds for t, and that $Y(t)$ is any member of J. Then by (9), $X(t)$, $Z(t')$, and Wt' as described in (10) exist, with wt' a chain of subsets of S. If Vt' is computed by (β), then $Z(t') \notin P^h_{\hat{\ell}}\ [\bar{A}_1, \sigma_1, \ldots, \bar{A}_n, \sigma_n]$ and $Z(t') \in \bar{A}_i$ implies $Z(t') \notin R^h_{\hat{\ell}}\ [\bar{A}_1, \sigma_1, \ldots, \bar{A}_i, \sigma_i]$ by (9). Since $\pi_1(Z(t')) \in A_i \subset A_{i-1} \subset \ldots \subset A_1$, A_j is still in $\mathcal{U} \cap Wt'$ for $1 \le j \le i$. Thus, (11) holds with t replaced by t'. It can easily be seen that (11) will also hold with t replaced by t' if Vt is computed by (α). Formulas (10) therefore define ω-sequences Z, W, V, and X, given ω-sequence Y, and (10) implies (11).

Note that the I-sequence X is produced by (10) in a 1-shift fashion from the J-sequence Y. Since Z, W, and V have only finitely many possible values, (10) in fact defines a 1-shift FAO mapping J-sequences into I-sequences. Note also that by (11), the sequence of states Z of the automaton $\mathfrak{A}^h$ is always forced not to be a member of a specific $R^h_{\hat{\ell}}$. It is this property that insures that machine $\mathfrak{A}^h$ will not ultimately cycle through some member of $\mathcal{U}^h$.

If Z is defined by (10), we must show that $\inf \pi_1(Z) \notin \mathcal{U}$, and so (10) defines a 1-shift FAO that solves $\neg C^h(X,Y)$ for X. Suppose that $\inf Z = \bar{D} \in \mathcal{U}^h$, with $\pi_1(\bar{D}) = D \in \mathcal{U}$. Then there exists some $t_1 \ge h$ such that

(12) $\quad t \ge t_1$ implies $Z(t) \in \bar{D}$, and $\bar{u} \in \bar{D}$ implies

$$(\forall a)(\exists t)\ [t \ge a \wedge Z(t) = \bar{u}].$$

That is, from some time t_1 on, Z continues to traverse the same set $\bar{D} \in \mathcal{U}^h$. Now W keeps track of all sets traversed by $\pi_1(Z)$ from each time t; i.e., Wt is the set

$$\{\{\pi_1(Z(0)), \ldots, \pi_1(Z(t))\}, \{\pi_1(Z(1)), \ldots, \pi_1(Z(t))\}, \ldots, \{\pi_1(Z(t-1)), \pi_1(Z(t))\}, \{\pi_1(Z(t))\}\}.$$

Since $D \in \mathcal{U}$, it is clear from the definition of Wt that there is some $t_2 \geq t_1$ such that

(13) $$t \geq t_2 \text{ implies } D \in \mathcal{U} \cap Wt.$$

Assume $Vt = [\bar{A}_1, \sigma_1, \ldots, \bar{A}_n, \sigma_n]$. Then by (10), $Z(t') \not\in Q^h_{\hat{\ell}}[\bar{A}_1, \sigma_1, \ldots, \bar{A}_n, \sigma_n]$. Hence, by (9), (12), and $\bar{D} \in \mathcal{U}^h$, we have:

$$[t \geq t_2 \wedge Vt = [\bar{A}_1, \sigma_1, \ldots, \bar{A}_n, \sigma_n] \wedge \bar{D} \subsetneq \bar{A}_n \wedge D \subsetneq A_n]$$
$$\supset \bigvee_{\bar{u} \in \bar{D}} \bar{u} \not\in R^h_{\hat{\ell}}[A_1, \sigma_1, \ldots, \bar{A}_n, \sigma_n, \bar{D}, \bar{D}(\bar{u})],$$

which yields by (12),

(14) $$[t \geq t_2 \wedge Vt = [\bar{A}_1, \sigma_1, \ldots, \bar{A}_n, \sigma_n] \wedge \bar{D} \subsetneq \bar{A}_n \wedge D \subsetneq A_n]$$
$$\supset (\exists a)[a \geq t \wedge Z(a') \not\in R^h_{\hat{\ell}}[\bar{A}_1, \sigma_1, \ldots, \bar{A}_n, \sigma_n, \bar{D}, \bar{D}(Z(a'))]].$$

Define the partial ordering $<$ on strictly decreasing chains $A_1 \supsetneq \ldots \supsetneq A_p$ of members of $\mathcal{U}$ by:

$$[B_1, \ldots, B_q] < [A_1, \ldots, A_p] \quad .\equiv. \bigvee_{1 \le i \le p,q} [\bigwedge_{1 \le j \le i} A_j = B_j \wedge$$

$$A_i \supsetneq B_i] \quad \wedge \bigwedge_{i \le j \le q} [A_j = B_j \wedge p > q].$$

