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BOUNDARY VALUE CONTROL OF THE HIGHER DIMENSIONAL WAVE EQUATION

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1. Introduction

Many important control processes can be described approximately by means of partial differential equations with control parameters appearing in the boundary conditions. For example, a triangular airplane wing may be equipped with ailerons on the trailing edge. An idealized model for such a plant would involve the partial differential equations which describe the motion of a plate with arbitrary control functions appearing in the boundary conditions along one of the sides of the triangle. Many other examples could be given.

Linear hyperbolic problems in one space dimension have been studied rather extensively, see, e.g. [1], [2], [3], [4], [5]. Here the theory is relatively uncomplicated. One can study questions of controllability using the geometric techniques based on characteristic curves or the more algebraic techniques based on the theory of non-harmonic Fourier series. One obtains

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not only theorems asserting the existence of controls transferring one state to another within a finite time period but also constructive proofs of these theorems which can be adapted to yield numerical techniques whereby the appropriate control functions can be calculated. The papers of Grainger [4] and Cirina [5] are noteworthy in this respect. Cirina's paper shows that such methods can even be used for quasilinear systems.

The theory is not nearly as complete for problems involving two or more space variables. The reasons why this should be so become apparent when one compares Chapters V and VI of the treatise [6] of Courant-Hilbert. Some results have been obtained in this area by Fattorini [7] who considers, for the most part, boundary value control problems wherein the controls can be described by finitely many functions of the time t - physically the most realistic situation.

The purpose of the present paper is to study hyperbolic problems in several space dimensions using certain uniqueness theorems due to Holmgren [8] and John [9]. Using these results we can obtain very explicit estimates on the length of time required to transfer a given state into an arbitrarily small neighborhood of any other state using boundary value controls restricted to a subset of the boundary of the region in question. More specifically, we are able to show for the wave equations in 3 or fewer space variables that the system can be controlled in any time T which exceeds twice the wave propagation time from the boundary set where controls are applied to the rest of the physical medium. It should be noted that such a result is in agreement with known results [1], [2] for the case of a single space dimension.

I should like to express my appreciation to Professor J. L. Lions of the University of Paris whose suggestions in a 1966 letter provided the germinal idea for the proofs presented in this paper.

2. The Control Problem

Let Ω be a bounded, open connected domain in R^n whose boundary is an analytic surface Γ of dimension $n-1$. We indicate points in Ω by

$$x = \begin{pmatrix} x^1 \\ x^2 \\ \vdots \\ x^n \end{pmatrix}.$$

The boundary surface Γ is parametrized by an $n-1$ dimensional vector variable s. Integrals over Ω will be denoted by $\int_\Omega (\quad) dx$ while integrals over Γ will be written $\int_\Gamma (\quad) ds$. If we wish to indicate a point in R^n which lies on Γ we will write $x(s)$. The surface Γ, being analytic, has everywhere a unique unit outward normal vector which we will indicate by $\eta(s)$.

We consider a second order linear hyperbolic partial differential equation

$$L(w) = \rho(x) w_{tt} - \sum_{i,j=1}^{n} (\alpha_{ij}(x) w_i)_j = 0. \tag{2.1}$$

The subscripts i,j indicate partial differentiation with respect to x^i, x^j, respectively. The coefficients $\rho(x)$, $\alpha_{ij}(x)$ are real analytic in some open subset of R^n which includes $\Omega \cup \Gamma$. Moreover, if $A(x)$ is the nxn matrix with entries $\alpha_{ij}(x)$, $A(x)$ is symmetric and there are positive numbers ρ_o and α_o such that

$$\rho(x) \geq \rho_o, \; x \in \Omega$$
$$v'A(x)v \geq \alpha_o \|v\|^2, \; x \in \Omega, \; v \in R^n.$$

Let Γ_1 be a relatively open subset of Γ. For $T > 0$ we denote by F the space of all C^∞ functions $f: \Gamma \otimes [0,T] \to R^1$ with the property that f vanishes outside some compact subset of $\Gamma_1 \otimes [0,T]$ (this set varies with f). We pose for (2.1) the initial-boundary value problem

$$
\begin{aligned}
&w(x,o) \equiv w_t(x,o) \equiv 0\,,\ x \in \Omega\,, \\
&w_x(x(s),t)\,A(x(s))\eta(s) \equiv f(s,t),\ (x(s),t) \in \Gamma \otimes [0,T).
\end{aligned}
\tag{2.2}
$$

The symbol w_x denotes the row vector of spatial partial derivatives of w:

$$w_x = (w_1, w_2, \ldots, w_n)\ .$$

With these assumptions it is known that the initial boundary value problem (2.1), (2.2) has a unique solution $w^f(x,t)$ which lies in the class $C^\infty((\Omega \cup \Gamma) \otimes [0,T])$. The reader is referred to the papers of Friedrichs, Lax and Duff [10], [11], [12].

If $f(s,t) \equiv 0$ for $t_1 \le t \le t_2$ then $w^f(\cdot,t)$ can be considered as a vector valued function with range in $L^2(\Omega)$ which solves the evolution equation

$$\frac{d^2 w}{dt^2} + Bw = 0\,,\ t_1 \le t \le t_2, \tag{2.3}$$

where B is the unbounded operator on $L^2(\Omega)$ which is the unique self-adjoint extension of the operator $\sum_{i,j=1}^{n} (\alpha_{ij}(x)w_i)_j$ defined on twice continuously differentiable functions $w(x)$ satisfying the boundary conditions

$$w_x(x(s))\,A(x(s))\eta(s) = 0,\ x(s) \in \Gamma\,.$$

Let H_1 denote the space of pairs of real valued functions $w(x)$, $w_t(x)$ defined on Ω with $w_t(x)$ square integrable and $w(x)$ having square integrable derivatives:

$$\int_\Omega (w_t(x)^2 + \sum_{i=1}^{n} w_i(x)^2)dx < \infty .$$

Let H_E denote the space of equivalence classes of H_1 modulo the zero energy states $w_t(x) \equiv 0$, $w(x) \equiv \text{const}$. The "energy"

$$\mathcal{E}(w,w_t) = \int_\Omega (\rho(x)(w_t(x))^2 + w_x(x)\, A(x) w_x(x)')\, dx$$

is a constant on each such equivalence class. H_E is a Hilbert space with the inner product

$$\langle (w,w_t);(v,v_t)\rangle_E = \int_\Omega (\rho(x) w_t(x) v_t(x) + w_x(x) A(x) v_x(x)\ )dx$$

and resulting norm

$$\|(w,w_t)\|_E = \sqrt{\langle (w,w_t);(w,w_t)\rangle_E} = \sqrt{\mathcal{E}(w,w_t)}$$

We will not stress the distinction between H_1 and H_E where unnecessary and we will say "$(w,w_t) \in H_E$" if the equivalence class of (w,w_t) is a member of H_E .

