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ON AN ALGEBRAIC IDENTITY WITH APPLICATIONS TO OPERATOR THEORY

by

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1. Let $\mathcal{A}$ be a commutative algebra on the field $\mathbb{C}$ of complex numbers. Let superscripts $^+$ and $^-$ denote a pair of operations on $\mathcal{A}$ to $\mathcal{A}$, such that, for any κ, $\kappa_1, \kappa_2, \ldots, \kappa_n$ in $\mathcal{A}$ and $a_1, a_2, \ldots, a_n$ in $\mathbb{C}$,

$$(\kappa^+)^+ = (\kappa^+)^- = \kappa^+ \text{ and } (\kappa^-)^+ = (\kappa^-)^- = \kappa^-, \tag{1}$$

$$(\Sigma_{j=1}^n a_j \kappa_j)^{\pm} = \Sigma_{j=1}^n a_j (\kappa_j^{\pm}), \tag{2}$$

and

$$(\Pi_{j=1}^n \kappa_j)^{\pm} = \Pi_{j=1}^n (\kappa_j^{\pm}). \tag{3}$$

Let Δ denote the corresponding difference-operator,

$$\Delta \kappa = \kappa^+ - \kappa^-; \tag{4}$$

so that, by (1) - (3),

$$\Delta(\kappa^{\pm}) = \Delta(\Delta\kappa) = 0 \text{ and } (\Delta\kappa)^{\pm} = \Delta\kappa, \tag{5}$$

$$\Delta(\Sigma_{j=1}^n a_j \kappa_j) = \Sigma_{j=1}^n a_j (\Delta \kappa_j), \tag{6}$$

and

$$\Delta(\Pi_{j=1}^n \kappa_j) = \Sigma_{j=1}^n (\Pi_{p=1}^{j-1} \kappa_p^-)\Delta\kappa_j (\Pi_{q=j+1}^n \kappa_q^+), \tag{7}$$

where we note that, in (7), the order of the factors κ_j is arbitrary, since $\mathcal{A}$ is commutative, but the order is the same in every term on the right-hand side of (7). Let

$$\mathcal{N} = \{\varphi \in \mathcal{A} \colon \varphi^{+} = \varphi\}. \tag{8}$$

Then it is easily verified that, equivalently,

$$\mathcal{N} = \{\varphi \in \mathcal{A} \colon (\exists \kappa \in \mathcal{A})\, \varphi = \kappa^{+} \text{ or } \varphi = \kappa^{-}\}; \tag{9}$$

that $\mathcal{N}$ is a subalgebra of $\mathcal{A}$; and that, for all $\varphi \in \mathcal{N}$,

$$\varphi^{\pm} = \varphi \text{ and } \Delta\varphi = 0. \tag{10}$$

2. As an example of the foregoing abstract structure, we may take $\mathcal{A}$ to be the class of all complex-valued functions of two real variables (x, ξ); such that, as functions of x, they are Hölder-continuous in an interval R of the real line $\mathbb{R}$ (where R may be all of $\mathbb{R}$), and so are in $L^2(R)$, and, as functions of ξ, their limits, as $\xi \to 0$ from above and from below, exist for each x. Such functions will be denoted by the alternative notations $\kappa(x, \xi)$ and $\kappa_{\xi}(x)$. The operations

$$\kappa^{\pm}(x) = [\kappa(x, \xi)]^{\pm} = \lim_{\eta \downarrow 0} \kappa(x, \pm\eta), \tag{11}$$

for all κ in $\mathcal{A}$ and all x in R, clearly have all the required properties (1) - (3); and then $\mathcal{N}$ is the class of all functions in $\mathcal{A}$ which are constant with respect to ξ (that is, do not depend on ξ):

$$\mathcal{N} = \{\varphi \in \mathcal{A} \colon \varphi(x, \xi) = \varphi(x)\}, \tag{12}$$

where, therefore,

$$\varphi(x) = \lim_{\eta \downarrow 0} \varphi(x, \eta). \tag{13}$$

We note, too, that the Plemelj formulae ([9]; or [5], Section 74) yield that

$$\Lambda \left[\int_R \frac{f(x, u)\, du}{x + i\xi - u} \right] = -2\pi i\; f(x, x). \tag{14}$$

3. If $\underset{\sim}{M}$ denotes an $(n \times n)$ matrix with elements

$$(\underset{\sim}{M})_{ij} = \mu_{ij} \in \mathcal{A}, \tag{15}$$

then we can define its determinant in the usual way:

$$D = \det \underset{\sim}{M} = |\mu_{i1}\, \mu_{i2} \cdots \mu_{in}| = \begin{vmatrix} \mu_{11} & \mu_{12} & \cdots & \mu_{1n} \\ \mu_{21} & \mu_{22} & \cdots & \mu_{2n} \\ \vdots & \vdots & & \vdots \\ \mu_{n1} & \mu_{n2} & \cdots & \mu_{nn} \end{vmatrix}$$

$$= \sum_{\rho \in P_n} \varepsilon_\rho\, \mu_{\rho(1)1}\, \mu_{\rho(2)2} \cdots \mu_{\rho(n)n}; \tag{16}$$

where P_n denotes the set of all permutations ρ of $N = \{1, 2, \ldots, n\}$ and ε_ρ is the parity-index of the permutation ρ (taking values ± 1). Thus it is clear that $D \in \mathcal{A}$ also. Further, by (6), (7), and (16), we see that

$$\Lambda D = \sum_{j=1}^{n} |\mu_{i1}^{-}\, \mu_{i2}^{-} \cdots \mu_{i(j-1)}^{-}\, (\Lambda \mu_{ij})\, \mu_{i(j+1)}^{+} \cdots \mu_{in}^{+}|. \tag{17}$$

Suppose now that $\underset{\sim}{N}$ is another $(n \times n)$ matrix with elements

$$(\underset{\sim}{N})_{ij} = \nu_{ij} \in \mathcal{A}, \tag{18}$$

and suppose further that, for all $i, j \in N$,

$$\Delta \mu_{ij} = \Delta \nu_{ij}. \tag{19}$$

If we write

$$F = \det(\underset{\sim}{M} - \underset{\sim}{N}); \tag{20}$$

then, by (6), (17), and (19), we see that

$$\Delta F = 0. \tag{21}$$

We shall seek various representations of F and consequent identities arising from (21).

If there exists a matrix $\underset{\sim}{H}$ with elements

$$(\underset{\sim}{H})_{ij} = \eta_{ij} \in \mathcal{A}, \tag{22}$$

which satisfies the matrix equation

$$\underset{\sim}{M}\underset{\sim}{H} = \underset{\sim}{N}, \tag{23}$$

then $\underset{\sim}{M} - \underset{\sim}{N} = \underset{\sim}{M}(\underset{\sim}{I} - \underset{\sim}{H})$; so that, since the determinant of a product of square matrices equals the product of their respective determinants, we get, by (16), that

$$F = D \det(\underset{\sim}{I} - \underset{\sim}{H}). \tag{24}$$

To expand $\det(\underset{\sim}{I} - \underset{\sim}{H})$, we observe that a determinant is a linear function of each of its columns (compare (16)): thus $\det(\underset{\sim}{I} - \underset{\sim}{H})$ is the sum, over all ways of selecting certain columns (say the p columns indexed with $j_1, j_2, \ldots, j_p \in N$; where

