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NATURAL CONVECTION IN AN ENCLOSURE WITH LOCALIZED HEATING FROM BELOW

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1. Introduction

The flow of a gas or of a liquid in a closed cavity is of widespread interest in fluid mechanics (see, e.g., references [1] - [4], [7], [8], and the references contained therein). In this paper we will develop a fast, economical numerical method for describing the motion of a fluid in a circular cylinder with a heat source located centrally on the bottom. The method is a finite difference boundary value technique which will be developed for steady state problems and then adapted to a special class of nonsteady state problems. The entire development extends in a natural way to any problem with an arbitrary axially-symmetric container. Convergence for steady state problems has been achieved for Prandtl numbers in the range 0.023 to 9884.0 and for Grashof numbers in the range 10 to $4(10)^6$. As applied to time dependent problems, the method is free of roundoff error amplification, which is inherent in all step-ahead methods.

2. Problem Statement

Consider a fluid in a circular cylinder of unit height and of radius b, and assume that a circular heat source of radius c is centrally located in the base.

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Since any resulting convective flow must be axially symmetric, the motion can be studied as a two dimensional problem. In particular, let us take the mathematical formulation of Torrance and Rockett [8], which, in their terminology, can be summarized as follows.

As shown in Figure 1, let OBCD be a rectangle with OB = b, OD = 1. Let A be between O and B with OA = c. The problem is to find three functions ψ, θ and Ω which, inside OBCD, satisfy the partial differential equations

$$(1) \quad \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial R^2} - \frac{1}{R}\frac{\partial \psi}{\partial R} = -R\Omega$$

$$(2) \quad \frac{\partial \theta}{\partial \tau} - \frac{\partial^2 \theta}{\partial x^2} - \frac{1}{R}\frac{\partial \theta}{\partial R} - \frac{\partial^2 \theta}{\partial R^2} + \frac{1}{R}\left(\frac{\partial \psi}{\partial R}\frac{\partial \theta}{\partial x} - \frac{\partial \psi}{\partial x}\frac{\partial \theta}{\partial R}\right) = 0$$

$$(3) \quad \frac{\partial \Omega}{\partial \tau} + \frac{1}{R^2}\frac{\partial \psi}{\partial x}\Omega + \frac{1}{R}\left[\frac{\partial \psi}{\partial R}\frac{\partial \Omega}{\partial x} - \frac{\partial \psi}{\partial x}\frac{\partial \Omega}{\partial R}\right] + G P^2 \frac{\partial \theta}{\partial R}$$

$$- P\left[\frac{\partial^2 \Omega}{\partial x^2} - \frac{1}{R^2}\Omega + \frac{1}{R}\frac{\partial \Omega}{\partial R} + \frac{\partial^2 \Omega}{\partial R^2}\right] = 0,$$

and, in addition, satisty the initial conditions

$$(4) \quad \psi = \theta = \Omega = 0; \quad \tau < 0, \ 0 \le x \le 1, \ 0 \le R \le b$$

and, for $\tau \ge 0$, the boundary conditions

$$(5) \quad \psi = \frac{\partial \psi}{\partial x} = 0; \quad 0 \le R \le b, \ x = 0$$