For each $f \in F$ the corresponding solution $w^f(x,t)$ of (2.1), (2.2) is such that $(w^f(\cdot,T), w_t^f(\cdot,T)) \in H_E$. In fact, if we put

$$R_T = \{ (w^f(\cdot,T),\ w_t^f(\cdot,T)) \mid f \in F \}$$

then R_T is a subspace of H_E which we will call the reachable space. Following others we make the

Definition: The control system $\{(2.1), (2.2), f \in F\}$ is approximately controllable in time $T > 0$ if R_T is dense in H_E relative to the topology induced by the norm $\| \ \|_E$.

We will conclude this section with a theorem which relates approximate controllability to "observability". (cf. parallel results for o.d.e.'s [13].)

Theorem 1 Let $(\hat{v}, \hat{v}_t) \in H_E$ be such that both $\hat{v}$ and $\hat{v}_t$ lie in $C^\infty(\Omega \cup \Gamma)$ and $\hat{v}$ satisfies the consistency conditions

$$\hat{v}_x(x(s))\, A(x(s))\, \eta\, (x(s)) = 0,\ \hat{v}_{tx}(x(s))A(x(s))\eta(x(s)) = 0,\ x(s) \varepsilon \Gamma .$$

Let $v(x,t)$ be the unique C^∞ solution of $L(v) = 0$ which satisfies the boundary conditions

(2.4) $v_x(x(s),t)\, A(x(s))\, \eta\, (x(s)) \equiv 0,\ (x(s),t) \in \Gamma \otimes [0,T]$

and the terminal conditions

$$v(x,T) \equiv \hat{v}(x),\quad v_t(x,T) \equiv \hat{v}_t(x) . \tag{2.5}$$

Then $(\hat{v}, \hat{v}_t) \in (R_T)^\perp$ in H_E if and only if $v_t(x(s),t) \equiv 0,\ (x(s),t) \in \Gamma_1 \otimes [0,T]$.

Proof: For $f \in F$ we have

$$0 = \int_{\Omega \otimes [0,T]} (V_t L(w^f) + w^f_t\ L(v))dxdt \tag{2.6}$$

$$= \int_{\Omega \otimes [0,T]} \operatorname*{div}_{x,t} \begin{pmatrix} -v_t(w^f_x a_1) - w^f_t(v_x a_1) \\ \vdots \\ -v_t(w^f_x a_n) - w^f_t(v_x a_n) \\ \rho w^f_t v_t + w^f_x A\, v'_x \end{pmatrix} dxdt,$$

where, for convenience, we have suppressed the arguments in the integrand and the column vectors $a_i(x)$ are the columns of the symmetric matrix $A(x)$:

$$A(x) = (a_1(x),\ a_2(x),\ \ldots,\ a_n(x)).$$

Applying the divergence theorem to the second member of (2.6) we obtain

$$(2.7)\qquad 0 = \int_{\Omega\otimes\{T\}} (\rho\, w_t^f\, v_t + w_x^f\, A\, v_x')\, dx$$

$$- \int_{\Omega\otimes\{0\}} (\rho\, w_t^f\, v_t + w_x^f\, A\, v_x')\, dx$$

$$- \int_{\Gamma\otimes[0,T]} (v_t\, (w_x^f A\, \eta) + w_t^f (v_x A\, \eta))\, ds dt.$$

Using (2.5), (2.2), respectively, in the first two members of (2.7) and (2.2), (2.4) in the third member we obtain

$$(2.8)\quad \int_{\Omega} (\rho\,(x)\, w_t^f(x,T)\hat{v}_t(x) + w_x^f(x,T)A(x)\,\hat{v}_x(x)')\, dx = \int_{\Gamma_1\otimes[0,T]} (v_t\, f)\, ds dt.$$

From the definition of $\langle\, ,\, \rangle_E$ we see that (2.8) becomes

$$(2.9)\qquad \langle (w^f(\cdot\,,T),\ w_t^f(\cdot\,,T));(\hat{v},\hat{v}_t)\rangle_E = \int_{\Gamma_1\otimes[0,T]} (v_t f) ds dt$$

The right hand side of (2.9) vanishes for all $f \in F$ if and only if $v_t(x(s),t) \equiv 0$, $(x(s),t) \in \Gamma_1 \otimes [0,T]$ and thus the proof is complete.

Theorem 1 is fundamental in the proofs of the controllability theorems of the subsequent sections.

3. The Time T_o

In order to state and prove our theorems on approximate controllability of (2.1), (2.2) we must employ the concept of a characteristic surface for (2.1) in R^{n+1}. This concept is treated in detail in [6], for example, but we give a brief description to make our presentation somewhat self-contained.

Let S be a surface in R^{n+1} given by

$$S = \{(x,t) \mid \Phi(x,t) = 0\}$$

where $\Phi(x,t)$ is a smooth real valued function of $n+1$ variables. We define the characteristic form

$$x(\Phi,x,t) = \rho(x)\,(\Phi_t(x,t))^2 - \Phi_x(x,t)A(x)\,\Phi_x{}'(x,t)$$

The surface S is: characteristic if $x(\Phi,x,t) \equiv 0$ for $(x,t) \in S$; uniformly space-like if $\exists\ \delta > 0$ such that $x(\Phi,x,t) \geq \delta$ for $(x,t) \in S$; uniformly time-like if $\exists\ \delta > 0$ such that $x(\Phi,x,t) \leq -\delta$ for $(x,t) \in S$. For what is usually called the wave equation $\rho(x) \equiv 1$ and $A(x) \equiv I$, the $n \times n$ identity matrix, a surface is characteristic if and only if it everywhere makes an angle of 45^o with any intersecting surface $t = \text{const}$. It is this special case that we will use in our diagrams since it is less confusing than the general case.