$$1 \leq j_1 < j_2 < \cdots < j_p \leq n, \tag{25}$$

to be specific) from $\underset{\sim}{H}$ and the remaining columns from $\underset{\sim}{I}$, of $(-1)^p$ (to allow for the fact that $-\underset{\sim}{H}$ occurs in the original determinant) times a determinant of the form

$$\begin{vmatrix}
1 & 0 & \cdots & 0 & \eta_{1j_1} & 0 & \cdots & 0 & \eta_{1j_2} & 0 & \cdots & 0 & \eta_{1j_p} & 0 & \cdots & 0 \\
0 & 1 & \cdots & 0 & \eta_{2j_1} & 0 & \cdots & 0 & \eta_{2j_2} & 0 & \cdots & 0 & \eta_{2j_p} & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\
0 & 0 & \cdots & 1 & \eta_{(j_1-1)j_1} & 0 & \cdots & 0 & \eta_{(j_1-1)j_2} & 0 & \cdots & 0 & \eta_{(j_1-1)j_p} & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & \eta_{j_1j_1} & 0 & \cdots & 0 & \eta_{j_1j_2} & 0 & \cdots & 0 & \eta_{j_1j_p} & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & \eta_{(j_1+1)j_1} & 1 & \cdots & 0 & \eta_{(j_1+1)j_2} & 0 & \cdots & 0 & \eta_{(j_1+1)j_p} & 0 & \cdots & 0 \\
\vdots & \vdots & & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\
0 & 0 & \cdots & 0 & \eta_{(j_2-1)j_1} & 0 & \cdots & 1 & \eta_{(j_2-1)j_2} & 0 & \cdots & 0 & \eta_{(j_2-1)j_p} & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & \eta_{j_2j_1} & 0 & \cdots & 0 & \eta_{j_2j_2} & 0 & \cdots & 0 & \eta_{j_2j_p} & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & \eta_{(j_2+1)j_1} & 0 & \cdots & 0 & \eta_{(j_2+1)j_2} & 1 & \cdots & 0 & \eta_{(j_2+1)j_p} & 0 & \cdots & 0 \\
\vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & & \vdots \\
0 & 0 & \cdots & 0 & \eta_{(j_p-1)j_1} & 0 & \cdots & 0 & \eta_{(j_p-1)j_2} & 0 & \cdots & 1 & \eta_{(j_p-1)j_p} & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & \eta_{j_pj_1} & 0 & \cdots & 0 & \eta_{j_pj_2} & 0 & \cdots & 0 & \eta_{j_pj_p} & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & \eta_{(j_p+1)j_1} & 0 & \cdots & 0 & \eta_{(j_p+1)j_p} & 0 & \cdots & 0 & \eta_{(j_p+1)j_p} & 1 & \cdots & 0 \\
\vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & \eta_{nj_1} & 0 & \cdots & 0 & \eta_{nj_2} & 0 & \cdots & 0 & \eta_{nj_p} & 0 & \cdots & 1
\end{vmatrix} . \tag{26}$$

If we now expand every determinant (26) by each of the columns selected from $\underset{\sim}{I}$, and note that the p-rowed minors of a matrix vanish whenever p exceeds the rank of the matrix [4,6], we obtain

Theorem 1. *If* $\underset{\sim}{H}$ *is any* $(n \times n)$ *matrix with elements* $(\underset{\sim}{H})_{ij} = \eta_{ij}$, *then*

$$\det(\underset{\sim}{I} - \underset{\sim}{H}) = \sum_{p=0}^{\operatorname{rank}(\underset{\sim}{H})} (-1)^p \sum_{J \in Q_p} \eta_{JJ}^{(p)}; \tag{27}$$

where Q_p *denotes the set of all* $\binom{n}{p}$ *distinct unordered selections of* p *indices from* $N = \{1,2,\ldots,n\}$, $J = \{j_1, j_2, \ldots, j_p\}$ *satisfies* (25), *and* $\eta_{JJ}^{(p)}$ *is the corresponding p-rowed principal minor of* $\underset{\sim}{H}$,

$$\eta_{JJ}^{(p)} = \begin{vmatrix} \eta_{j_1 j_1} & \eta_{j_1 j_2} & \cdots & \eta_{j_1 j_p} \\ \eta_{j_2 j_1} & \eta_{j_2 j_2} & \cdots & \eta_{j_2 j_p} \\ \vdots & \vdots & & \vdots \\ \eta_{j_p j_1} & \eta_{j_p j_2} & \cdots & \eta_{j_p j_p} \end{vmatrix} \in \mathcal{A}. \tag{28}$$

It follows immediately from (24) and (27) that

$$F = D \sum_{p=0}^{\operatorname{rank}(\underset{\sim}{H})} (-1)^p \sum_{J \in Q_p} \eta_{JJ}^{(p)}. \tag{29}$$

4. We proceed by demonstrating an explicit ordering of the sets in Q_p. We define two integer-valued functions on Q_p:

$$\ell_p(J) = \ell_p(j_1, j_2, \ldots, j_p) = j_1 + j_2 n + j_3 n^2 + \cdots + j_p n^{p-1}, \tag{30}$$

and

$$\lambda_p(J) = \lambda_p(j_1, j_2, \ldots, j_p) = 1 + \binom{j_1 - 1}{1} + \binom{j_2 - 1}{2} + \cdots + \binom{j_p - 1}{p}; \tag{31}$$

where the set J satisfies the relation (25).

Theorem 2. The function λ_p defined in (31) puts the sets $J \in Q_p$ in one-to-one correspondence with the integers $1, 2, \ldots, \binom{n}{p}$.

Proof. By (25), no two distinct sets in Q_p have the same index (30). Thus the function ℓ_p puts the sets of Q_p in one-to-one correspondence with a certain set of positive integers (however, these integers are not consecutive.) The ordering of Q_p corresponding to increasing numerical order under ℓ_p is that which we shall impose: it is the lexical ordering of the 'words' $j_p j_{p-1} \cdots j_2 j_1$. The ordering condition

$$\ell_p(I) < \ell_p(J), \tag{32}$$

for any $I, J \in Q_p$, both ordered as in (25), holds if there is an r (necessarily unique) taking one of the values $1, 2, \ldots, p$, such that

$$i_r < j_r \text{ and } (\forall s > r)\ i_s = j_s. \tag{33}$$

For a given r, the number of sets I satisfying (33) for a fixed J is equal to the number of ways of selecting the r indices $i_1, i_2, \ldots, i_r$ all (by (25) for I) less than j_r; namely, $\binom{j_r - 1}{r}$. Thus, the total number of sets I satisfying (32) (that is, preceding J in the imposed ordering of Q_p) is clearly $\lambda_p(J) - 1$, by (31). The assertion of the theorem follows. ╟

Let

$$c = c(n, p) = \binom{n}{p} \text{ and } q = q(n, p) = \binom{n-1}{p-1}. \tag{34}$$