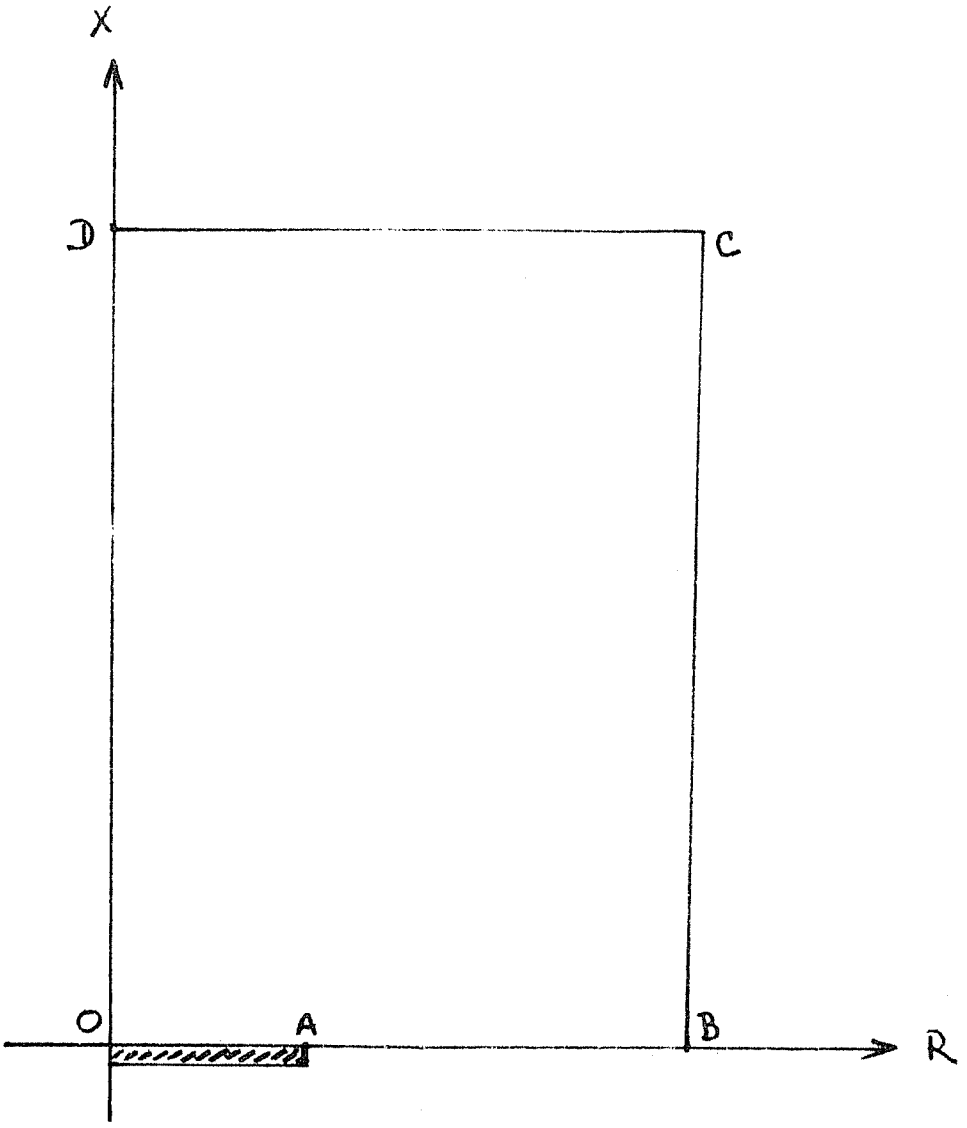


FIGURE 1

$$(6)\qquad \begin{cases} \theta = 1 \;\; ; & 0 \le R \le c,\ x = 0 \\ \theta = 0.5 \; ; & R = c \quad ,\ x = 0 \\ \theta = 0 \;\; ; & c < R \le b,\ x = 0 \end{cases}$$

$$(7)\qquad \psi = \frac{\partial \psi}{\partial x} = \theta = 0;\ \ 0 \le R \le b,\ x = 1$$

$$(8)\qquad \psi = \Omega = \frac{\partial \theta}{\partial R} = 0\ ;\ \ R = 0,\ 0 \le x \le 1$$

$$(9)\qquad \psi = \frac{\partial \psi}{\partial R} = \theta = 0\ ;\ \ R = b,\ 0 \le x \le 1.$$

In (1) - (9), ψ is the stream function, θ is the temperature, Ω is the vorticity, P is the Prandtl number, and G is the Grashof number.

3. Difference Formulas

Let us develop, first, the basic difference formulas which will be needed for our numerical method. Throughout, it will be assumed that h and $\Delta\tau$ are positive space and time grid constants, respectively, and that the points (R,x,τ), $(R+h,x,\tau)$, $(R,x+h,\tau)$, $(R-h,x,\tau)$, $(R,x-h,\tau)$, $(R,x,\tau+\Delta\tau)$, $(R,x,\tau-\Delta\tau)$ are represented by $0,1,2,3,4,5,6$, respectively. Then, in the usual way [4], the difference approximation for differential equation (1) is taken to be

$$(10)\qquad \frac{-4\psi_0 + \psi_1 + \psi_2 + \psi_3 + \psi_4}{h^2} - \frac{1}{R}\,\frac{\psi_1 - \psi_3}{2h} = -R\Omega_0 .$$

In order to preserve diagonal dominance, equation (2) is differenced in the following special manner ([5], [7]):

$$(11)\quad \frac{-3\theta_6 + 4\theta_0 - \theta_5}{2\Delta\tau} - \frac{-4\theta_0 + \theta_1 + \theta_2 + \theta_3 + \theta_4}{h^2} - \frac{1}{R}\,\frac{\theta_1 - \theta_3}{2h}$$

$$+ \frac{1}{R}\left[\frac{\psi_1 - \psi_3}{2h} F - \frac{\psi_2 - \psi_4}{2h} E\right] = 0,$$

where

$$(12)\quad \begin{cases} F = \dfrac{\theta_0 - \theta_4}{h}, & \text{if } \psi_1 - \psi_3 \geq 0 \\ F = \dfrac{\theta_2 - \theta_0}{h}, & \text{if } \psi_1 - \psi_3 < 0 \\ E = \dfrac{\theta_1 - \theta_0}{h}, & \text{if } \psi_2 - \psi_4 \geq 0 \\ E = \dfrac{\theta_0 - \theta_3}{h}, & \text{if } \psi_2 - \psi_\psi < 0. \end{cases}$$

In a similar manner, equation (3) is approximated by

$$(13)\quad \frac{-3\Omega_6 + 4\Omega_0 - \Omega_5}{2\Delta t} + \frac{1}{R^2}\,\frac{\psi_2 - \psi_4}{2h}\,\Omega_0 + \frac{1}{R}\left[\frac{\psi_1 - \psi_3}{2h} F - \frac{\psi_2 - \psi_4}{2h} E\right]$$

$$+ GP^2\,\frac{\theta_1 - \theta_3}{2h} - P\left[\frac{-4\Omega_0 + \Omega_1 + \Omega_2 + \Omega_3 + \Omega_4}{h^2}\right.$$

$$\left. - \frac{1}{R^2}\,\Omega_0 + \frac{1}{R}\,\frac{\Omega_1 - \Omega_3}{2h}\right] = 0,$$

where F and E are defined as in (12), but with θ replaced by Ω.