Let $(x_o,t_o) \in (\Omega \cup \Gamma) \otimes [0,T]$. We define the forward cone of influence of (x_o,t_o) to be the subset $K^+(x_o,t_o)$, the largest closed subset of $(\Omega \cup \Gamma) \otimes [t_o,T]$ which contains (x_o,t_o) and does not meet any uniformly space-like surface passing through (x_o,t_o). Similarly we define $K^-(x_o,t_o)$, the backward cone of influence of (x_o,t_o), by replacing $[t_o,T]$ with $[0,t_o]$. It is easy to see that

$K^+(x_o,t_o)$ and $K^-(x_o,t_o)$ have characteristic boundary surfaces. When $\rho(x) \equiv 1$, $A(x) \equiv I$, we have $K^+(x_o,0) = \{(x,t) \in (\Omega \cup \Gamma) \otimes [0,T] \mid t^2 - \|x-x_o\|^2 \geq 0\}$

If G is a subset of $\Omega \cup \Gamma$ we define forward and backward cones of influence of (G,t_o) by

$$K^+(G,t_o) = \overline{\bigcup_{x_o \in G} K^+(x_o,t_o)}$$

$$K^-(G,t_o) = \overline{\bigcup_{x_o \in G} K^-(x_o,t_o)} \, .$$

Let t_o, t_1 lie in $[0,T]$ and let $G \subseteq \Omega \cup \Gamma$. We define

$$K(G,t_o,t_1) = K^+(G,t_o) \cap K^-(G,t_1).$$

Since the coefficients of the operator L do not depend upon t, $K(G,t_o,t_1)$ is symmetric about the plane $t = \frac{1}{2}(t_o + t_1)$.

The fact that $A(x)$ is uniformly positive definite can be used to prove that there is a least time $T_o > 0$ such that $K^+(\Gamma_1,0)$ includes the set $\Omega \otimes \{T_o\}$. Then $K(\Gamma_1,0,2T_o)$ also includes $\Omega \otimes \{T_o\}$. If $T > 2T_o$ there is a $\alpha > 0$ such that $K(\Gamma_1,0,T)$ includes $\Omega \otimes \{t\}$ for $|t - \frac{T}{2}| \leq \alpha$. If $T < 2T_o$ the set

$$J(\tfrac{T}{2}) = \Omega \otimes \{\tfrac{T}{2}\} - K(\Gamma_1,0,T)$$

is a non-empty set. For $\rho(x) \equiv I$, Ω = unit disc in R^2, Figures 1-4 illustrate the geometry of the situations described above both when $\Gamma_1 = \Gamma$ and when Γ_1 is a small sub-arc of Γ.

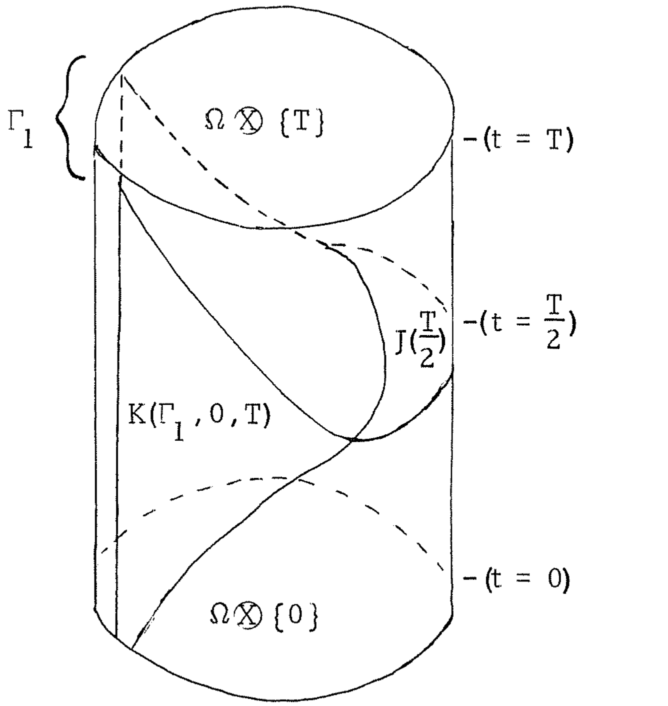


Fig. 1 Γ_1 is a subarc of Γ. $T < 2T_o$

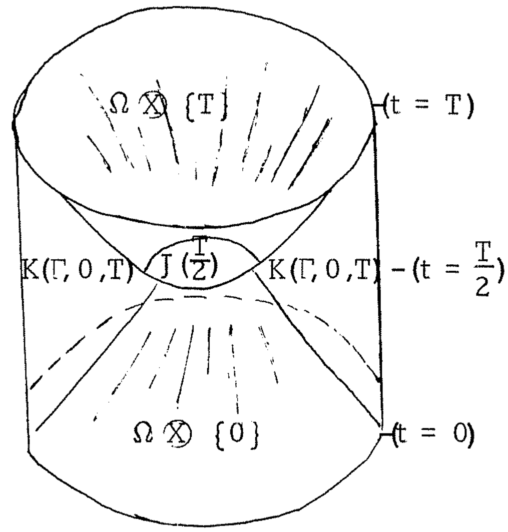
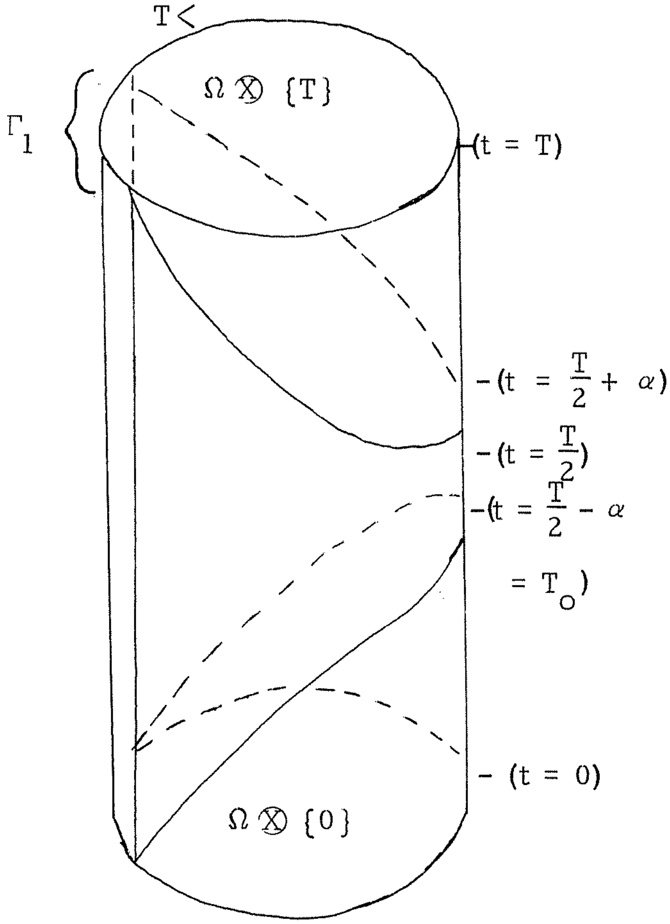