The $(c \times c)$ matrix $\underset{\sim}{H}^{(p)}$ with elements

$$\eta_{IJ}^{(p)} = \begin{vmatrix} \eta_{i_1 j_1} & \eta_{i_1 j_2} & \cdots & \eta_{i_1 j_p} \\ \eta_{i_2 j_1} & \eta_{i_2 j_2} & \cdots & \eta_{i_2 j_p} \\ \vdots & \vdots & & \vdots \\ \eta_{i_p j_1} & \eta_{i_p j_2} & \cdots & \eta_{i_p j_p} \end{vmatrix} \in \mathcal{A}, \tag{35}$$

which are p-rowed minors of $\underset{\sim}{H}$; where the sets I and J of indices are ordered by λ_p; is called the p-th compound matrix of $\underset{\sim}{H}$ (so that $\underset{\sim}{H}^{(1)} = \underset{\sim}{H}$ and $\underset{\sim}{H}^{(n)} = \det \underset{\sim}{H}$.) The minors $\eta_{JJ}^{(p)}$ defined in (28) and occurring in (27) and the expansion (29) of F are obviously the diagonal elements of $\underset{\sim}{H}^{(p)}$. Thus we may write (27) and (29) in the forms

$$\det (\underset{\sim}{I} - \underset{\sim}{H}) = \sum_{p=0}^{\operatorname{rank}(\underset{\sim}{H})} (-1)^p \operatorname{trace} \underset{\sim}{H}^{(p)} \tag{36}$$

and

$$F = D \sum_{p=0}^{\operatorname{rank}(\underset{\sim}{H})} (-1)^p \operatorname{trace} \underset{\sim}{H}^{(p)}. \tag{37}$$

The Binet-Cauchy theorem [4] asserts that, if $\underset{\sim}{M}\underset{\sim}{H} = \underset{\sim}{N}$ as in (23), then

$$\underset{\sim}{M}^{(p)}\underset{\sim}{H}^{(p)} = \underset{\sim}{N}^{(p)}, \tag{38}$$

where $\underset{\sim}{M}^{(p)}$ and $\underset{\sim}{N}^{(p)}$ are the p-th compounds of $\underset{\sim}{M}$ and $\underset{\sim}{N}$, respectively. Write

$$D^{(p)} = \det \underset{\sim}{M}^{(p)}. \tag{39}$$

Let D_{ij} denote the determinant obtained by replacing the i-th column of D by the j-th column of $\underset{\sim}{N}$; and similarly, let $D^{(p)}_{IJ}$ denote the determinant obtained by replacing the $\lambda_p(I)$-th column of $D^{(p)}$ by the $\lambda_p(J)$-th column of $\underset{\sim}{N}^{(p)}$(these are the columns respectively indexed by the sets I and J in Q_p.) Then the Leibnitz-Cramer rule tells us that, if $D = \det \underset{\sim}{M} \neq 0$, then the solution $\underset{\sim}{H}$ of (23) is given by

$$\eta_{ij} = D_{ij}/D; \tag{40}$$

and similarly, if $D^{(p)} = \det \underset{\sim}{M}^{(p)} \neq 0$, then the solution $\underset{\sim}{H}^{(p)}$ of (38) is given by

$$\eta^{(p)}_{IJ} = D^{(p)}_{IJ}/D^{(p)}. \tag{41}$$

From this, we derive yet another form of Theorem 1 and of the expansion of F: by (27), (29) and (41),

$$\det(\underset{\sim}{I} - \underset{\sim}{H}) = \Sigma_{p=0}^{\mathrm{rank}(H)} (-1)^p \, \Sigma_{J \in Q_p} D_{JJ}^{(p)}/D^{(p)} \qquad (42)$$

and

$$F = [D/D^{(p)}] \, \Sigma_{p=0}^{\mathrm{rank}(\underset{\sim}{H})} (-1)^p \, \Sigma_{J \in Q_p} D_{JJ}^{(p)} \, . \qquad (43)$$

It is well-known [4,6,7] that a determinant can be expanded by any column or row;

$$\left.\begin{aligned} D &= \Sigma_{k=1}^{n} (-1)^{i+k} \mu_{ki} C(\mu_{ki}), \\ &\text{and} \\ D &= \Sigma_{i=1}^{n} (-1)^{i+k} \mu_{ki} C(\mu_{ki}); \end{aligned}\right\} \qquad (44)$$

where $C(\mu_{ki})$ denotes the complementary minor to μ_{ki} in D; so that (if ^{c} denotes the complement in N)

$$C(\mu_{ki}) = \mu_{\{k\}^c \{i\}^c}^{(n-1)} . \qquad (45)$$

It is further well-known that (44) can be extended to yield that

$$\left.\begin{aligned} &\Sigma_{k=1}^{n} (-1)^{i+k} \mu_{ki'} C(\mu_{ki}) = \delta_{ii'} D, \\ &\text{and} \\ &\Sigma_{i=1}^{n} (-1)^{i+k} \mu_{k'i} C(\mu_{ki}) = \delta_{kk'} D. \end{aligned}\right\} \qquad (46)$$

The <u>Laplace expansion theorem</u> [4,6,7] states that, if

$$\sigma_p(J) = \Sigma_{s=1}^{p} j_s \, , \qquad (47)$$

then

$$\left.\begin{aligned} D &= \Sigma_{K \in Q_p} (-1)^{\sigma_p(I)+\sigma_p(K)} \mu_{KI}^{(p)} C(\mu_{KI}^{(p)}), \\ \text{and} \qquad & \\ D &= \Sigma_{I \in Q_p} (-1)^{\sigma_p(I)+\sigma_p(K)} \mu_{KI}^{(p)} C(\mu_{KI}^{(p)}); \end{aligned}\right\} \qquad (48)$$

where $C(\mu_{KI}^{(p)})$ is the (n-p)-rowed minor complementary to the p-rowed minor $\mu_{KI}^{(p)}$ of D; so that

$$C(\mu_{KI}^{(p)}) = \mu_{K^c I^c}^{(n-p)} \quad . \qquad (49)$$

It is easily shown, by a proof analogous to that used to extend (44) to (46), that we can extend (48) to yield that

$$\left.\begin{aligned} \Sigma_{K \in Q_p} (-1)^{\sigma_p(I)+\sigma_p(K)} \mu_{KI'}^{(p)} C(\mu_{KI}^{(p)}) &= \delta_{II'} D, \\ \text{and} \qquad & \\ \Sigma_{I \in Q_p} (-1)^{\sigma_p(I)+\sigma_p(K)} \mu_{K'I}^{(p)} C(\mu_{KI}^{(p)}) &= \delta_{KK'} D; \end{aligned}\right\} \qquad (50)$$

where, because of the internal ordering (25) imposed on the sets in Q_p,

$$\delta_{II'} = \delta_{i_1 i'_1} \delta_{i_2 i'_2} \cdots \delta_{i_p i'_p} . \qquad (51)$$

Finally, we observe that, since $\mu_{KI}^{(p)}$ is an entry in the compound determinant $D^{(p)}$, we may apply (46) to $D^{(p)}$, with the notation $C^{(p)}(\mu_{KI}^{(p)})$ for the minor of $D^{(p)}$ complementary to $\mu_{KI}^{(p)}$, to obtain that

$$\left.\begin{aligned} &\Sigma_{K\in Q_p}(-1)^{\lambda_p(I)+\lambda_p(K)}\mu^{(p)}_{KI'}\,C^{(p)}(\mu^{(p)}_{KI}) = \delta_{II'}\,D^{(p)}, \\ \text{and} \quad &\Sigma_{I\in Q_p}(-1)^{\lambda_p(I)+\lambda_p(K)}\mu^{(p)}_{K'I}\,C^{(p)}(\mu^{(p)}_{KI}) = \delta_{KK'}\,D^{(p)}. \end{aligned}\right\} \quad (52)$$

Theorem 3. The minors complementary to $\mu^{(p)}_{KI}$ in D and in $D^{(p)}$ are related by

$$\frac{C^{(p)}(\mu^{(p)}_{KI})}{C(\mu^{(p)}_{KI})} = \frac{D^{(p)}}{D}(-1)^{\sigma_p(I)+\lambda_p(I)+\sigma_p(K)+\lambda_p(K)} \quad . \qquad (53)$$