Define the principal part of the chain $A_1 \supsetneq \ldots \supsetneq A_n$ (of the sequence $[\bar{A}_1, \sigma_1, \ldots, \bar{A}_n, \sigma_n]$) to be the chain $A_1 \supsetneq \ldots \supsetneq A_p$ (the sequence $[\bar{A}_1, \sigma_1, \ldots, \bar{A}_p, \sigma_p]$) where p is the largest number i such that $A_i \supseteq D$, or $p = 0$ if there is no such i. Let $[\bar{A}_1, \sigma_1, \ldots, \bar{A}_p, \sigma_p]$ be the principal part of $Vt = [\bar{A}_1, \sigma_1, \ldots, \bar{A}_n, \sigma_n]$. Then by examining the construction (10), it can be seen that the only case in which the principal part of Vt' is not equal to or greater than (with respect to $<$) the principal part of Vt is where Vt' is constructed from (β) with $i < p$. But if $t \ge t_2$, so that by (12) $Z(t') \in \bar{D} \subseteq \bar{A}_p$, then ($\beta$) with $i < p$ cannot be used. Hence, for $t \ge t_2$, the principal part of Vt either stays the same or increases. Since $<$ is a partial ordering on a finite set, there must be some t_3 such that $t \ge t_3$ implies the principal part of Vt' is equal to the principal part of Vt. That is, there is some $m \ge 0$, $\bar{\hat{A}}_1, \hat{\sigma}_1, \ldots, \bar{\hat{A}}_m, \hat{\sigma}_m$, such that $m = 0$ or $\hat{A}_m \supseteq D$, and if $t \ge t_3$,

(15) $$Vt = [\bar{\hat{A}}_1, \hat{\sigma}_1, \ldots, \bar{\hat{A}}_m, \hat{\sigma}_m, \ldots, \bar{A}_n, \sigma_n]$$

where $n = m$ or $D \not\subseteq A_{m+1}$. Assume that for all $t \geq t_3$, Vt has the form $[\bar{\hat{A}}_1, \hat{\sigma}_1, \ldots, \bar{\hat{A}}_m, \hat{\sigma}_m, \bar{A}_{m+1}(t), \sigma_{m+1}(t), \ldots]$. By (10), $\bar{A}_{m+1}(t) \subseteq \bar{A}_{m+1}(t')$ for $t \geq t_3$. Hence for some $u \geq t_3$, $t \geq u$ implies that $\bar{A}_{m+1}(t) = \bar{A}_{m+1}(t')$, and Vt is of the form $[\bar{\hat{A}}_1, \sigma_1, \ldots, \bar{\hat{A}}_m, \sigma_m, \bar{A}, \ldots]$, where $\bar{A} = \bar{A}_{m+1}(u)$.

By (15), $D \not\subseteq A$, and since Z traverses $\bar{D}$ by (12), there exists some $v \geq u$ with $Z(v') \notin \bar{A}$. But this would cause case (β) with $i = m$ to come into play, and Vv' would equal $[\bar{\hat{A}}_1, \hat{\sigma}_1, \ldots, \bar{\hat{A}}_m, \hat{\sigma}_m]$ contrary to assumption. Hence, there must be some $t_4 \geq t_3$ such that

(16) $$Vt_4 = [\bar{\hat{A}}_1, \hat{\sigma}_1, \ldots, \bar{\hat{A}}_m, \hat{\sigma}_m].$$

Now if $\hat{A}_m \supsetneq D$ or $m = 0$, by (14) and (16) there is an $a \geq t_4$ such that $Z(a') \notin R^h_{\hat{\ell}}[\bar{\hat{A}}_1, \hat{\sigma}_1, \ldots, \bar{\hat{A}}_m, \hat{\sigma}_m, \bar{D}, \bar{D}(Z(a'))]$. By (15), either (i) $Va = [\bar{\hat{A}}_1, \hat{\sigma}_1, \ldots, \bar{\hat{A}}_m, \hat{\sigma}_m]$, or (ii) $Va = [\bar{\hat{A}}_1, \hat{\sigma}_1, \ldots, \bar{\hat{A}}_m, \hat{\sigma}_m, \bar{A}_{m+1}, \ldots]$ and $D \subsetneq A_{m+1}$. If (ii) is the case, then $D \in \mathcal{U} \cap Wa$ by (13), and $A_{m+1} \in \mathcal{U} \cap Wa$ by (10). Since Wt is a chain and $D \not\subseteq A_{m+1}$, we have $A_{m+1} \subsetneq D$. Thus in both (i) and (ii), $\bar{D} \in \mathcal{U}^h$ is a possible value for $\bar{B}$ in case (α) of (10). Hence, Va' would be calculated using some $\bar{B} \supseteq \bar{D}$, and

$$Va' = [\bar{\hat{A}}_1, \hat{\sigma}_1, \ldots, \bar{\hat{A}}_m, \hat{\sigma}_m, \bar{B}, \bar{B}(Z(a'))]$$

where $\bar{\bar{A}}_m \supsetneq \bar{B} \supseteq \bar{D}$ and $\hat{A}_m \supsetneq B \supseteq D$, contradicting (15). Therefore,

(17) $\qquad m \geq 1$ and $\hat{A}_m = \bar{D}$.