Finally, in order to generate Ω at grid points of either OB, BC, or CD, at each such point we will use

$$\Omega = \frac{-2}{h^2 R}\,\tilde{\psi}, \tag{14}$$

where $\tilde{\psi}$ is the value of the stream function at the interior grid point nearest to the given boundary grid point ([4], [7]).

4. Numerical Solution of The Steady State Problem.

When steady state is physically realizable, it can be described mathematically by (1) - (3) and (5) - (9) merely by dropping the time derivatives in (2) and (3). We will show how to approximate the solution of such a problem first.

For positive grid size h, construct and number, in the usual way, the set of interior grid points I_h and the set of boundary grid points S_h. (Though not essential, it will be assumed, for simplicity, that $\frac{l}{h}$, $\frac{b}{h}$ and $\frac{c}{h}$ are integers.) The problem then is to approximate ψ, ω and Ω on those points of $R_h + S_h$ at which they are not already given by (5) - (9).

The procedure is to guess $\psi^{(0)}$, $\omega^{(0)}$ and $\Omega^{(0)}$ on these point sets and to proceed recursively from each $\psi^{(k)}$, $\omega^{(k)}$, $\Omega^{(k)}$ to $\psi^{(k+1)}$, $\omega^{(k+1)}$, $\Omega^{(k+1)}$ as follows. At each point of I_h which is further than h from all points of S_h, write down (10) with $\Omega_0 = \Omega_0^{(k)}$. At each point (R,x) of I_h which is

exactly a distance h from at least one point (R^*, x^*) of S_h, write down

$$-4\psi_i + \psi_j = 0, \tag{15}$$

where ψ_i is the value of ψ at (R,x), while ψ_j is the value of ψ at the point (R',x') which is $2h$ from (R^*,x^*) and is collineal with (R^*,x^*) and (R,x). Inserting the known ψ boundary values from (7) - (9), one then solves the resulting linear algebraic system by successive over-relaxation and denotes the solution by $\bar{\psi}^{(k+1)}$. Then $\psi^{(k+1)}$ is defined by

$$\psi^{(k+1)} = \rho\psi^{(k)} + (1-\rho)\bar{\psi}^{(k+1)}, \tag{16}$$

where ρ is a smoothing parameter in the range $0 \le \rho \le 1$.

To obtain the iterate $\vartheta^{(k+1)}$, apply (11) without the time dependent term at each interior point of I_h and with $\psi_i = \psi_i^{(k+1)}$, $i = 1,2,3,4$. Also, at each point $(0,x)$ of S_h, write down

$$\vartheta(0,x) = \vartheta(h,x). \tag{17}$$

Into the system generated by (11) and (17), insert the known values of ϑ given by (6), (7) and (9). Solve the resulting system by successive over-relaxation and denote the solution by $\bar{\vartheta}^{(k+1)}$. Then, $\vartheta^{(k+1)}$ is defined by

$$\vartheta^{(k+1)} = \mu\vartheta^{(k)} + (1-\mu)\bar{\vartheta}^{(k+1)}, \quad 0 \le \mu \le 1. \tag{18}$$

To obtain the iterate $\Omega^{(k+1)}$, first generate Ω at points of S_h by (14), where the values of $\psi^{(k+1)}$ are used for $\bar{\psi}$. Equation (13), without the time

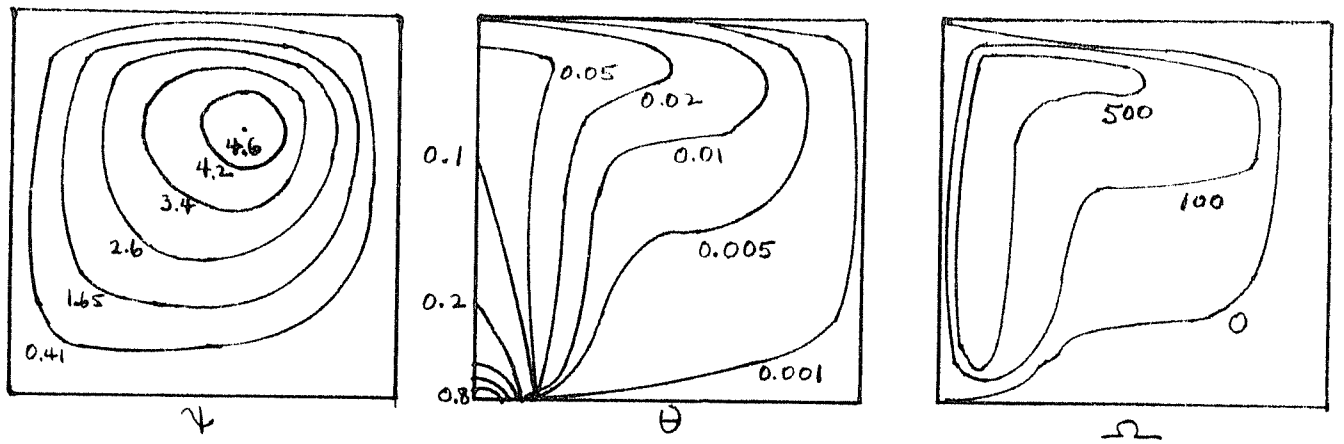
dependent term, and with $\psi_i = \psi_i^{(k+1)}$, $i = 0,1,2,3,4$, is then applied at each point of I_h and the resulting system is solved by successive over-relaxation. If the solution of this system and the values generated by (14) are denoted by $\bar{\Omega}^{(k+1)}$, then $\Omega^{(k+1)}$ is defined by