Proof. We use the second equation in (50) (the sum by rows) and the first equation in (52) (the sum by columns) to derive the relation:

$$(-1)^{\lambda_p(I)+\lambda_p(K)}DC^{(p)}(\mu^{(p)}_{KI}) = \Sigma_{K'\in Q_p}(-1)^{\lambda_p(I)+\lambda_p(K')}\delta_{KK'}DC^{(p)}(\mu^{(p)}_{K'I})$$

$$= \Sigma_{I',K'\in Q_p}(-1)^{\lambda_p(I)+\lambda_p(K')+\sigma_p(I')+\sigma_p(K)}\mu^{(p)}_{K'I'}\,C(\mu^{(p)}_{KI'})C^{(p)}(\mu^{(p)}_{K'I})$$

$$= \Sigma_{I'\in Q_p}(-1)^{\sigma_p(I')+\sigma_p(K)}\delta_{II'}D^{(p)}C(\mu^{(p)}_{KI'}) = (-1)^{\sigma_p(I)+\sigma_p(K)}D^{(p)}C(\mu^{(p)}_{KI}).$$

From the extreme members of this chain of equalities, the relation (53) follows at once. ‖—

The Sylvester-Franke theorem [4,7] asserts that

$$D^{(p)} = D^q, \qquad (54)$$

where q is defined as in (34). This means that we may replace $D^{(p)}/D$ by D^{q-1} in (42), (43), and (53), for instance.

Consider now the sum by columns (the first equation) in (52). In view of the definition of $D_{IJ}^{(p)}$, it is clear that

$$\Sigma_{K \in Q_p} (-1)^{\lambda_p(I)+\lambda_p(K)} \nu_{KJ}^{(p)} C^{(p)}(\mu_{KI}^{(p)}) = D_{IJ}^{(p)} . \tag{55}$$

Similarly, if D_{IJ} denotes the determinant obtained by replacing the i_s-th column of D by the j_s-th column of $\underset{\sim}{N}$, for $s = 1, 2, \ldots, p$, the sum by columns (the first equation) in the Laplace expansion (48) of D yields the equation

$$\Sigma_{K \in Q_p} (-1)^{\sigma_p(I)+\sigma_p(K)} \nu_{KJ}^{(p)} C(\mu_{KI}^{(p)}) = D_{IJ}. \tag{56}$$

Theorem 4. With the notation defined above,

$$D_{IJ}^{(p)}/D^{(p)} = D_{IJ}/D. \tag{57}$$

Proof. If we apply (53) to the left-hand side of (55), we obtain $D^{(p)}/D$ times the left-hand side of (56). From the corresponding relation of the right-hand sides of (55) and (56), the result (57) follows immediately. ||—

From (41) and (57), we get

$$\eta_{IJ}^{(p)} = D_{IJ}/D; \tag{58}$$

and (42) and (43), by (57), become

$$\det(\underset{\sim}{I} - \underset{\sim}{H}) = \Sigma_{p=0}^{\operatorname{rank}(\underset{\sim}{H})} (-1)^p \; \Sigma_{J \in Q_p} D_{JJ}/D \tag{59}$$

and

$$F = \Sigma_{p=0}^{\operatorname{rank}(\underset{\sim}{H})} (-1)^p \; \Sigma_{J \in Q_p} D_{JJ}. \tag{60}$$

5. The concept of a p-th <u>compound matrix</u> extends to arbitrary rectangular matrices: if $\underset{\sim}{H}$ were defined as an $(m \times n)$ matrix, then $\underset{\sim}{H}^{(p)}$ would be the $(\binom{m}{p} \times \binom{n}{p})$ matrix whose elements are $\eta_{IJ}^{(p)}$, just as in (35), with $I = \{i_1, i_2, \ldots, i_p\}$ $(1 \le i_1 < i_2 < \cdots < i_p \le m)$ a p-index subset of $\{1, 2, \ldots, m\}$, ordered in $\underset{\sim}{H}^{(p)}$ as a row-index by λ_p; and with $J = \{j_1, j_2, \ldots, j_p\}$ $(1 \le j_1 < j_2 < \cdots < j_p \le n)$ a p-index subset of $\{1, 2, \ldots, n\}$, ordered in $\underset{\sim}{H}^{(p)}$ as a column-index, also by λ_p. With this definition, the <u>Binet-Cauchy theorem</u> [4], invoked for the product of square matrices in (38), holds for arbitrary feasible products of rectangular matrices.

Certain important properties of matrices will be required below (see [4,6,7].) First, if $\underset{\sim}{X}$ and $\underset{\sim}{Y}$ are any two matrices, so dimensioned that both products $\underset{\sim}{XY}$ and $\underset{\sim}{YX}$ are feasible, then

$$\operatorname{trace} \underset{\sim}{XY} = \operatorname{trace} \underset{\sim}{YX}. \tag{61}$$

Secondly,

$$(-\underset{\sim}{H})^{(p)} = (-1)^p \underset{\sim}{H}^{(p)}. \tag{62}$$

Thirdly, if the superscript ^{T} denotes the transpose of a matrix, then

$$\text{trace }(c\underset{\sim}{H}^T) = \text{trace }(c\underset{\sim}{H}) = c \text{ trace } \underset{\sim}{H} \tag{63}$$

and

$$(\underset{\sim}{H}^T)^{(p)} = (\underset{\sim}{H}^{(p)})^T. \tag{64}$$

Fourthly, we may define the p-th compound of a matrix to have zero trace and zero determinant if p exceeds the dimensions of the matrix. In fact, it follows directly from the definition of the rank of a matrix that

$$\underset{\sim}{H}^{(p)} = \underset{\sim}{0} \text{ if } p > \text{rank }(\underset{\sim}{H}). \tag{65}$$

We can now derive

Theorem 5. *If $\underset{\sim}{X}$ and $\underset{\sim}{Y}$ are $(m \times n)$ and $(n \times m)$ matrices, respectively; then*

$$\det(\underset{\sim}{I}^{(m)} + \underset{\sim}{X}\underset{\sim}{Y}) = \det(\underset{\sim}{I}^{(n)} + \underset{\sim}{Y}\underset{\sim}{X}), \tag{66}$$

where $I^{(m)}$ and $I^{(n)}$ respectively denote the $(m \times m)$ and the $(n \times n)$ unit matrices.

Proof. By Theorem 1, in the form (36), with $-\underset{\sim}{X}\underset{\sim}{Y}$ for $\underset{\sim}{H}$, and by (62) with (63), we get that, if $u \geq \max(m,n)$,

$$\begin{aligned} \det(\underset{\sim}{I}^{(m)} + \underset{\sim}{X}\underset{\sim}{Y}) &= \Sigma_{p=0}^{m} (-1)^p \text{ trace }(-\underset{\sim}{X}\underset{\sim}{Y})^{(p)} \\ &= \Sigma_{p=0}^{u} \text{ trace }(XY)^{(p)}, \end{aligned} \tag{67}$$

since rank $(\underset{\sim}{X}\underset{\sim}{Y}) \leq m \leq u$. Similarly,

$$\det(\underset{\sim}{I}^{(n)} + \underset{\sim}{Y}\underset{\sim}{X}) = \Sigma_{p=0}^{u} \text{ trace } (\underset{\sim}{Y}\underset{\sim}{X})^{(p)}. \tag{68}$$

Now, by the Binet-Cauchy theorem,

$$(\underset{\sim}{X}\underset{\sim}{Y})^{(p)} = \underset{\sim}{X}^{(p)}\underset{\sim}{Y}^{(p)} \text{ and } (\underset{\sim}{Y}\underset{\sim}{X})^{(p)} = \underset{\sim}{Y}^{(p)}\underset{\sim}{X}^{(p)}. \tag{69}$$

Thus, by (61), the results (67)-(69) combine to yield (66). #

Note. The result (66) follows from an observation due to Noble [8], and the authors are grateful to him for bringing it to their attention. He states the following result as an exercise.