Now for $t \geq t_3$, $Vt' = [\bar{\bar{A}}_1, \hat{\sigma}_1, \ldots, \bar{\bar{A}}_m, \hat{\sigma}_m, \ldots]$ and $Z(t') \in \bar{\bar{A}}_m$ by (11). But by (10), this implies $\pi_1(Z(t')) \neq \pi_1 P(\hat{\sigma}_m)$. Hence, for some $s_m \in \hat{A}_m$, $\pi_1(Z(t')) \neq s_m$ for all $t \geq t_3$, and thus $\hat{A}_m \neq D = \inf \pi_1(Z)$. This contradicts (17) and completes the proof of the lemma. ∎

Lemma 3. $C^h(X,Y)$ has a 1-delay solution for Y implies $\langle s_0, \Lambda, \ldots, \Lambda \rangle = s_0^h \in R_{\hat{\ell}}^h[\ ]$.

Proof. The proof follows from lemma 2, by noting that a 1-delay solution is equivalent to a 0-shift solution, and from the fact that every finite state condition is determined. ∎

Lemma 4. Let $\bar{s} = \langle s, x_0, \ldots, x_{h-2} \rangle \in s^h$, $\bar{\alpha} = \bar{A}_1, \sigma_1, \ldots, \bar{A}_n, \sigma_n$, and $\alpha = A_1, s_1, \ldots, A_n, s_n$, where $s_i = \pi_i P(\sigma_i)$. Then if $x_0 \neq \Lambda$, $\bar{s} \in \mathscr{R}_k^h[\bar{\alpha}]$ implies $s \in \mathscr{R}_h[\alpha]$, $\bar{s} \in \mathscr{P}_k^h[\bar{\alpha}]$ implies $s \in \mathscr{P}_k[\alpha]$, and $\bar{s} \in \mathcal{Q}_k^h[\bar{\alpha}]$ implies $s \in \mathcal{Q}_k[\alpha]$, for all $k \geq 0$.

Proof. Proof is by inducation on k. We indicate the induction for $\mathscr{R}$, and leave the rest to the reader.

$$\bar{s} = \langle s, x_0, \ldots, x_{h-2}\rangle \in \mathscr{R}^h_{k+1}[\bar{A}_1, \sigma_1, \ldots, \bar{A}_n, \sigma_n]$$

$$.\equiv. \bigvee_{x \in I} \bigvee_{y \in J} [\delta^h(\bar{s}, x, y) =$$

$$\langle \delta(s, x_0, y),\ x_1, \ldots, x_{h-2}, x\rangle \in$$

$$\mathscr{P}^h_k[\bar{A}_1, {}_1, \ldots, \bar{A}_n, \sigma_n] \cup \mathscr{Q}^h_k[\bar{A}_1, \sigma_1, \ldots, \bar{A}_n, \sigma_n]$$

$$\vee \bigvee_{1 \le i \le n} [\delta^h(\bar{s}, x, y) \in \bar{A}_i \wedge \delta(s, x_0, y) = \pi_1 P(\sigma_i)]].$$

Hence, by the inductive assumption,

$$\bigvee_{y \in J} \delta(s, x_0, y) \in \{s_1, \ldots, s_n\} \cup \mathscr{P}_k[A_1, s_1, \ldots, A_n, s_n]$$

$$\cup\ \mathscr{Q}_k[A_1, s_1, \ldots, A_n, s_n];$$

and thus by (2), $s \in \mathscr{R}_{k+1}[A_1, s_1, \ldots, A_n, s_n]$. The remainder of the theorem follows easily. ▮

Note that Lemma 4 does not hold if the $\mathscr{P}$, $\mathscr{Q}$, and $\mathscr{R}$ predicates are replaced by P,Q, and R.

We now construct the graph G^h mentioned above, such that $C^h(X,Y)$ has a 1-delay solution for Y implies G^h is 1-delay solvable, and such that G^h has a 1-delay solution implies G has an h-delay solution. Recall that $\bar{\ell}$ is such that

$$\mathscr{R}^h_{\bar{\ell}}[\alpha] = \mathscr{R}^h_{\bar{\ell}-1}[\alpha],\ \mathscr{P}^h_{\bar{\ell}}[\alpha] = \mathscr{P}^h_{\bar{\ell}-1}[\alpha],\ \text{and}\ \ \mathscr{Q}^h_{\bar{\ell}}[\alpha] = \mathscr{Q}^h_{\bar{\ell}-1}[\alpha],$$

for all α. $G^h = \langle N^h, N^h_0, E^h\rangle$ is then defined by:

I. $N^h = \{[y, \bar{s}, k, \bar{v}]\}$, where

1. $y \in J$, $\bar{s} \in S^h$, $0 < k \leq \bar{\ell}$;