Fig. 2 $\Gamma_1 = \Gamma$

$T < 2T_o$

T<



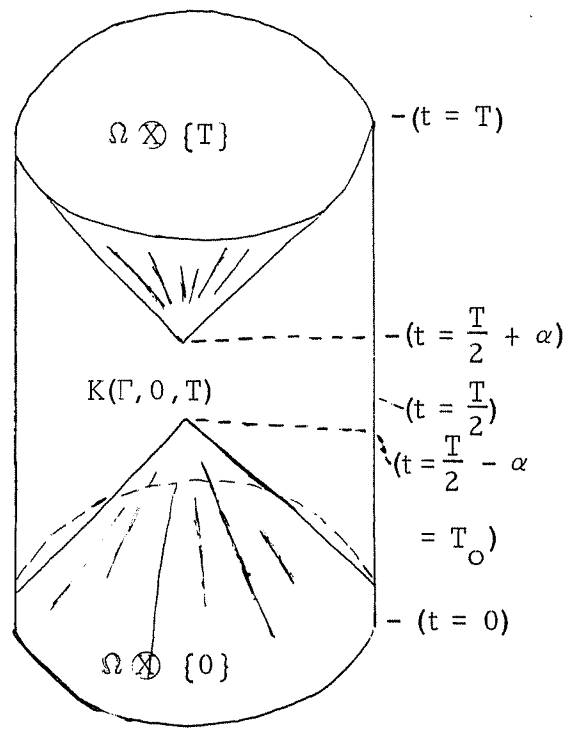


Fig. 3 Γ_1 is a subarc of Γ

$T > 2T_o$

Fig. 4 $\Gamma_1 = \Gamma$

$T > 2T_o$

4. Non-controllability for $T < 2T_0$

Since the reachable set R_T is a linear subspace of the Hilbert space H_E, R_T fails to be dense in H_E just in case there is a non-zero element of H_E which is orthogonal to all elements of R_T. In view of our definition of H_E in terms of equivalence classes of states in H_1 modulo the zero energy states, we see that R_T is dense in H_E if and only if the equations

$$(4.1) \quad \int_\Omega (\rho(x) w_t^f(x,T)\hat{v}_t(x) + w_x^f(x,T)A(x)\hat{v}_x(x)')dx = 0, \; f \in F,$$

where $\hat{v}$, $\hat{v}_t$ is a fixed element of H_1, imply that $\hat{v}_t \equiv 0$, $\hat{v}$ = constant.

When $T < 2T_0$ the subset J of Ω given by

$$J = \{x \mid (x, \tfrac{T}{2}) \in J(\tfrac{T}{2})\}$$

is a non-empty open set. Let $\tilde{v}(x)$, $\tilde{v}_t(x)$ be a state in H_1 such that (i) $\tilde{v}(x)$, $\tilde{v}_t(x)$ has non-zero energy norm (ii) $\tilde{v}(x)$, $\tilde{v}_t(x) \in C^\infty(\Omega)$ and vanish outside a compact subset of the interior of J.

We now permit the state $\tilde{v}(x)$, $\tilde{v}_t(x)$ to evolve, via the partial differential equation with

$$v(x,\tfrac{T}{2}) = \tilde{v}(x), \quad v_t(x,\tfrac{T}{2}) = \tilde{v}_t(x).$$

We put

$$(4.2) \qquad \hat{v}(x) = v(x,T), \quad \hat{v}_t(x) = v_t(x,T)$$

and note that (2.1), (2.4) imply conservation of energy so that

$$\|(\hat{v},\hat{v}_t)\|_E = \|(\tilde{v},\tilde{v}_t)\|_E \neq 0.$$

Now our requirement (ii) on $\tilde{v}$, $\tilde{v}_t$ guarantees that $v(x,t) \in C^{\infty}(\Omega \otimes [0,T])$, as does $w^f(x,t)$ for each $f \in F$. This enables us to apply Theorem 1 to show that (4.1) holds if and only if $v_t(x(s),t) \equiv 0$ for $(x(s), t) \in \Gamma_1 \otimes [0,T]$.

It is easily seen that the fact that $K(\Gamma_1, 0, T) \cap J = \Phi$ implies that the interior of $K(\Gamma_1, 0, T)$ does not meet $K^+(J, \frac{T}{2}) \cup K^-(J, \frac{T}{2})$, the cone of influence of $(J, \frac{T}{2})$. Well known results for hyperbolic partial differential equations [6] then show that $v(x,t) \equiv 0$, $v_t(x,t) \equiv 0$ in $K(\Gamma_1, 0, T)$. But $\Gamma_1 \otimes [0,T] \subseteq K(\Gamma_2, 0, T)$ so we conclude that $v_t(x(s),t) \equiv 0$, $(x(s),t) \in \Gamma_1 \otimes [0,T]$. Therefore (4.1) must hold for the non-zero energy state $\hat{v}, \hat{v}_t$ and we have proved

Theorem 2 The system (2.1), (2.2) is not approximately controllable in time T if $T < 2T_0$.

This theorem may be compared with comparable results [1],[2] for hyperbolic systems in one space dimension.