Theorem 6. (B. Noble.) If $\underset{\sim}{A}$, $\underset{\sim}{B}$, $\underset{\sim}{C}$, and $\underset{\sim}{D}$ are respectively $(m \times m)$, $(m \times n)$, $(n \times m)$, and $(n \times n)$ matrices, with $\underset{\sim}{A}$ and $\underset{\sim}{D}$ nonsingular; then

$$\det \underset{\sim}{A} \det(\underset{\sim}{D} + \underset{\sim}{C}\underset{\sim}{A}^{-1}\underset{\sim}{B}) = \det \underset{\sim}{D} \det(\underset{\sim}{A} + \underset{\sim}{B}\underset{\sim}{D}^{-1}\underset{\sim}{C}). \tag{70}$$

Proof. The determinantal identity,

$$\det\left\{\begin{bmatrix} \underset{\sim}{A} & \underset{\sim}{O} \\ \underset{\sim}{C} & \underset{\sim}{D} \end{bmatrix}\begin{bmatrix} \underset{\sim}{A}^{-1} & \underset{\sim}{O} \\ \underset{\sim}{O} & \underset{\sim}{D}^{-1} \end{bmatrix}\begin{bmatrix} \underset{\sim}{A} & \underset{\sim}{B} \\ -\underset{\sim}{C} & \underset{\sim}{D} \end{bmatrix}\right\}$$

$$= \det\left\{\begin{bmatrix} \underset{\sim}{A} & \underset{\sim}{B} \\ -\underset{\sim}{C} & \underset{\sim}{D} \end{bmatrix}\begin{bmatrix} \underset{\sim}{A}^{-1} & \underset{\sim}{O} \\ \underset{\sim}{O} & \underset{\sim}{D}^{-1} \end{bmatrix}\begin{bmatrix} \underset{\sim}{A} & \underset{\sim}{O} \\ \underset{\sim}{C} & \underset{\sim}{D} \end{bmatrix}\right\} \tag{71}$$

(where $\underset{\sim}{O}$ denotes blocks of zero elements of appropriate dimensions), holds because the determinant of a feasible product of square matrices equals the product of the respective determinants, and is therefore independent of the order of the matrices. On multiplying-out the block-matrices in (71), we get that

$$\det \begin{bmatrix} \underset{\sim}{A} & \underset{\sim}{B} \\ \underset{\sim}{O} & (\underset{\sim}{D} + \underset{\sim}{C}\underset{\sim}{A}^{-1}\underset{\sim}{B}) \end{bmatrix} = \det \begin{bmatrix} (\underset{\sim}{A} + \underset{\sim}{B}\underset{\sim}{D}^{-1}\underset{\sim}{C}) & \underset{\sim}{B} \\ \underset{\sim}{O} & \underset{\sim}{D} \end{bmatrix}; \tag{72}$$

and now the theorem follows, by the Laplace expansion of these determinants. $\#$

Our Theorem 5 is now seen to be a particular case of Theorem 6, with $\underset{\sim}{A} = \underset{\sim}{I}^{(m)}$, $\underset{\sim}{B} = \underset{\sim}{Y}$, $\underset{\sim}{C} = \underset{\sim}{X}$, and $\underset{\sim}{D} = \underset{\sim}{I}^{(n)}$.

6. We now return to the example presented in §2. Let

$$\mathcal{M} = \mathcal{N}^m; \tag{73}$$

so that, if $\underset{\sim}{\varphi}, \underset{\sim}{\psi} \in \mathcal{M}$, their inner product is given by

$$(\underset{\sim}{\varphi}, \underset{\sim}{\psi}) = \Sigma_{s=1}^{m} \int_R \varphi_s(y) \; \psi_s(y)^* \, dy, \tag{74}$$

where the asterisk * denotes the conjugate complex quantity. Thus $\mathcal{M}$ is the direct sum of m replicas $\mathcal{N}_s$ of $\mathcal{N}$:

$$\mathcal{M} = \sum_{s=1}^{m} \oplus \mathcal{N}_s. \tag{75}$$

We may now define the space $\mathcal{V}$, bearing the same relation to $\mathcal{M}$ as $\mathcal{A}$ does to $\mathcal{N}$: if $\underset{\sim}{\omega} \in \mathcal{V}$, it will have m components $\omega_r \in \mathcal{A}$, with $r = 1, 2, \ldots, m$, with values denoted by $\omega_{\xi r}(x)$; and $\underset{\sim}{\omega}$ itself will sometimes be written in the form $\underset{\sim}{\omega}_\xi$, to emphasize the dependence on the parameter ξ.

We can now define vectors and matrices whose elements are in $\mathcal{V}$, and linear operators mapping $\mathcal{V}$ into $\mathcal{V}$. If $\underset{\sim}{T}$ denotes such an operator (also written $\underset{\sim}{T}_\xi$), it will have a kernel function $T_{\xi rs}(x,y)$, such that

$$(\underset{\sim}{T}_\xi \underset{\sim}{\omega})_r(x) = \Sigma_{s=1}^{m} \int_R T_{\xi rs}(x,y)\,\omega_{\xi s}(y)\,dy, \tag{76}$$

for all $\underset{\sim}{\omega} = \underset{\sim}{\omega}_\xi \in \mathcal{V}$. We shall also define a simple linear operator Γ_ξ mapping linear operators on $\mathcal{V}$ to $\mathcal{V}$ into other such operators, by

$$(\Gamma_\xi \underset{\sim}{T})_{rs}(x,y) = \frac{T_{\xi rs}(x,y)}{x + i\xi - y}, \tag{77}$$

for all linear operators $\underset{\sim}{T}$. Related to this is the linear functional g_ξ, defined by

$$[g_\xi(\varphi)](x) = \int_R \frac{\varphi(y)\,dy}{x + i\xi - y} \quad [g_\xi(\varphi) \in \mathcal{A}], \tag{78}$$

for all $\varphi \in \mathcal{N}$, and more generally by

$$[g_\xi(\underset{\sim}{\Lambda})]_{ij}(x) = \int_R \frac{\lambda_{ij}(y)\,dy}{x + i\xi - y} = [g_\xi(\lambda_{ij})](x), \tag{79}$$

where $\underset{\sim}{\Lambda}$ is any matrix with elements $\lambda_{ij} \in \mathscr{N}$. We note, by the Plemelj formula (14), that

$$\left.\begin{aligned} \{\Delta[g_\xi(\varphi)]\}(x) &= -2\pi i\, \varphi(x) \\ \text{and} \qquad \{\Delta[g_\xi(\underset{\sim}{\Lambda})]\}_{ij}(x) &= -2\pi i\, \lambda_{ij}(x); \end{aligned}\right\} \tag{80}$$

or

$$\Delta[g_\xi(\varphi)] = -2\pi i\, \varphi \tag{81}$$

and

$$\Delta[g_\xi(\underset{\sim}{\Lambda})] = -2\pi i\, \underset{\sim}{\Lambda}\,; \tag{82}$$