2. $\bar{v}$ is of the form $[\bar{A}_1, \sigma_1, h_1, \ldots, \bar{A}_n, \sigma_n, h_n]$

with $n \geq 0$; $\bar{A}_i \in \mathscr{U}^h$, $\sigma_i \in \mathrm{In}\ (\bar{A}_i)$, $1 \leq i \leq n$;

$\bar{A}_1 \supsetneq \bar{A}_2 \supsetneq \cdots \supsetneq \bar{A}_n$; $\pi_1(\bar{A}_1) \supsetneq \cdots \supsetneq \pi_1(\bar{A}_n)$;

$0 < k \leq h_n < \ldots < h_1 \leq \ell$;

3. $\bigwedge_{\bar{u} \in \bar{A}_i} \quad \bar{u} \in R^h_{h_i}[\bar{A}_1, \sigma_1, \ldots, \bar{A}_i, \bar{A}_i(\bar{u})]$.

II. $N^h_0 = \{[y, s^h_0, k, \bar{v}] \in N^h\}$.

III. If $\bar{s} = \langle s, x_0, \ldots, x_{h-2} \rangle$, $\bar{v} = [\bar{A}_1, \sigma_1, h_1, \ldots, \bar{A}_n, \sigma_n, h_n]$,

$x \in I$, then $([y, \bar{s}, k, \bar{v}], [\hat{y}, \hat{\bar{s}}, \hat{k}, \hat{\bar{v}}], x) \in E^h$ just in case one of the following holds:

(α) $\delta^h(\bar{s}, x, y) = \hat{\bar{s}} \in \bar{A}_i$, and $\delta(s, x_0, y) = \pi_1 P(\sigma_i)$, for some $1 \leq i \leq n$;

$\hat{\bar{v}} = [\bar{A}_1, \sigma_1, h_1, \ldots, \bar{A}_i, \bar{A}_i(\sigma_i), h_i]$;

$\hat{k} = h_i$.

(β) $\delta^h(\bar{s}, x, y) = \hat{\bar{s}} \in \mathscr{P}^h_{k-1}[\bar{A}_1, \sigma_1, \ldots, \bar{A}_n, \sigma_n]$;

$\hat{\bar{v}} = [\bar{A}_1, \sigma_1, h_1, \ldots, \bar{A}_j, \sigma_j, h_j]$, where $\hat{\bar{s}} \in \bar{A}_j \cap \mathscr{R}^h_{k-1}[\bar{A}_1, \sigma_1, \ldots, \bar{A}_j, \sigma_j]$;

$\hat{k} = k - 1$.

(γ) $\delta^h(\bar{s},x,y) = \hat{\bar{s}} \in \mathcal{Q}^h_{k-1}[\bar{A}_1,\sigma_1,\ldots,\bar{A}_n,\sigma_n]$;

$\hat{\bar{v}} = [\bar{A}_1,\sigma_1,h_1,\ldots,\bar{A}_n,\sigma_n,h_n,\bar{B},\bar{B}(\hat{\bar{s}}),\ k-1]$, where

$\bar{B} \in \mathcal{U}^h$, $\hat{\bar{s}} \in \bar{B} \subsetneq \bar{A}_n$, $B \subsetneq A_n$, and $\bigwedge_{\bar{u}\in\bar{B}} \bar{u} \in \mathcal{R}^h_{k-1}[\bar{A}_1,\sigma_1,\ldots,\bar{A}_n,$ $\sigma_n,\bar{B},\bar{B}(\bar{u})]$;

$\hat{k} = k - 1$.

Note that just as in the case of G, G^h can be effectively constructed.

Theorem 3. If $C^h(X,Y)$ has a 1-delay solution for Y, then G^h is 1-delay solvable.

Proof. A node $[y,\bar{s} = \langle s,x_0,\ldots,x_{h-2}\rangle,k,v = [\bar{A}_1,\sigma_1,h_1,\ldots,\bar{A}_n,\sigma_n,h_n]]$ of graph G^h is said to have property $w(x)$ for $x \in I$, if

(i) $\bar{s} \in R^h_k[\bar{A}_1,\sigma_1,\ldots,\bar{A}_n,\sigma_n]$; and

(19) (ii) $\delta^h(\bar{s},x,y) \in P^h_{k-1}[\bar{A}_1,\sigma_1,\ldots,\bar{A}_n,\sigma_n] \cup Q^h_{k-1}[\bar{A}_1,\sigma_1,\ldots,\bar{A}_n,\sigma_n]$,

or $\bigvee_{1\le i\le n} \delta^h(\bar{s},x,y) \in \bar{A}_i$ and $\delta(s,x_0,y) = \pi_1 P(\sigma_i)$.

We will show that:

(a) given $X(0) \in I$, we can find a node $N0 \in N^h_0$ with property $w(X(0))$;

(b) given node Nt with property $w(X(t))$ and $X(t') \in I$, we can find a node Nt' with property $w(X(t'))$ such that $X(t)$ connects Nt to Nt'; (i.e., $(Nt,\ Nt',\ Xt) \in E^h$).

Thus G^h will have been shown to be 1-delay solvable.