5. The Holmgren-Fritz John Uniqueness Theorem

The uniqueness theorems of Holmgren and Fritz John [8],[9], applied to the case we have in mind, reduce to

Theorem 3 Let $u(x,t)$ be a twice continuously differentiable solution of $L(u) = 0$ (cf. (2.1)) in $K(\Gamma_1, t_o, t_1)$, $[t_o, t_1] \subseteq [0,T]$, with

$$\left.\begin{aligned} &u_x(x(s),t)A(x(s))\,\eta\,(x(s)) \equiv 0 \\ &u(x(s),t) \equiv 0 \end{aligned}\right\} \qquad (x(s),t) \in \Gamma_1 \otimes [t_o, t_1]. \tag{5.1}$$

Then

$$u(x,t) \equiv 0 \quad \text{in} \quad K(\Gamma_1, t_o, t_1) \tag{5.2}$$

The proof of this theorem is detailed in the works cited. However, we need to strengthen the theorem somewhat for our needs and this strengthening requires that we have some details of the proof. For this reason we give a short proof of Theorem 3. An important part of the proof is the lemma stated below, which we do not prove. See [9] for details in certain cases.

Lemma If $(\bar{x},\bar{t})$ lies in the interior of $K(\Gamma_1, t_o, t_1)$ there is a uniformly time-like family of surfaces $S(\lambda)$, $0 \leq \lambda \leq 1$, with the following properties:

(i) $S(\lambda)$ is a compact subset of a relatively open analytic $n-1$ dimensional surface;

(ii) $S(\lambda)$ varies analytically with respect to λ, $0 \leq \lambda \leq 1$;

(iii) $S(\lambda) \subseteq K(\Gamma_1, t_o, t_1)$, $0 \leq \lambda \leq 1$, and $S(0)$ is a subset of the interior of $\Gamma_1 \otimes [0,T]$;

(iv) <u>If</u> $0 < \lambda \leq 1$ <u>then</u> $S(0) \cup S(\lambda)$ <u>is the boundary of an open subset</u> $D(\lambda) \subseteq K(\Gamma_1, t_o, t_1)$ <u>and</u> $(\bar{x}, \bar{t}) \varepsilon D(1)$.

Assuming this lemma, we proceed with the

<u>Proof of Theorem 3</u>. The uniformly time-like character of the surfaces $S(\lambda)$ together with the analyticity of these surfaces enables one to employ the Cauchy-Kowalewski theorem [6] to show that there are n dimensional neighborhoods $N(\lambda)$ of the surfaces $S(\lambda)$ such that if analytic Cauchy data for z are prescribed on $S(\lambda)$ there will be a corresponding unique analytic solution $z(x,t)$ of $L(z) = 0$ in $N(\lambda)$. Since the equation $L(z) = 0$ is linear, $N(\lambda)$ depends only on $S(\lambda)$, not on the particular Cauchy data. Moreover, $N(\lambda)$ varies continuously with λ, $0 \leq \lambda \leq 1$. Thus for sufficiently small $\lambda > 0$, $S(0) \subseteq N(\lambda)$ and the analytic solution $z(x,t)$ is defined throughout the domain $D(\lambda)$ which is bounded by $S(0)$ and $S(\lambda)$. We consider the identity

$$(5.3) \qquad 0 = \int_{D(\lambda)} (uL(z) - zL(u)),\ dxdt$$

$$= \int_{D(\lambda)} \underset{x,t}{\mathrm{div}} \begin{pmatrix} -uz_x a_1 + zu_x a_1 \\ \vdots \qquad\quad \vdots \\ -uz_x a_n + zu_x a_n \\ \rho uz_t - \rho zu_t \end{pmatrix} dxdt$$

$$= \int_{S(0) \cup S(\lambda)} (-uz_x A\eta + zu_x A\eta + \eta_o \rho uz_t - \eta_o \rho zu_t)\, d\sigma,$$

where $d\sigma$ denotes integration with respect to surface area on $S(0) \cup S(\lambda)$ and

$\binom{\eta}{\eta_o}$ is the unit outward normal to $S(0) \cup S(\lambda)$ in R^{n+1}, defined in the relative interiors of $S(0)$ and $S(\lambda)$. We observe that $\eta_o \equiv 0$ on $S(0)$, and that (5.1) holds on $S(0)$ since $S(0) \subseteq \Gamma_1 \otimes [0,T]$. Thus (5.3) reduces to

$$(5.4) \qquad \int_{S(\lambda)} (u(\eta_o \rho z_t - z_x A\eta) + z(\eta_o \rho u_t - u_x A\eta)) d\sigma = 0.$$

We choose analytic Cauchy data for z on $S(\lambda)$ as follows: we put $z \equiv 0$ on $S(\lambda)$ and we take the normal derivative of z across $S(\lambda)$ to be an arbitrary real valued analytic function α, i.e.,

$$\left.\begin{aligned} &(5.5) \qquad z(x(\sigma), t(\sigma)) = 0 \\ &(5.6) \qquad z_x(x(\sigma), t(\sigma))\eta(\sigma) + z_t(x(\sigma), t(\sigma))\eta_o(\sigma) = \alpha(\sigma) \end{aligned}\right\} \quad (x(\sigma), t(\sigma) \varepsilon S(\lambda).$$

The equations (5.5), (5.6) together imply that

$$(5.7) \quad (z_x(x(\sigma), t(\sigma)), z_t(x(\sigma), t(\sigma)) \equiv \alpha(\sigma)(\eta'(\sigma), \eta_o(\sigma)), \ (x(\sigma), t(\sigma)) \varepsilon S(\lambda).$$

Substituting (5.7) and (5.5) in (5.4) we obtain

$$(5.8) \qquad \int_{S(\lambda)} u\,\alpha(\rho(\eta_o)^2 - \eta' A\eta) d\sigma = 0.$$