$$\Omega^{(k+1)} = b\Omega^{(k)} + (1 - b)\bar{\Omega}^{(k+1)}, \quad 0 < b < 1. \tag{19}$$

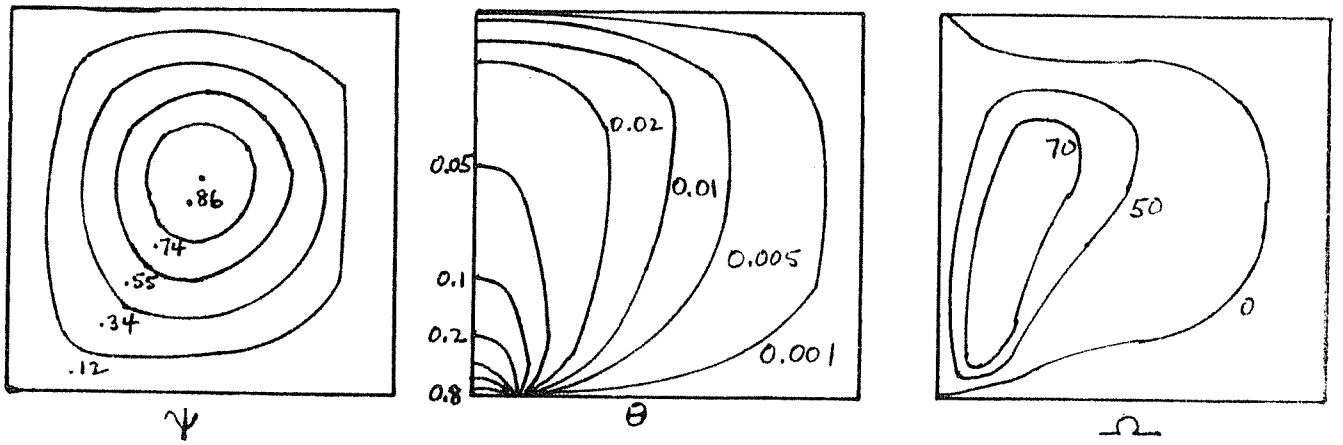
Finally, the iteration is terminated when, for a given positive tolerance ε, one has uniformly on $R_h + S_h$

$$|\psi^{(k+1)} - \psi^{(k)}| < \varepsilon$$
$$|\vartheta^{(k+1)} - \vartheta^{(k)}| < \varepsilon$$
$$|\Omega^{(k+1)} - \Omega^{(k)}| < \varepsilon .$$

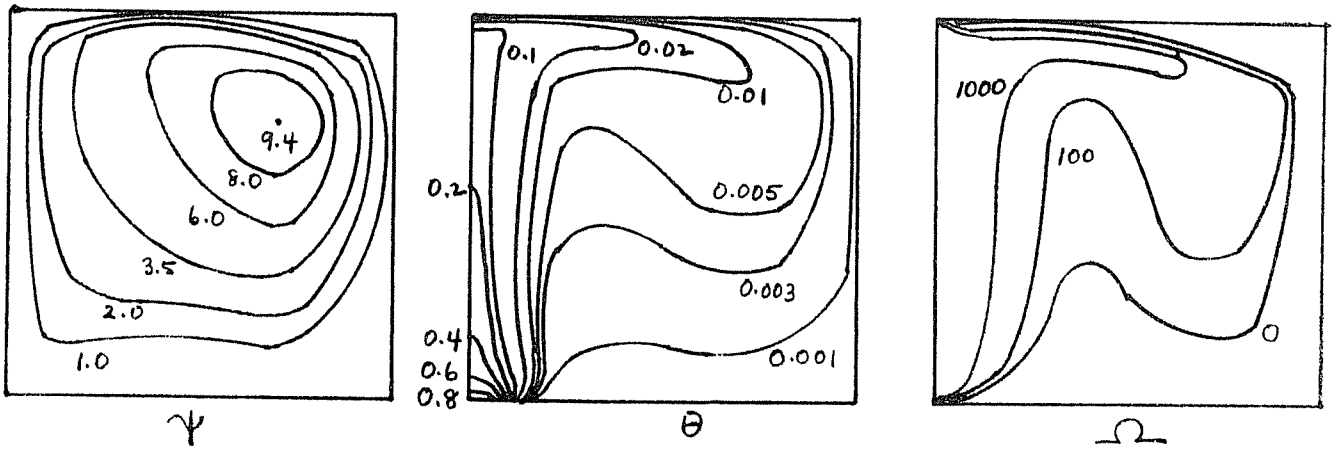
Using the above algorithm with $b = 1.0$ and $c = 0.1$, a variety of examples were run. For $P = 0.7$, which was the only parameter choice of Torrance and Rockett, our results are shown, typically, in Figure 2 and are in complete agreement with those of Torrance of Rockett. However, in addition, our method was also applied to the two extreme cases $P = 0.023$, which corresponds to liquid mercury, and to $P = 9884.0$, which corresponds to aircraft engine oil, and typical results are shown in Figure 3. For $P = 0.023$ the maximum value of ψ varies from 0.0001 when $G = 1000$ to $\psi = 0.019$ when $G = 50000$, while for $P = 9884$ the maximum of ψ is 3.6 for $G = 10$.