Since $C^h(X,Y)$ is assumed to have a 1-delay solution, $\langle s_0,\Lambda,\ldots,\Lambda\rangle = s_0^h \in R^h_{\hat{\ell}}[\ ] .\equiv. \bigwedge_{x\in I} \bigvee_{y\in J} [\delta^h(\langle s_0,\Lambda,\ldots,\Lambda\rangle, x, y) \in P^h_{\hat{\ell}-1}[\ ] \cup Q^h_{\hat{\ell}-1}[\ ]]$, by Lemma 3. Assume that $X(0) \in I$ is given. Then let $Y(0)$ be such that

$$\delta^h(\langle s_0,\Lambda,\ldots,\Lambda\rangle,\ X(0),\ Y(0)) \in P^h_{\hat{\ell}-1}[\ ] \cup Q^h_{\hat{\ell}-1}[\ ],$$

and take

$$N0 = [Y(0),\ \langle s_0,\Lambda,\ldots,\Lambda\rangle, \hat{\ell}, [\ ]].$$

$N0 \in N_0^h$ is a node of G^h and clearly has property $w(X(0))$.

Now assume that node $Nt = [Y(t),\ \bar{s}t = \langle st, x_0,\ldots,x_{h-2}\rangle,\ Kt,$ $\bar{v}t = [\bar{A}_1,\sigma_1,h_1,\ldots,\bar{A}_n,\sigma_n,h_n]]$ has property $w(X(t))$, and that $X(t') \in I$ is given. By (19), since Nt has property $w(X(t))$,

$$\delta^h(\bar{s}t,\ X(t),\ Y(t)) \in P^h_{Kt-1}[\bar{A}_1,\sigma_1,\ldots,\bar{A}_n,\sigma_n] \cup$$

$$Q^h_{Kt-1}[\bar{A}_1,\sigma_1,\ldots,\bar{A}_n,\sigma_n] \vee \bigvee_{1\le i\le n}$$

$$(\delta^h(\bar{s}t, X(t), Y(t)) \in \bar{A}_i \wedge \delta(st, x_0, Y(t)) = \pi_1 P(\sigma_i)).$$

There are three cases to be considered.

<u>Case 1</u>. $\delta^h(\bar{s}t,\ X(t),\ Y(t)) \in \bar{A}_i$ and $\delta(st,x_0,Y(t)) = \pi_1 P(\sigma_i)$. Let $\delta^h(\bar{s}t,X(t),Y(t)) = \bar{s}t' = \langle \pi_1 P(\sigma_i),\ x_1,\ldots,x_{h-2},\ X(t)\rangle$. Since $\bar{s}t' \in \bar{A}_i$ and Nt is a node of G^h, we have by the definition of G^h(18):

$$\bar{s}t' \in R^h_{h_i}[\bar{A}_1,\sigma_1,\ldots,\bar{A}_i,\bar{A}_i(\bar{s}t')] \;.\equiv.$$

(20)
$$\bigwedge_{x\in I}\ \bigvee_{y\in J}\ [\delta^h(\bar{s}t',x,y) \ \in\ P^h_{h_i-1}[\bar{A}_1,\sigma_1,\ldots,\bar{A}_i,\bar{A}_i(\bar{s}t')]$$
$$\cup\ Q^h_{h_i-1}[\bar{A}_1,\sigma_1,\ldots,\bar{A}_i,\bar{A}_i(\bar{s}t')]\ \vee$$
$$\bigvee_{1\le j<i}[\delta^h(\bar{s}t',x,y)\ \in\ \bar{A}_j\ \wedge\ \delta(\pi_1P(\sigma_i),x_1,y) =$$
$$\pi_1P(\sigma_j)]\ \vee\ [\delta^h(\bar{s}t',x,y)\ \in\ \bar{A}_i\ \wedge$$
$$\delta(\pi_1P(\sigma_i),\ x_1,y) = \pi_1P(\bar{A}_i(\bar{s}t'))].$$

Choose $Y(t')$ such that (20) holds for $X(t')$. Take $Nt' = [Y(t'),\ \delta^h(\bar{s}t,X(t),Y(t)),$ $h_i,\ [\bar{A}_1,\sigma_1,h_1,\ldots,\bar{A}_i,\bar{A}_i(\sigma_i),\ h_i]]$. Now since Nt was a node, $\bigwedge_{\bar{u}\in A_j}\ \bar{u} \in R^h_{h_i}[\bar{A}_1,\sigma_1,\ldots,\bar{A}_j,\bar{A}_j(\bar{u})],\ 1\le j\le i$. Hence, Nt' is a node of G^h, and by case (α) of (18), $(Nt,\ Nt',\ X(t)) \in E^h$. By (20), Nt' has property $w(X(t'))$.