Since $S(\lambda)$ is uniformly timelike we have

$$(5.9) \qquad \rho(x(\sigma))(\eta_o(\sigma))^2 - \eta'(\sigma)A(x(\sigma))\eta(\sigma) \equiv \beta(\sigma) \leq -\beta_o < 0$$

for all values of the vector σ parametrizing $S(\lambda)$. The equation (5.8) becomes

$$\int_{S(\lambda)} u\,\alpha\beta\, d\sigma = 0.$$

Since this equation holds for all real analytic functions α, we conclude

$$u(x(\sigma), t(\sigma))\beta(\sigma) \equiv 0, \ (x(\sigma), t(\sigma)) \varepsilon S(\lambda)$$

and, since (5.9) shows that β never vanishes we have

$$u(x(\sigma),t(\sigma)) \equiv 0, \quad (x(\sigma),t(\sigma)) \in S(\lambda).$$

Repeating this argument on surfaces $S(\mu)$, $0 < \mu < \lambda$, which sweep out the interior of $D(\lambda)$, we conclude

$$u(x,t) \equiv 0, \ (x,t) \in D(\lambda).$$

We now let I denote the largest subinterval of $[0,1]$ which includes 0 and has the property that $u \equiv 0$ on $S(\lambda)$ if $\lambda \in I$. We have seen above that I is non-empty. Essentially the same technique can be used to show that I is open. But it is obvious that I is closed and we conclude, from the connectedness of $[0,1]$ that $I = [0,1]$. Thus

$$u(x,t) \equiv 0, \quad (x,t) \in D(1)$$

and, since $(\bar{x},\bar{t}) \in D(1)$, we have

$$u(\bar{x},\bar{t}) \equiv 0.$$

Since (x,t) is an arbitrary point in the interior of $K(\Gamma_1, t_0, t_1)$ and since u is continuous in $K(\Gamma_1, t_0, t_1)$ we see that (5.2) follows and the proof is complete.

6. Controllability for $T > 2T_o$, $n \le 3$.

Let the state $(\hat{v}, \hat{v}_t)$ lie in the finite energy space H_E and suppose that for all $f \in F$ we have

$$(6.1) \qquad \int_\Omega (\rho(x) w_t^f(x,T) \hat{v}_t(x) + W_x^f(x) A(x) \hat{v}'_x(x)) dx = 0.$$

If this implies $\hat{v}_t(x) \equiv 0$, $\hat{v}(x) \equiv$ const. then R_T is dense in the Hilbert space H_E and we have approximate controllability.

Let $v(x,t)$ be the generalized solution in $\Omega \otimes [0,T]$ of the partial differential equation $L(v) = 0$ corresponding to homogeneous boundary conditions (2.4). If $v(x,t)$ were smooth, say $v \in C^3((\Omega \cup \Gamma) \otimes [0,T]$, the proof of our controllability result would not be difficult. Applying Theorem 1 we would get $v_t(x(s), t) \equiv 0$ for $(x(s),t) \in \Gamma_1 \otimes [0,T]$. Putting $u(x,t) \equiv v_t(x,t)$ we would have a solution of $L(u) = 0$ satisfying the hypotheses (5.1) of Theorem 3 and we could conclude $v_t \equiv u \equiv 0$ in $K(\Gamma_1, 0, T)$. If $T > 2T_o$ the set $K(\Gamma_1, 0, T)$ includes $\Omega \otimes [\frac{T}{2} - \varepsilon, \frac{T}{2} + \varepsilon]$ for some $\varepsilon > 0$. If v_t vanishes in $\Omega \otimes [\frac{T}{2} - \varepsilon, \frac{T}{2} + \varepsilon]$ then $v_{tt}(x, \frac{T}{2}) \equiv 0$ which would imply $\sum_{i,j=1}^{n} (\alpha_{ij}(x) v_i(x, \frac{T}{2}))_j \equiv 0$. Thus $v(x, \frac{T}{2})$ would be a solution of the elliptic boundary value problem

$$(6.2) \qquad \sum_{i,j=1}^{n} (\alpha_{ij}(x) v_i(x, \tfrac{T}{2}))_j = 0, \quad x \in \Omega$$

(6.3) $$v_x(x(s), \frac{T}{2})\, A(x(s))\, \eta(s) = 0, \quad x(s) \in \Gamma.$$

It is clear that the only solutions of (6.2), (6.3) have the form $v \equiv$ const. Thus we would have $v_t(x, \frac{T}{2}) \equiv 0$, $v_x(x, \frac{T}{2}) \equiv 0$ which would show that $(v(\cdot, \frac{T}{2}), v_t(\cdot, \frac{T}{2})) = 0$. Since solutions of $L(v) = 0$ with boundary conditions (2.4) conserve energy, we could then conclude $\mathcal{E}(v(\cdot, T), v_t(\cdot, T)) = \mathcal{E}(\hat{v}, \hat{v}_t) = 0$, so that $\hat{v}_t \equiv 0$, $\hat{v} \equiv$ const. and the proof would be complete. In fact, under these conditions we could obtain the result for $T = 2T_0$ also and there is nothing special about $n \leq 3$.

Unfortunately we are not at all justified in assuming such smoothness for $v(x,t)$. A rigorous proof requires that we allow $(\hat{v}, \hat{v}_t)$ to be an arbitrary finite energy state. All this gives us is that $\hat{v}_t \in L^2(\Omega)$ and $\hat{v}_i \in L^2(\Omega)$, $i = 1, 2, \ldots, n$. The generalized solution $v(x,t)$ is no smoother. For this reason it becomes a non-trivial task to justify the argument presented above. In the present paper we will give such justification only for $n \leq 3$, $T > 2T_0$. The result undoubtedly remains true for larger values of n and for $T = 2T_0$ but rather involved arguments seem to be required. Fortunately, $n \leq 3$ includes most cases of physical interest.

<u>Theorem 4</u>. <u>If</u> $(\hat{v}, \hat{v}_t) \in H_E$ <u>is such that</u> (6.1) <u>holds for all</u> $f \in F$ <u>then</u> $\hat{v}_t \equiv 0$, $\hat{v} \equiv$ a const, <u>provided</u> $n \leq 3$, $T > 2T_0$. <u>Thus the system</u> (2.1), (2.2) <u>is approximately controllable in time</u> $T > 2T_0$ <u>when</u> $n \leq 3$.