or, more abstractly,

$$\Delta g_\xi = -2\pi i. \tag{83}$$

With these preliminaries, let us define the $(n \times n)$ matrix $\underset{\sim}{K}$ with elements $(\underset{\sim}{K})_{ij} = \kappa_{ij} \in \mathscr{A}$, defined by

$$\kappa_{\xi ij}(x) = \delta_{ij} - [g_\xi(\underset{\sim}{A}\underset{\sim}{B}^\dagger)]_{ij} = \delta_{ij} - \sum_{r=1}^{m} \int_R \frac{\alpha_{ir}(y)\beta_{jr}^*(y)\,dy}{x + i\xi - y}\,; \tag{84}$$

where $\dagger$ denotes the Hermitian conjugate-transpose, $\underset{\sim}{B}^\dagger = (\underset{\sim}{B}^*)^T$ or $(\underset{\sim}{B}^\dagger)_{rj} = \beta_{jr}^*$, and $\underset{\sim}{A}$ and $\underset{\sim}{B}$ are $(n \times m)$ matrices with elements

$$(\underset{\sim}{A})_{ir} = \alpha_{ir} \in \mathscr{N} \text{ and } (\underset{\sim}{B})_{js} = \beta_{js} \in \mathscr{N}. \tag{85}$$

Let $\underset{\sim}{\Phi}$ be an $(n \times n)$ diagonal matrix with elements

$$(\underset{\sim}{\Phi})_{ij} = \delta_{ij}\, \varphi_j \quad (\varphi_j \in \mathscr{N}), \tag{86}$$

and write

$$\underset{\sim}{M} = \underset{\sim}{K}\,\underset{\sim}{\Phi} \tag{87}$$

and

$$\underset{\sim}{N} = -\underset{\sim}{A}\, g_\xi(\underset{\sim}{B}^\dagger \underset{\sim}{\Phi}). \tag{88}$$

Then the formulae (81) - (83) give

$$[\Delta(\mu_{\xi ij})](x) = 2\pi i\ \Sigma_{t=1}^{m}\ \alpha_{it}(x)\beta_{jt}(x)\overset{*}{\varphi}_j(x) = [\Delta(\nu_{\xi ij})](x),$$

in accordance with (19).

Let us suppose that, for all sufficiently small values of ξ and for almost all values of x in R, the matrix $\underset{\sim}{K}$ defined in (84) is non-singular. Then the equation

$$\underset{\sim}{K}\underset{\sim}{\Theta} = \underset{\sim}{\Theta} - g_\xi(\underset{\sim}{A}\underset{\sim}{B}^\dagger)\underset{\sim}{\Theta} = \underset{\sim}{A} \tag{89}$$

has a unique solution $\underset{\sim}{\Theta} = \underset{\sim}{K}^{-1}\underset{\sim}{A}$, for almost all choices of $x \in R$. $\underset{\sim}{\Theta}$ will have elements $\theta_{\xi ir} \in \mathcal{A}$, with $i = 1, 2, \ldots, n$ and $r = 1, 2, \ldots, m$. Let us further suppose that the matrix $\underset{\sim}{\Phi}$ is also non-singular (that is, $\varphi_1(x)\varphi_2(x)\cdots\varphi_n(x) \neq 0$) for almost all values of $x \in R$. Then we may define the $(n \times n)$ matrix

$$\underset{\sim}{H} = -\underset{\sim}{\Phi}^{-1}\underset{\sim}{\Theta}\, g_\xi(\underset{\sim}{B}^\dagger\underset{\sim}{\Phi}), \tag{90}$$

for almost all $x \in R$ and all sufficiently small values of ξ.

Now, by (87) - (90),

$$\underset{\sim}{M}\,\underset{\sim}{H} = \underset{\sim}{K}\underset{\sim}{\Phi}\underset{\sim}{H} = -\underset{\sim}{K}\underset{\sim}{\Theta}\, g_\xi(\underset{\sim}{B}^\dagger\underset{\sim}{\Phi}) = -\underset{\sim}{A}\, g_\xi(\underset{\sim}{B}^\dagger\underset{\sim}{\Phi}) = \underset{\sim}{N};$$

that is, we have

<u>Theorem 7.</u> <u>If</u> $\underset{\sim}{\Phi}$ <u>is non-singular for almost all</u> $x \in R$ <u>and if</u> $\underset{\sim}{K}$ (defined in (84)) <u>is non-singular for almost all</u> $x \in R$ <u>and for all</u>

sufficiently small ξ; then $\underset{\sim}{H}$ (defined by (89) and (90)) is the solution of (23), for $\underset{\sim}{M}$ and $\underset{\sim}{N}$ defined in (87) and (88).

We now return to $\mathcal{M}$ and $\mathcal{N}$, and define operators $\underset{\sim}{V}$ and $\underset{\sim}{G}_\xi$ by

$$\underset{\sim}{V}(x,y) = \underset{\sim}{A}(x)^T \underset{\sim}{B}(y)^* \text{ or } V_{rs}(x,y) = \Sigma_{i=1}^n \alpha_{ir}(x)\,\beta_{is}(y)^* \tag{91}$$

and

$$\underset{\sim}{G}_\xi(x,y) = \underset{\sim}{\Theta}_\xi(x)^T \underset{\sim}{B}(y)^* \text{ or } G_{\xi rs}(x,y) = \Sigma_{i=1}^n \theta_{\xi ir}(x)\,\beta_{is}(y)^*. \tag{92}$$

Then (89) yields

$$\underset{\sim}{B}(y)^\dagger \underset{\sim}{\Theta}_\xi(x) - \underset{\sim}{B}(y)^\dagger [g_\xi(\underset{\sim}{A}\underset{\sim}{B}^\dagger)](x)\underset{\sim}{\Theta}_\xi(x) = \underset{\sim}{B}(y)^\dagger \underset{\sim}{A}(x);$$

or, by (79), (91), and (92),

$$G_{\xi rs}(x,y) - \Sigma_{i=1}^n \Sigma_{t=1}^m \Sigma_{j=1}^n \beta_{is}(y) \int_R \frac{\alpha_{it}(u)\beta_{jt}(u)^*}{x + i\xi - u}\, du\, \theta_{\xi jr}(x)$$

$$= V_{rs}(x,y)\ ,$$

which reduces, by (76), (77), (91), and (92), to the equation

$$G_{\xi rs}(x,y) - \Sigma_{t=1}^m \int_R (\Gamma_\xi \underset{\sim}{G}_\xi)_{rt}(x,u)\, V_{ts}(u,y)\, du = V_{rs}(x,y);$$

that is, we get

Theorem 8. With $\underset{\sim}{V}$ defined as in (91), the operator equation

$$\underset{\sim}{G}_\xi = (1 + \Gamma_\xi \underset{\sim}{G}_\xi)\underset{\sim}{V} \tag{93}$$

has the solution (92) (in terms of $\underset{\sim}{\Theta}$ defined by (84) and (89).)

The equation (93) is called Friedrichs' equation [2,3].

We note from (89) and (92) that G_ξ is not dependent on Φ.