<u>Case 2.</u> $\delta^h(\bar{s}t,\ X(t),\ Y(t)) \in P^h_{Kt-1}[\bar{A}_1,\sigma_1,\ldots,\bar{A}_n,\sigma_n]$. Suppose j is the least number such that

(21)
$$\bar{s}t' = \delta^h(\bar{s}t,\ X(t),\ Y(t)) = \langle\delta(st,x_0,Y(t)),$$
$$x_1,\ldots,x_{h-2},\ X(t)\rangle\ \in \bar{A}_j\ \cap\ R^h_{Kt-1}[\bar{A}_1,\sigma_1,\ldots,\bar{A}_j,\sigma_j].$$

Then by (8),

$$(22)\quad \bigwedge_{x\in I} \bigvee_{y\in J} [\delta^h(\bar{s}t',x,y) = \langle \delta(\delta(st,x_0,Y(t)),x_1,y), x_2,\ldots,x_{h-2}, X(t),x\rangle \in P^h_{Kt-2}[\bar{A}_1,\sigma_1,\ldots,\bar{A}_j,\sigma_j] \cup Q^h_{Kt-2}[\bar{A}_1,\sigma_1,\ldots,\bar{A}_j,\sigma_j] \vee \bigvee_{1\le i\le j} (\delta^h(\bar{s}t',x,y) \in \bar{A}_i \wedge \delta(\delta(st,x_0,Y(t))) = \pi_1 P(\sigma_i))].$$

Let $Y(t')$ be such that (22) holds for $X(t')$, and let $Nt' = [Y(t), \bar{s}t', Kt-1, [\bar{A}_1,\sigma_1,h_1,\ldots,\bar{A}_j,\sigma_j,h_j]]$. Again, since Nt is a node, $\bigwedge_{\bar{u}\in\bar{A}_i} \bar{u} \in R^h_{k_i} [\bar{A}_1,\sigma_1,\ldots,\bar{A}_i,\bar{A}_i(\bar{u})]$, for $1 \le i \le j$, and thus Nt' is a node of G^h. By (21) and Lemma 1, $\delta^h(\bar{s}t,X(t),Y(t)) \in \bar{A}_j \cap \mathscr{R}^h_{Kt-1}[\bar{A}_1,\sigma_1,\ldots,\bar{A}_j,\sigma_j]$. Hence, by case ($\beta$) of (18), Nt is connected by $X(t)$ to Nt'. By (22), Nt' has property $w(X(t'))$.

<u>Case 3</u>. $\delta^h(\bar{s}t, X(t), X(t)) \in Q^h_{Kt-1}[\bar{A}_1,\sigma_1,\ldots,\bar{A}_n,\sigma_n]$. Suppose $\bar{B}$ is the first in some chosen order of $\mathscr{U}^h$ such that

$$(23)\quad \delta^h(\bar{s}t, X(t), Y(t)) = \bar{s}t' \in \bar{B} \in \mathscr{U}^h \wedge \bar{B} \subsetneq \bar{A}_n \wedge B \subsetneq A_n \wedge \bigwedge_{\bar{u}\in\bar{B}} \bar{u} \in R^h_{Kt-1} [\bar{A}_1,\sigma_1,\ldots,\bar{A}_n,\sigma_n,\bar{B}, \bar{B}(\bar{u})].$$

Then $\delta^h(\bar{s}t, X(t), Y(t)) = \langle \delta(st,x_0,Y(t)), x_1,\ldots,x_{h-2},X(t)\rangle = \bar{s}t'$, and

$$\bar{s}t' \in R^h_{Kt-1}[\bar{A}_1,\sigma_1,\ldots,\bar{A}_n,\sigma_n,\bar{B},\bar{B}(\bar{s}t')] \ .\equiv.$$

$$\bigwedge_{x\in I} \ \bigvee_{y\in J} [\delta^h(\bar{s}t,x,y) \in$$

$$P^h_{Kt-2}[\bar{A}_1,\sigma_1,\ldots,\bar{A}_n,\sigma_n,\bar{B},\bar{B}(\bar{s}t')] \cup$$

(24) $$Q^h_{Kt-2}[\bar{A}_1,\sigma_1,\ldots,\bar{A}_n,\sigma_n,\bar{B},\bar{B}(\bar{s}t')] \vee \bigvee_{1\le i\le n}$$

$$[\bar{\delta}^h(\bar{s}t',x,y) \in \bar{A}_i \ \wedge \ \delta(\delta(st,x_0,Y(t)),\ x_1,y) = \pi_1 P(\sigma_i)] \vee$$

$$[\delta(\bar{\delta}^h(\bar{s}t',x,y) \in \bar{B} \ \wedge \ \delta(\delta(st,x_0,Y(t)),\ x_1,y) = \pi_1 P(\bar{B}(\bar{s}t'))]].$$