<u>Proof</u> Let $v(x,t)$ be the generalized solution of $L(v) = 0$ with homogeneous boundary conditions (2.4) and satisfying the terminal conditions $v(x,t) \equiv \hat{v}(x)$, $v_t(x,T) \equiv \hat{v}_t(x)$. Let $(\hat{v}, \hat{v}_t)$ be a sequence of states in H_E converging to $(\hat{v}, \hat{v}_t)$ in the energy norm

as $k \to \infty$. Moreover, $\hat{v}^k(x)$, $\hat{v}_t^k(x)$ ε $C^\infty(\Omega \cup \Gamma)$ and satisfy the consistency conditions

$$\hat{v}^k(x(s))\, A(x(s))\, \eta(s) = 0, \quad x(s) \in \Gamma. \tag{6.4}$$

One way in which this could be done is to expand $\hat{v}(x)$, $\hat{v}_t(x)$ in terms of the eigenfunctions $\varphi_j(x)$ of the operator B introduced in (2.3):

$$\hat{v}(x) = \sum_{j=0}^{\infty} \alpha_j \varphi_j(x), \quad \hat{v}_t(x) = \sum_{j=0}^{\infty} \beta_j \varphi_j(x)$$

and take

$$\hat{v}^k(x) = \sum_{j=0}^{k} \alpha_j \varphi_j(x), \quad \hat{v}_t^k(x) = \sum_{j=0}^{k} \beta_j \varphi_j(x).$$

For $k = 0,1,2,\ldots$ let $v^k(x,t)$ be the C^∞ solutions of $L(v^k) = 0$ which satisfy the terminal and boundary conditions

$$(6.5) \quad \begin{aligned} & v^k(x,T) \equiv \hat{v}^k(x), \quad v_t^k(x,T) \equiv \hat{v}_t^k(x) \\ & v_x^k(x(s),t)\, A(x(s))\, \eta(s) \equiv 0, \quad (x(s),t) \in \Gamma \otimes [0,T]. \end{aligned}$$

It is known ([6], Chap. VI) that for each fixed t_1, $0 \le t_1 \le T$, the states $(v^k(\cdot,t_1), v_t^k(\cdot,t_1))$ converge to $(v(\cdot,t_1), v_t(\cdot,t_1))$ in the space H_E, in fact this is just a consequence of the energy conservation.

Since

$$\lim_{k \to \infty} \| (\hat{v}, \hat{v}_t) - (\hat{v}^k, \hat{v}_t^k) \|_E = 0 \tag{6.6}$$

and (6.1) is assumed to hold we have

$$\lim_{k \to \infty} \int_{\Omega} \left[\rho\, w_t^f \hat{v}^k + w_x^f A \hat{v}_x^{k\,\prime} \right] dx = 0.$$

Performing a computation similar to that and in the proof of Theorem 1 we conclude that for each fixed $f \in F$

(6.7) $$\lim_{k \to \infty} \int_{\Gamma_1 \otimes [0,T]} v_t^k f \;\; dsdt = 0.$$

Let us now define functions $D^{-j} v$, $j = 1,2,\ldots,$ for $x \in \Omega$ by

$$(D^{-1}v)(x,t) = \int_T^t v(x,\tau)d\tau ,$$

$$(D^{-(j+1)}v)(x,t) = \int_T^t (D^{-j}v)x,\tau)d\tau.$$

We define $D^{-j}v^k$ similarly and verify without difficulty, since the v^k satisfy the evolution equation (2.3), that

(6.8) $$\frac{d^2}{dt^2}(D^{-j}v^k) + B(D^{-j}v^k) = \hat{v}_t^k \frac{(t-T)^{j-1}}{(j-1)!} + \hat{v}^k \frac{(t-T)^{j-2}}{(j-2)!} ,$$

$$(D^{-j}v^k)(T) = (D^{-j}v^k)_t(T) = 0,$$

for $j \geq 2$.

A result proved in [14] (Theorem 1.19, p. 486) shows that the inhomogeneous linear initial value problem

$$\frac{d^2r}{dt^2} + Br = g(t), \quad r(T) = r_t(T) = 0,$$

where $g: [0,T] \to L^2(\Omega)$ is continuously differentiable with respect to t, has a unique solution r such that $r_{tt}(t_1) \in L^2(\Omega)$ and $r(t_1)$ lies in the domain $\Delta(B) \subseteq L^2(\Omega)$ of the unbounded self adjoint operator B for $0 \le t_1 \le T$. Moreover it is also shown in the theorem cited that there are constants M_o, M_1 such that

$$(6.9) \quad \|Br(t_1)\| \le M_o \sup_{0 \le t \le T} \|g(\tau)\| + M_1 \sup_{0 \le \tau \le T} \|g_t(\tau)\|$$

uniformly for all $t_1 \in [0,T]$, where $\| \ \|$ denotes the usual norm in $L^2(\Omega)$, and it is also shown that $Br(t)$ is continuous in t with respect to the norm $\| \ \|$.

Applying this theorem we see that for $j \ge 2$ $D^{-j} v$ lies in $\Delta(B)$ and solves

$$(6.10) \qquad \frac{d^2}{dt^2}(D^{-j}v) + B(D^{-j}v) = \hat{v}_t \frac{(t-T)^{j-1}}{(j-1)!} + \hat{v} \frac{(t-T)^{j-2}}{(j-2)!},$$

$$(D^{-j}v)(T) = (D^{-j}v)_t(T) = 0.$$

Further, (6.9) together with (6.8), (6.10) shows that

$$(6.11) \qquad \lim_{k \to \infty} \|B(D^{-j}v)(\cdot,t) - B(D^{-j}v^k)(\cdot,t)\| = 0$$

uniformly for $0 \le t \le T$.

Now the operator B is uniformly elliptic and we can apply known results from the theory of elliptic boundary value problems ([15], Theorem 9.11, p. 132 and remarks, p. 148) to show that the fact that $D^{-j} v \in \Delta(B)$ together with (6.6) implies that $D^{-j} v(\cdot,t)$ lies in the space $H_2(\Omega)$ (for definition of $H_m(\Omega)$, see [15]) for $0 \le t \le T$, $\|(D^{-j}v)(\cdot,t)\|_{2,\Omega}$ is continuous and uniformly bounded for $0 \le t \le T$, and

$$(6.12) \qquad \lim_{k \to \infty} \| (D^{-j}v)(\cdot, t) - (D^{-j}v^k)(\cdot, t) \|_{2,\Omega} = 0$$

uniformly for $0 \leq t \leq T$, where $\| \quad \|_{2,\Omega}$ is the norm in $H_2(\Omega)$, the sum of the integrals of the squares of the partial derivatives of order ≤ 2.