Let us choose for Φ the matrix

$$\Psi = \psi I^{(n)}, \tag{94}$$

and write similarly

$$\Lambda = \psi I^{(m)}; \tag{95}$$

where $I^{(n)}$ and $I^{(m)}$ are respectively $(n \times n)$ and $(m \times m)$ unit matrices, and we have taken all the $\varphi_i = \psi$. Following the definition (76) we shall write

$$(T_\xi M)_{rs}(x) = \Sigma_{t=1}^{m} \int_R T_{\xi rt}(x,y)\, u_{\xi ts}(y)\, dy. \tag{96}$$

Theorem 9. With the notation defined above, if F is defined as in (20), with M and N defined as in (87) and (88), and Φ takes the value Ψ; for any A, B, and ψ; then

$$F = (\det K)\, \psi^{n-m} \det[(I^{(m)} + \Gamma_\xi G_\xi)\Lambda]. \tag{97}$$

or

$$F = (\det K)\, \psi^{n}\, \Sigma_{p=0}^{m} \text{ trace } (\Lambda^{-1}\Gamma_\xi G_\xi \Lambda)^{(p)}, \tag{98}$$

Proof. By (16), (24), (87), (88), (90), and (94), with Theorem 5,

$$\begin{aligned} F &= \det(K\Psi) \det[I^{(n)} + \Psi^{-1}\Theta\, g_\xi(B^\dagger\Psi)] \\ &= (\det K)(\det \Psi) \det[I^{(m)} + g_\xi(B^\dagger\Psi)\Psi^{-1}\Theta]. \end{aligned} \tag{99}$$

Now, by (77), (79), (92), and (95),

$$[g_{\xi}(\underset{\sim}{B}^{\dagger}\underset{\sim}{\psi})\underset{\sim}{\psi}^{-1}\underset{\sim}{\Theta}]_{sr}(x) = \Sigma_{i=1}^{n}\int_R \frac{\beta_{is}(y)^{*}\psi(y)}{x + i\xi - y}\, dy\ [\psi(x)]^{-1}\theta_{\xi ir}(x)$$

$$= \int_R (\Gamma_{\xi}\underset{\sim}{G}_{\xi})_{rs}(x,y)\psi(y)\, dy\, [\psi(x)]^{-1}$$

$$= (\underset{\sim}{\Lambda}^{-1}\Gamma_{\xi}\underset{\sim}{G}_{\xi}\underset{\sim}{\Lambda})_{rs}(x). \tag{100}$$

Hence, by (99) and (100), and since the determinant of a matrix equals the determinant of its transpose,

$$F = (\det \underset{\sim}{K})\ (\det \underset{\sim}{\psi})\ \det\,[\underset{\sim}{\Lambda}^{-1}(\underset{\sim}{I}^{(m)} + \Gamma_{\xi}\underset{\sim}{G}_{\xi})\,\underset{\sim}{\Lambda}\,]$$

$$= (\det \underset{\sim}{K})\ (\det \underset{\sim}{\psi})(\det \underset{\sim}{\Lambda}^{-1})\ \det[(\underset{\sim}{I}^{(m)} + \Gamma_{\xi}\underset{\sim}{G}_{\xi})\underset{\sim}{\Lambda}]. \tag{101}$$

Since, by (94) and (95),

$$\det \underset{\sim}{\psi} = \psi^{n} \quad \text{and} \quad \det \underset{\sim}{\Lambda} = \psi^{m}, \tag{102}$$

it follows from (101) that (97) holds.

Now, by the formula (36), with $\underset{\sim}{H}$ replaced by $-\underset{\sim}{\Lambda}^{-1}\Gamma_{\xi}\underset{\sim}{G}_{\xi}\underset{\sim}{\Lambda}$, and using (62) and (63), we see that

$$\det[\underset{\sim}{\Lambda}^{-1}(\underset{\sim}{I}^{(m)} + \Gamma_{\xi}\underset{\sim}{G}_{\xi})\underset{\sim}{\Lambda}] = \Sigma_{p=0}^{\mathrm{rank}(\underset{\sim}{\Lambda}^{-1}\Gamma_{\xi}\underset{\sim}{G}_{\xi}\underset{\sim}{\Lambda})}\ \mathrm{trace}(\underset{\sim}{\Lambda}^{-1}\Gamma_{\xi}\underset{\sim}{G}_{\xi}\underset{\sim}{\Lambda})^{(p)}; \tag{103}$$

whence (98) is obtained, when we note that

$$\mathrm{rank}\ (\underset{\sim}{\Lambda}^{-1}\Gamma_{\xi}\underset{\sim}{G}_{\xi}\underset{\sim}{\Lambda}) = \mathrm{rank}\ (\Gamma_{\xi}\underset{\sim}{G}_{\xi}\underset{\sim}{\Lambda}) \leq m, \tag{104}$$

since the matrix is $(m \times m)$. ‡

Theorem 10. With the same notation as in Theorem 9,

$$(\det \underset{\sim}{K})^{+} \det[(\underset{\sim}{I}^{(m)} + \Gamma^{+}\underset{\sim}{G}^{+})\underset{\sim}{\Lambda}] = (\det \underset{\sim}{K})^{-} \det[(\underset{\sim}{I}^{(m)} + \Gamma^{-}\underset{\sim}{G}^{-})\underset{\sim}{\Lambda}]. \quad (105)$$

Further, if $m = 1$, we have the operator equation

$$(\det \underset{\sim}{K})^{+} (1 + \Gamma^{+}G^{+}) = (\det \underset{\sim}{K})^{-} (1 + \Gamma^{-}G^{-}). \quad (106)$$

Proof. We simply apply equations (4) and (21) to (97), to get (105). If $m = 1$, the equation simplifies to

$$(\det \underset{\sim}{K})^{+} (1 + \Gamma^{+}G^{+})\psi = (\det \underset{\sim}{K})^{-} (1 + \Gamma^{-}G^{-})\psi, \quad (107)$$

where G_{ξ} is now a scalar operator. Since ψ is an arbitrary function, this yields the operator equation (106). ‡

Note. The particular case represented by (106) was proved, independently and by a different argument, by Carey [1].

7. We now turn to another question associated with the example examined in §§2 and 6. With the notation of §6 and the assumption that $\underset{\sim}{K}_\xi(x)$ is invertible, we seek an $(\ell \times m)$ matrix $\underset{\sim}{Z}_\xi(x)$, with elements

$$(\underset{\sim}{Z})_{ur} = \zeta_{ur} \in \mathcal{A}, \tag{108}$$

such that, for <u>all</u> m-vectors $\underset{\sim}{\varphi}(x)$, with elements $\varphi_r \in \mathcal{N}$,

$$\Delta\{\underset{\sim}{Z}_\xi(1 + \Gamma_\xi \underset{\sim}{G}_\xi)\underset{\sim}{\varphi}\} = 0 , \tag{109}$$

that is

$$\Delta\{\underset{\sim}{Z}_\xi(x)\ \underset{\sim}{\varphi}(x) + \underset{\sim}{Z}_\xi(x) \int_R \frac{\underset{\sim}{G}_\xi(x,y)\underset{\sim}{\varphi}(y)}{x + i\xi - y}\, dy\} = 0 . \tag{110}$$

By (89) and (92),

$$\underset{\sim}{G}_\xi(x,y) = \underset{\sim}{A}(x)^T \underset{\sim}{K}_\xi(x)^W \underset{\sim}{B}(y)^* , \tag{111}$$

where ^{W} denotes the transposed inverse, $\underset{\sim}{K}^W = (K^{-1})^T$; so (110) becomes

$$\Delta\{\underset{\sim}{Z}_\xi(x)\underset{\sim}{\varphi}(x) + \underset{\sim}{Z}_\xi(x)\underset{\sim}{A}(x)^T \underset{\sim}{K}_\xi(x)^W \int_R \frac{\underset{\sim}{B}(y)^*\underset{\sim}{\varphi}(y)}{x + i\xi - y}\, dy\} = 0. \tag{112}$$