$h = 0.05, G = 400000, P = 0.7$

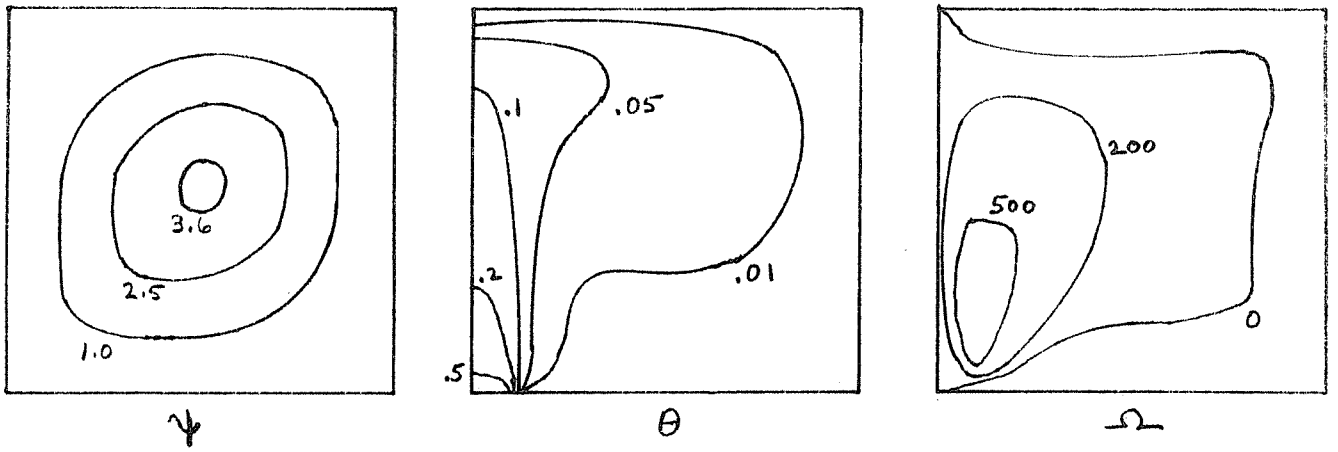


$h = 0.05, G = 40000, P = 0.7$

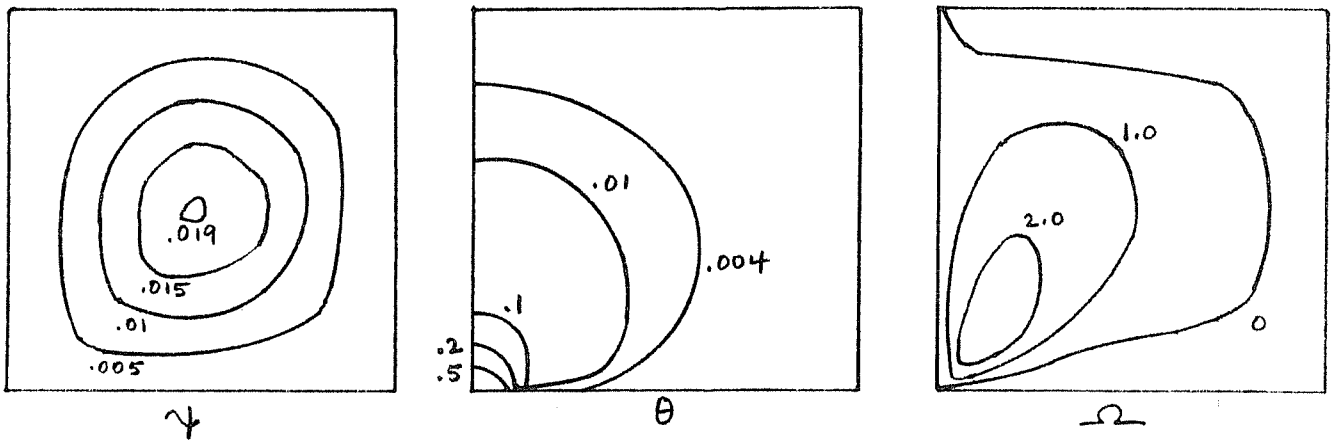


$h = 0.02, G = 4000000, P = 0.7$

FIGURE 2



$h = 0.05,\ G = 10,\ P = 9884$



$h = 0.05,\ G = 50000,\ P = 0.023$

FIGURE 3

To illustrate the speed of the method, the three cases h = 0.1, G = 40000; h = 0.1, G = 400000; and h = 0.05, G = 40000, with P = 0.7 and $\varepsilon = 10^{-6}$ required a total computing time of two minutes and seven seconds on the UNIVAC 1108.