Let $Y(t')$ be such that (24) holds for $X(t')$, and $Nt' = [Y(t'),\ \bar{s}t',\ Kt-1,$ $[\bar{A},\sigma_1,h_1,\ldots,\bar{A}_n,\sigma_n,h_n,\bar{B},\ \bar{B}(\bar{s}t'),\ Kt-1]$. Then by (23) and the fact that Nt is a node of G^h, $\bigwedge_{\bar{u}\in A_i} \bar{u} \in R^h_{h_i}[\bar{A}_1,\sigma_1,\ldots,\bar{A}_i,\bar{A}_i(\bar{u})]$, $1 \le i \le n+1$ (where $\bar{A}_{n+1} = \bar{B}$ and $h_{n+1} = Kt-1$). Hence, Nt' is a node of G^h. By (23) and Lemma 1, $\bigwedge_{\bar{u}\in\bar{B}} u \in \mathcal{R}^h_{Kt-1}[\bar{A}_1,\sigma_1,\ldots,\bar{A}_n,\sigma_n,\bar{B},\ \bar{B}(\bar{u})]$ and

$\delta^h(\bar{s}t,\ X(t),\ Y(t)) = \bar{s}t' \in \mathcal{Q}^h_{Kt-1}[\bar{A}_1,\sigma_1,\ldots,\bar{A}_n,\sigma_n]$. Hence, $(Nt,\ Nt',\ X(t)) \in E^h$ by (γ) of (18). By (24), Nt' has property $w(X(t'))$.

This completes the proof of Theorem 3. ∎

<u>Theorem 4</u>. G^h has a 1-delay solution implies G has an h-delay solution.

Proof. We must show that given $X(0)\ldots X(h-1) \in I^*$, we can obtain a node $N0 \in N_0$ of G, and for every t, assuming that we have obtained a node Nt of G and that $X(t)\ldots X(t+h) \in I^*$ has been given, we can find a node Nt' of G such that $[Nt, Nt', X(t)] \in E$.

Assume $X(0)\ldots X(h-1)$ given. Since G^h has a 1-delay solution, nodes $M0,\ldots,M(h-1)$ of G^h can be determined such that $M0 \in N_0^h$ and $[Mk, M(k+1), X(k)] \in E^h$ for $0 \le k < h-1$. Recall that $([y_0,\bar{s}_0,k_0,\bar{v}_1], [y_1,\bar{s}_1,k_1,\bar{v}_1],x) \in E^h$ implies $\delta^h(\bar{s}_0,x,y_0) = \bar{s}_1$. Then since $M0 \in N_0^h$ and $\delta^h(<s,\Lambda,x_1,\ldots,x_{h-2}>,x,y) = <s,x_1,\ldots,x_{h-2},x>$, $M0,\ldots,M(h-1)$ have the form:

$$M0 = [Y(0), <s_0,\Lambda,\ldots,\Lambda>, K0, \bar{v}0];$$

$$M1 = [Y(1), <s_0,\Lambda,\ldots,\Lambda,X(0)>, K0, \bar{v}0];$$

$$\vdots$$

$$M(h-1) = [Y(h-1), <s_0,X(0),\ldots,X(h-2)>,K(h-1),\bar{v}(h-1)].$$

Take $N0$ to be

$$N0 = [Y(h-1),s_0,K(h-1),\pi_1(\bar{v}(h-1))],$$

where if $\bar{v} = [\bar{A}_1,\sigma_1,h_1,\ldots,\bar{A}_n,\sigma_n,h_n]$, then $\pi_1(\bar{v})$ is

$[A_1,\pi_1P(\sigma_1),h_1,\ldots,A_n,\pi_1P(\sigma_n),h_n]$. $N0 \in N_0$, by definition of N_0.

Now assume that $X(t)\ldots X(t+h)$ have been given, and that node Nt of G and nodes $Mt, \ldots, M(t+h)$ of G^h have been determined so that $(Mk, M(k+1), X(k)) \in E^h$, $t \le k < t + h$. Assume

$$Mt = [Y(t), \langle st, x_0, \ldots, x_{h-2}\rangle, Kt, \bar{v}t];$$

$$M(t+1) = [Y(t+1), \langle s(t+1), x_1, \ldots, x_{h-2}, X(t)\rangle, K(t+1), \bar{v}(t+1)];$$

$$\vdots$$

$$M(t+h-1) = [Y(t+h-1), \langle s(t+h-1), X(t), \ldots, X(t+h-2)\rangle,$$

$$K(t+h-1), \bar{v}(t+h-1) = [\bar{A}_1, \sigma_1, \ldots, A_n, \sigma_n, h_n]];$$

$$M(t+h) = [Y(t+h), \langle s(t+h) = \delta(s(t+h-1), X(t), Y(t+h-1),$$

$$X(t+1), \ldots, X(t+h-1)\rangle, K(t+h), \bar{v}(t+h)];$$

$$Nt = [Y(t+h-1), s(t+h-1), K(t+h-1), \pi_1(\bar{v}(t+h-1))].$$

Take Nt' to be

$$Nt' = [Y(t+h), s(t+h) = \delta(s(t+h-1), X(t), Y(t+h-1),$$

$$K(t+h), \pi_1(\bar{v}(t+h))].$$

By assumption, $M(t+h-1)$ is connected to $M(t+h)$ by $X(t+h-1)$, and hence, either (α), (β), or (γ) of (18) holds.