Suppose now that $\underset{\sim}{A}(x)^* \underset{\sim}{A}(x)^T$ is invertible almost everywhere in R, and consider

$$\underset{\sim}{Z}_\xi(x) = \underset{\sim}{P}(x)\underset{\sim}{K}_\xi(x)^T [\underset{\sim}{A}(x)^* \underset{\sim}{A}(x)^T]^{-1} \underset{\sim}{A}(x)^*, \tag{113}$$

for some $(\ell \times n)$ matrix $P(x)$ with elements $(P)_{ui} = \rho_{ui} \in \mathcal{N}$. Then (112) reduces, by (81) - (84), and since $K^T K^W = I^{(n)}$, to

$$0 = \wedge \{ P(x) K_\xi(x)^T [A(x)^* A(x)^T]^{-1} A(x)^* \varphi(x)$$

$$+ P(x) K_\xi(x)^T [A(x)^* A(x)^T]^{-1} A(x)^* A(x)^T K_\xi(x)^W \int_R \frac{B(y)^* \varphi(y)}{x + i\xi - y} dy \}$$

$$= 2\pi i \; P(x) B(x)^* A(x)^T [A(x)^* A(x)^T]^{-1} \; A(x)^* \varphi(x)$$

$$- 2\pi i \; P(x) \; B(x)^* \varphi(x)$$

$$= 2\pi i \; P(x) B(x)^* \{ A(x)^T [A(x)^* A(x)^T]^{-1} A(x)^* - I^{(m)} \} \varphi(x). \quad (114)$$

Thus (114) will be satisfied, for all $\varphi(x)$, if $B(x)^T B(x)^*$ is invertible almost everywhere in R and if we choose

$$P(x) = A(x)^* [B(x)^T B(x)^*]^{-1} \; B(x)^T; \quad (115)$$

and so, by (113),

$$Z_\xi(x) = A(x)^* [B(x)^T B(x)^*]^{-1} \; B(x)^T K_\xi(x)^T [A(x)^* A(x)^T]^{-1} A(x)^*. \quad (116)$$

However, if both $A^* A^T$ and $B^T B^*$ are to be invertible, necessary and sufficient conditions are that both A and B be of full rank; so that both $n \leq m$ and $m \geq n$, that is, $m = n$. But then A and B will be invertible square matrices, and a simpler solution suffices, since (114) holds exactly: we may take $P = I^{(n)}$

and so

$$\underset{\sim}{Z}_\xi(x) = \underset{\sim}{K}_\xi(x)^T \underset{\sim}{A}(x)^W . \tag{117}$$

Note that the invertibility of $\underset{\sim}{B}$ is no longer required, here. Thus we obtain

Theorem 11. *Sufficient conditions, for the existence of a matrix $\underset{\sim}{Z}_\xi(x)$ satisfying (109) or (110), are that $m = n$ and that $\underset{\sim}{A}(x)$ and $\underset{\sim}{K}_\xi(x)$ be invertible for almost all x in R and all sufficiently small ξ. Then (117) provides the solution.*

In this case,

$$\det \underset{\sim}{Z}_\xi(x) = \det \underset{\sim}{K}_\xi(x) / \det \underset{\sim}{A}(x). \tag{118}$$

We have thus found solutions to our problem when $m = 1$ (Theorem 10) and when $m = n$ (Theorem 11). One more case readily yields a solution: when $n = 1$. In that case, $\underset{\sim}{A}$ and $\underset{\sim}{B}$ are respectively, the m-dimensional row-vectors $\underset{\sim}{\alpha}^T$ and $\underset{\sim}{\beta}^T$ with elements

$$(\underset{\sim}{A})_{1r} = (\underset{\sim}{\alpha})_r = \alpha_r \in \mathcal{N} \text{ and } (\underset{\sim}{B})_{1r} = (\underset{\sim}{\beta})_r = \beta_r \in \mathcal{N}, \tag{119}$$

and $\underset{\sim}{K}_\xi(x)$ is a scalar,

$$\underset{\sim}{K}_\xi(x) = \kappa_\xi(x) = 1 - \int_R \frac{\underset{\sim}{\alpha}(y)^T \underset{\sim}{\beta}(y)^*}{x + i\xi - y}\, dy; \tag{120}$$

and, further, we note that the scalar quantity

$$\underset{\sim}{\alpha}(y)^T \underset{\sim}{\beta}(y)^* = \sum_{r=1}^{m} \alpha_r(y)\, \beta_r(y)^* = \underset{\sim}{\beta}^\dagger \underset{\sim}{\alpha} . \tag{121}$$

We now get

<u>Theorem 12.</u> <u>When</u> $n = 1$, <u>a solution of</u> (109), <u>for</u> $\kappa_\xi(x)$ <u>non-zero almost everywhere in</u> R <u>for all sufficiently small</u> ξ, <u>is given by</u>

$$\underset{\sim}{Z}_\xi(x) = \kappa_\xi(x)\, \underset{\sim}{\beta}^\dagger(x). \tag{122}$$

<u>Proof.</u> Substituting (122) in the left-hand side of (112), for the case of $n = 1$, yields

$$\Delta \{\kappa_\xi(x) \underset{\sim}{\beta}^\dagger(x) \underset{\sim}{\varphi}(x) + \underset{\sim}{\beta}^\dagger(x)\, \underset{\sim}{\alpha}(x) \int_R \frac{\underset{\sim}{\beta}(y)^\dagger \underset{\sim}{\varphi}(y)}{x + i\xi - y}\, dy\}$$

$$= 2\pi i\, \underset{\sim}{\alpha}(x)^T \underset{\sim}{\beta}(x)^* \underset{\sim}{\beta}^\dagger(x) \underset{\sim}{\varphi}(x) - 2\pi i\, \underset{\sim}{\beta}^\dagger(x) \underset{\sim}{\alpha}(x)\, \underset{\sim}{\beta}^\dagger(x) \underset{\sim}{\varphi}(x)$$

$$= 2\pi i\, [\underset{\sim}{\alpha}(x)^T \underset{\sim}{\beta}(x)^* - \underset{\sim}{\beta}^\dagger(x) \underset{\sim}{\alpha}(x)]\, \underset{\sim}{\beta}^\dagger(x)\, \underset{\sim}{\varphi}(x) = 0 ,$$

where we have used the commutativity of scalar multiplication and the identity (121). ╫

In this case, we have, by (91), that

$$\underset{\sim}{V}(x,y) = \underset{\sim}{A}(x)^T \underset{\sim}{B}(y)^* = \underset{\sim}{\alpha}(x)\, \underset{\sim}{\beta}(y)^\dagger . \tag{123}$$

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*Computer Sciences Department, Univ. of Wis., Madison, Wis.
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<tr><td colspan="4">16. Abstracts
In this preliminary report on an investigation of the Friedrichs equation in operator theory, certain algebraic relations, connecting the Friedrichs intertwining operators and perturbation determinants, are presented together with some generalizations.
These relations underlie the construction of eigenfunction expansions occurring in the theory of non-self-adjoint perturbations of operators with continuous spectra.</td></tr>
<tr><td colspan="4">17. Key Words and Document Analysis. 17a. Descriptors
algebra, matrix, determinant, identity, Friedrichs' equation, operator theory, perturbation, eigenfunction expansion, continuous spectra, non-self-adjoint.

17b. Identifiers/Open-Ended Terms

17c. COSATI Field/Group</td></tr>
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