With regard to convergence, it is of interest to note that, as observed previously ([4], [5], [7]), any choice of ρ, μ, or δ outside the ranges $0 \leq \rho \leq 0.4$, $0 \leq \mu \leq 0.4$, $0.6 \leq \delta \leq 1$ resulted, invariably, in divergent behavior. In addition, a method analogous to that described here, but in which the natural selection of central difference formulas was made for all derivatives in (2) and (3), was explored and, though the results agreed with those already obtained, the method was of limited viability due to the loss of diagonal dominance.

5. Numerical Solution of the Time Dependent Problem.

Time dependent problems are solved usually by initial value techniques. Such methods often suffer from numerical error accumulation with the advance in time and from severe stability limitations when the dynamical equations are nonlinear. Since boundary value techniques are not prone to these particular shortcomings, let us show how to extend the method of Section 4 to initial value problems by first converting the given problem (1) - (9) to a boundary value problem. In particular, this can be done readily if (1) - (9) has a steady state solution $\psi(R,x,\infty)$, $v(R,x,\infty)$, $\Omega(R,x,\infty)$ which is

independent of the initial conditions. Under the assumption that such a solution exists, it is prescribed at $\tau = \infty$ and condition (4) is dropped.

The numerical method can be formulated now as follows ([5], [7]). Solve the steady state problem numerically as described in Section 4. Denote the numerical solution by

$$(20) \qquad \psi^{(k+1)} = \psi(R,x,\infty), \quad \theta^{(k+1)} = \theta(R,x,\infty), \quad \Omega^{(k+1)} = \Omega(R,x,\infty).$$

Next, fix a value $\tau = T$ and define $\psi(R,x,T)$, $\theta(R,x,T)$, $\Omega(R,x,T)$ such that at the points $(b - nh, 1 - mh, T)$, $n = 1,2,\ldots,\frac{b-1}{h}$, $m = 1,2,\ldots,\frac{2}{h}$, one has

$$(21) \quad \psi(R,x,T) = \psi(R,x,\infty), \quad \theta(R,x,T) = \theta(R,x,\infty), \quad \Omega(R,x,T) = \Omega(R,x,\infty).$$

Consider then the boundary value problem defined by (1) - (3), (5) - (9) and (21) on the rectangular parallelopiped defined by $0 \le R \le b$, $0 \le x \le 1$, $0 \le \tau \le T$. Apply the techniques described in Section 4 to this three dimensional problem by employing the full equations (11) and (13) with space time step h and time grid step $\Delta\tau$, thus yielding the numerical solution.

Using the above method with $b = 1.0$ and $c = 0.1$, a variety of examples again were run efficiently and economically. Typical problems and results are shown in Figures 4-6. Interestingly enough, the numerical results themselves always indicated whether or not a choice of T was either too small or too large, for, when T was too small, the numerical solution

showed an erratic change only between $T - \Delta t$ and T, whereas, when T was too large, there would be no change in the numerical solution for many time steps preceding T. Running times usually varied between 3 and 20 minutes with Δt in the range from 0.001 to .025 and T in the range from 0.007 to 2.0. One could use such relatively small values of T because, both mathematically and physically, steady state, when it exists, is usually reached rapidly.