<u>Case (α)</u>. $\delta(s(t+h-1), X(t), Y(t+h-1)) = \pi_1 P(\sigma_i)$,

$$\bar{v}(t+h) = [\bar{A}_1, \sigma_1, h_1, \ldots, \bar{A}_i, \bar{A}_i(\sigma_i), h_i]$$

$K(t+h) = h_i$, for some $1 \le i \le n$.

Then since $\pi_1 (\bar{v}(t+h-1)) = [A_1, \pi_1 P(\sigma_1), h_1, \ldots A_n, \pi_1 P(\sigma_n), h_n]$, and by (6) $\pi_1 P(\bar{A}_i(\sigma_i)) = A_i(\pi_1 P(\sigma_i))$, case ($\alpha$) of (3) holds, and $(Nt, Nt', X(t)) \in E$.

<u>Case (β)</u>. $\delta^h(\langle s(t+h-1), X(t), \ldots, X(t+h-2)\rangle,$

$$X(t+h-1), Y(t+h-1)) = \langle \delta(s(t+h-1), X(t), Y(t+h-1)),$$

$$X(t+1), \ldots, X(t+h-2), X(t+h-1)\rangle$$

$$\in \bar{A}_j \cap \mathscr{R}^h_{h_j} [\bar{A}_1, \sigma_1, \ldots, \bar{A}_j, \sigma_j],$$

$$\bar{v}(t+h) = [\bar{A}_1, \sigma_1, h_1, \ldots, \bar{A}_j, \sigma_j, h_j],$$

$$K(t+h) = K(t+h-1) - 1, \text{ for some } 0 \le j \le n.$$

Then by Lemma 4, $\delta(s(t+h-1), X(t), Y(t+h-1)) \in A_j \cap \mathscr{R}_{h_j} [A_1, \pi_1 P(\sigma_1), \ldots, A_j, \pi_1 P(\sigma_j)]$, and by ($\beta$) of (3), $(Nt, Nt', X(t)) \in \dot{E}$.

<u>Case (γ).</u> $\delta^h(\langle s(t+h-1), X(t), \ldots, X(t+h-2)\rangle$

$$X(t+h-1), Y(t+h-1)) = \langle \delta(s(t+h-1),$$

$$X(t), Y(t+h-1)), X(t+1), \ldots, X(t+h-1)\rangle$$

$$\in \mathscr{Q}^h_{K(t+h-1)-1}[\bar{A}_1, \sigma_1, \ldots, \bar{A}_n, \sigma_n].$$

Suppose $\bar{B} \in \mathscr{U}^h$, $B \subsetneq A_n$, $\bar{B} \subsetneq \bar{A}_n$, and

$$\delta^h(\bar{s}(t+h-1), X(t+h-1), Y(t+h-1)) \in \bar{B},$$

$$\bigwedge_{\bar{u} \in \bar{B}} \bar{u} \in \mathscr{R}^h_{K(t+h-1)-1}[\bar{A}_1, \sigma_1, \ldots, \bar{A}_n, \sigma_n, \bar{B}, \bar{B}(\bar{u})],$$

$$\bar{v}(t+h) = [\bar{A}_1, \sigma_1, h_1, \ldots, \bar{A}_n, \sigma_n, h_n, \bar{B}, \bar{B}(\bar{s}(t+h-1)), K(t+h-1) - 1];$$

$$K(t+h) = K(t+h-1) - 1.$$

Then by Lemma 4,

$$\bigwedge_{u \in B} u \in \mathcal{R}_{K(t+h-1)-1}[A_1, \pi_1 P(\sigma_1), \ldots, A_n, \pi_1 P(\sigma_n), B, B(u)],$$

and by (γ) of (3), Nt is connected to Nt' by X(t).

This completes the proof of Theorem 4. ∎

Theorems 2, 3, 4, and the definition of $C^h(X,Y)$ now yield that $C(X,Y)$ has an h-delay solution for Y if and only if G is h-delay solvable. The result of Even and Meyer (Theorem 1) thus gives the desired algorithm.

6. Further Problems

Rabin [11] among others has used various types of finite automata to give decision procedures for second order theories other than SC. To our knowledge, there have been no successful formulations of the synthesis problem for these theories. For example, the monadic second-order theory of two successor functions studied by Rabin [11] can be considered to describe conditions on infinite trees in the same way that SC expresses conditions on ω-sequences. Can a meaningful class of operators on infinite trees be formulated such that a synthesis algorithm can be given with respect to these operators?

Concerning sequential calculus, Büchi and Landweber [4] pose the problem of giving for any finite state condition $C(X,Y)$, a set of recursions with parameters which by proper specification of the parameters will yield any deterministic operator that solves C for Y.

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15. Supplementary Notes

16. Abstracts An algorithm is presented for deciding whether or not a condition C(X,Y) stated in sequential calculus admits a finite delay solution. This solves a problem stated by Buchi and Landweber concerning the existence of h-shift solutions for finite state conditions. The method is a combination of the methods of Buchi and Landweber for the non-delay case and a graph theoretic algorithm developed by Even and Meyer in studying Boolean conditions.

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