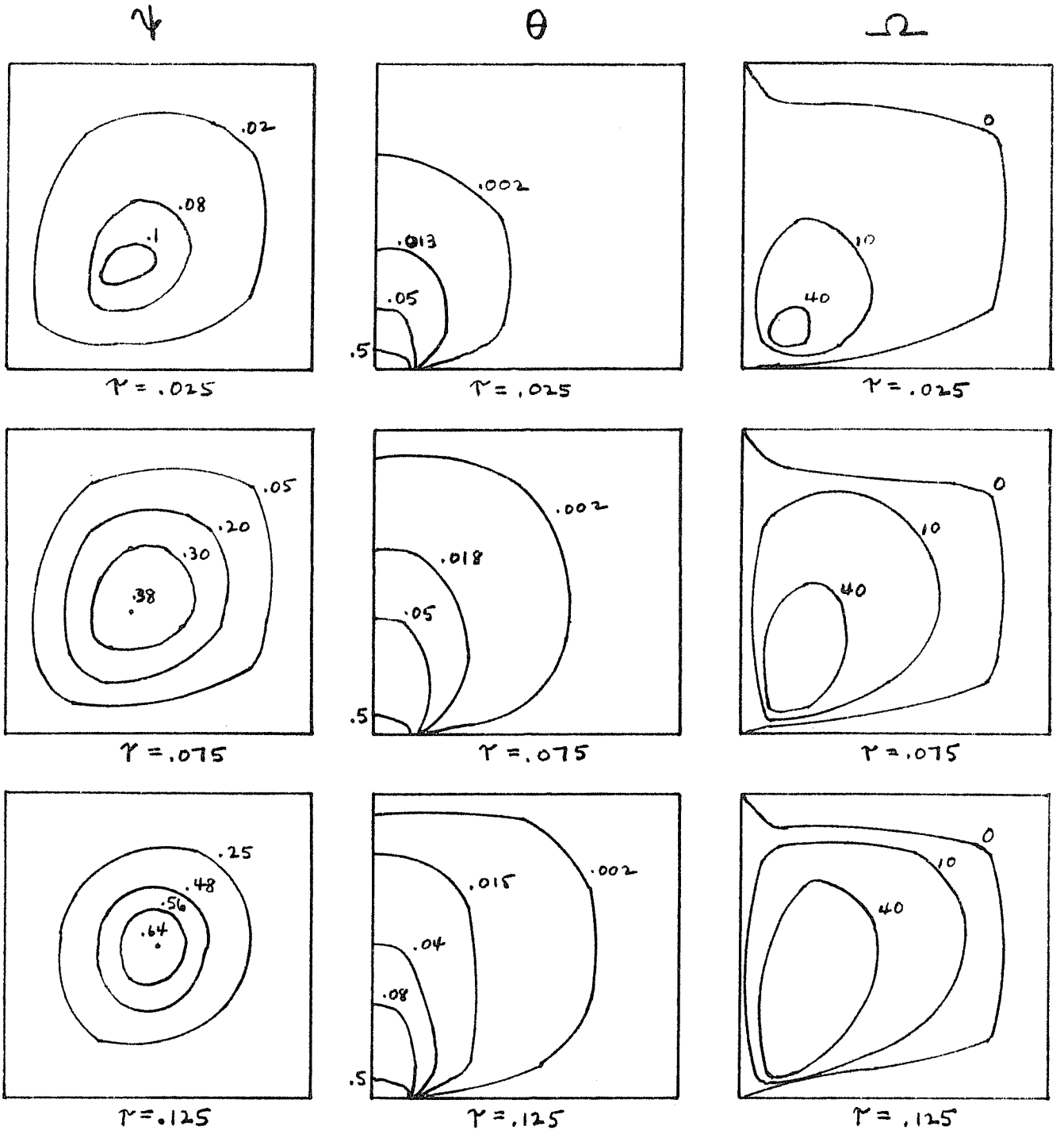


FIGURE 4. h = 0.1, Δt = 0.001, G = 40000, P = 0.7

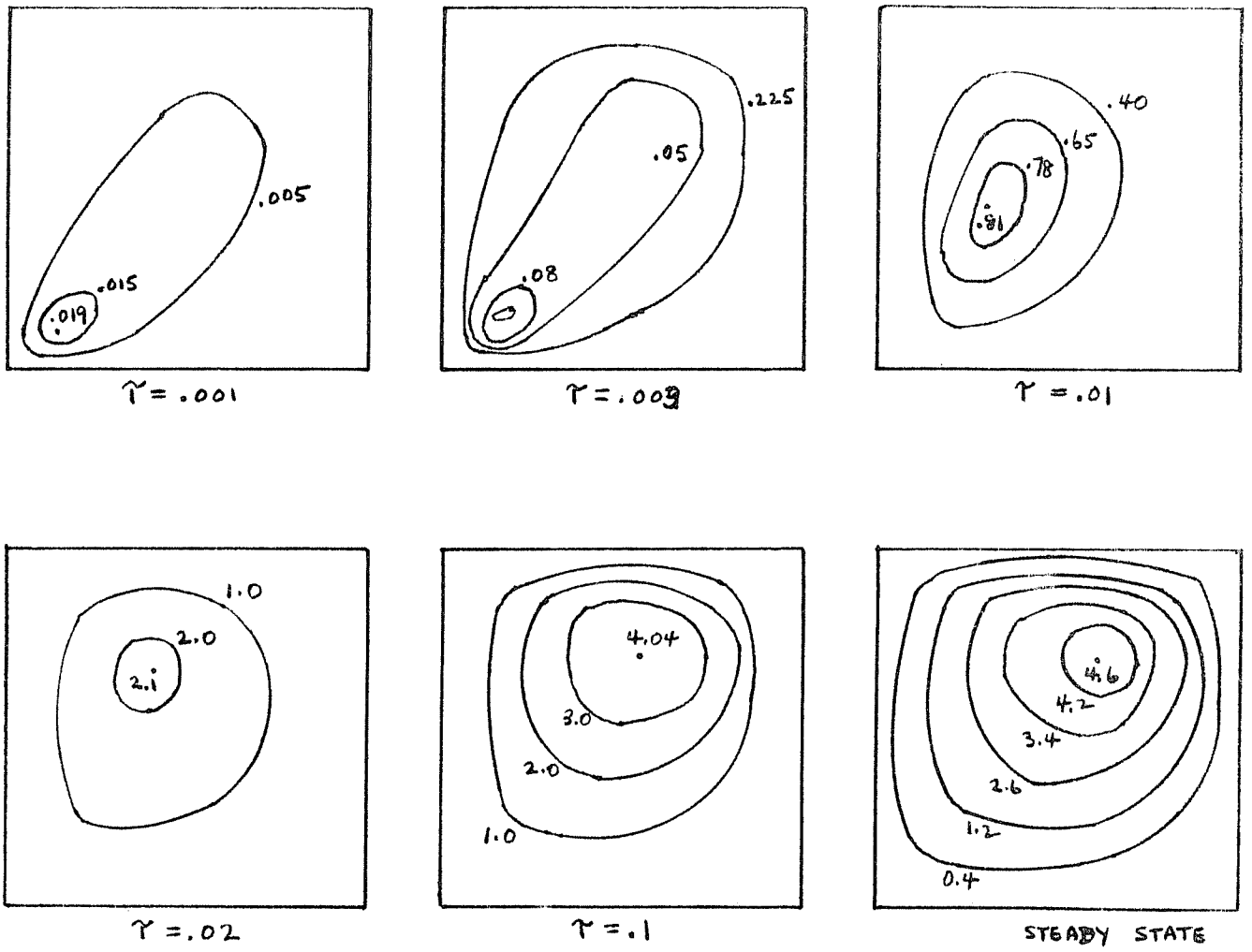


FIGURE 5. ψ-curves, h = 0.05, $\Delta\tau$ = 0.001, G = 400000, P = 0.7

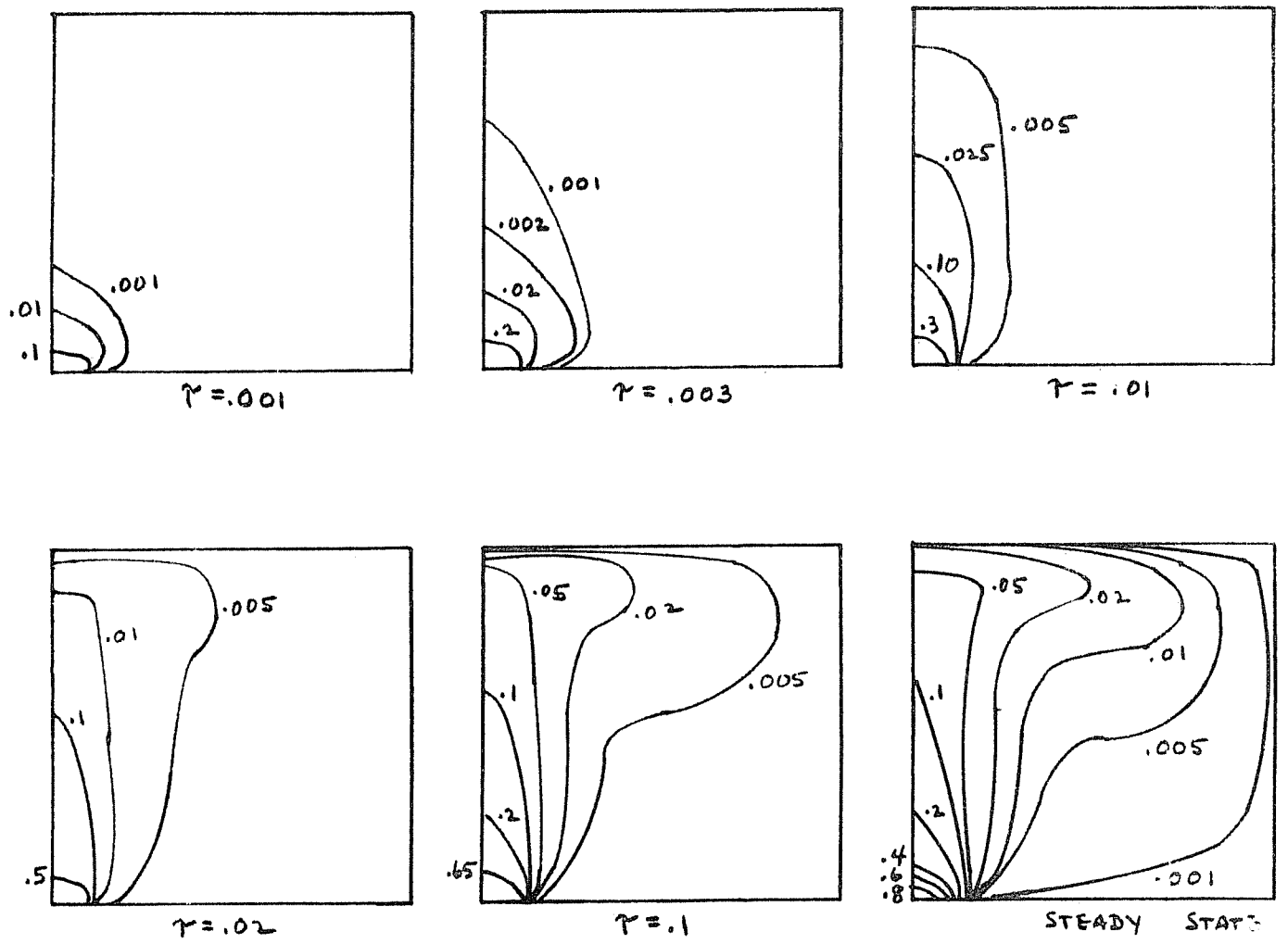


FIGURE 6. θ-curves, h = 0.05, $\Delta\tau$ = 0.025, G = 400000, P = 0.7

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