The theorem of Sobolev ([15], Theorem 3.9, p. 32) states that if $r \varepsilon H_m(\Omega)$ then r can be modified on a set of measure zero so that $r \varepsilon \; C^l(\Omega \cup \Gamma)$, provided ℓ is an integer such that $\ell < \tau - \frac{n}{2}$. For $m = 2$, we have $0 < 2 - \frac{n}{2}$ when $n \leq 3$ so, for such n, $r \varepsilon H_2(\Omega)$ implies $r \varepsilon C^o(\Omega \cup \Gamma)$. Moreover, if $\| \quad \|_s$ denotes the usual "sup" norm in $C^o(\Omega \cup \Gamma)$, we have

$$\| r \|_s \leq \alpha_o \| r \|_{2,\Omega} + \alpha_1 \| r \| .$$

Applying these results with $r = D^{-j}v$, $j \geq 2$, the uniform boundedness and continuity of $\| (D^{-j}v)(\cdot, t) \|_{2,\Omega}$ and (6.11) we conclude that $(D^{-j}v)(x,t)$ is continuous for $(x,t) \varepsilon (\Omega \cup \Gamma) \otimes [0,T]$ and

$$(6.13) \qquad \lim_{k \to \infty} (D^{-j}v^k)(x,t) = (D^{-j}v)(x,t)$$

uniformly for such (x,t).

Having now obtained the continuity of $(D^{-2}v)(x,t)$ we return to (6.7). Integrating by parts three times we conclude that for all $f \varepsilon F$

$$\lim_{k \to \infty} \int_{\Gamma_1 \otimes [0,T]} (D^{-2}v^k) f_{ttt} \, ds \, dt = 0$$

which with (6.13) implies

$$(6.14) \qquad \int_{\Gamma_1 \otimes [0,T]} (D^{-2}v) f_{ttt} \, ds dt = 0, \; f \varepsilon F.$$

Taking account of the fact that f and all its derivatives vanish outside a compact subset of $\Gamma_1 \otimes [0,T]$, (6.14) implies that $(D^{-2}v)(x(s), t)$ is a polynomial in t of degree at most 2 whose coefficients are continuous functions of $x(s)$, for all $(x(s),t) \in \Gamma_1 \otimes [0,T]$.

Let δ be a small positive number. We define the third order difference

$$\Delta^3 r(x,t) = r(x,t + 3\delta) - 3r(x,t + 2\delta) + 3r(x,t + \delta) - r(x,t)$$

for any function r defined on $\Omega \otimes [0,T - 3\delta]$ and possess all smoothness properties of $r(x,t)$. Applying this difference operator to $D^{-2}v$ and $D^{-2}v^k$ we obtain functions

(6.15)
$$\hat{u}(x,t) = \Delta^3 (D^{-2})(x,t).$$
$$u^k(x,t) = \Delta^3 (D^{-2}v^k)(x,t), \; k = 0,1,2,\ldots.$$

From (6.8), (6.10) we see that

$$L(\hat{u}) = \frac{d^2\hat{u}}{dt^2} + B\hat{u} = 0$$

$$L(u^k) = \frac{d^2u^k}{dt^2} + Bu^k = 0,$$

the u^k satisfy the homogeneous boundary conditions

(6.16) $\quad u^k_x(x(s),t)\, A(x(s))\, \eta(s) = 0, \; (x(s),t) \in \Gamma \otimes [0,T-3\delta]$

while $\hat{u}(x,t)$ is continuous, $\hat{u}(\cdot,t)$ lies in $\Delta(B)$ (which means $\hat{u}(x,t)$ satisfies the boundary conditions $\hat{u}_x(x(s),t)A(x(s))\eta(s) = 0$, $(x(s),t) \in \Gamma \otimes [0,T - 3\delta]$ in some sense which we need not specify) and, from the fact that $D^{-2}v$ is a polynomial of degree at most 2 on $\Gamma_1 \otimes [0,T]$, we have

(6.17) $\hat{u}(x(s),t) \equiv 0, \quad (x(s),t) \in \Gamma_1 \otimes [0, T-3\delta]$.

Now we refer back to Theorem 3, or, more precisely, its proof. We let $t_o = 0$, $t_1 = T - 3\delta$ and define the surfaces $S(\lambda)$ as we did there. We define z as we did there and put u^k in place of the function u of Theorem 3. Repeating the calculations following (5.3) we see that

$$\int_{S(0) \cup S(\lambda)} (-u^k z_x A\eta + z u_x^k A\eta + \eta_o \rho u^k z_t - \eta_o \rho z u_t^k)\, d\varphi = 0$$

On $S(0)$ we have $\eta_o(s) \equiv 0$ and $u_x^k A\eta \equiv 0$. Defining α and β as in the proof of Theorem 3 and recalling $z \equiv 0$ on $S(\lambda)$ we have

$$\int_{S(\lambda)} u^k \alpha\beta \, d\varphi = \int_{s(0)} u^k z_x A\eta \, dsdt$$

Now u^k converges uniformly to $\hat{u}$ in $(\Omega \cup \Gamma) \otimes [0, T-3\delta]$, so we have

$$\int_{S(\lambda)} \hat{u}\, \alpha\beta \, d\delta = \int_{S(0)} \hat{u}\, z_x A\eta \, dsdt = 0$$

since $\hat{u}$ obeys (6.17). As in Theorem 3 we conclude that $\hat{u} \equiv 0$ on $S(\lambda)$. A continuation process similar to that described in Theorem 3 can be used to show that $\hat{u} \equiv 0$ on every surface $S(\lambda)$, $0 \leq \lambda \leq 1$. Then, just as in Theorem 3, we conclude that

$$\hat{u}(x,t) \equiv 0, \quad (x,t) \in K(\Gamma_1, 0, T - 3\delta).$$

Now if $T > 2T_o$, we have

$$\Omega \otimes \left[\frac{T}{2} - \varepsilon, \frac{T}{2} + \varepsilon\right] \subseteq K(\Gamma_1, 0, T - 3\delta)$$

if ε and δ are both chosen sufficiently small. (We need $\varepsilon + 3\delta < \frac{T}{2} - T_o$.) Thus we have, for small ε and δ,

$$\hat{u}(x,t) \equiv 0,\ x \varepsilon \Omega,\ \frac{T}{2} - \varepsilon \leq t \leq \frac{T}{2} + \varepsilon . \tag{6.18}$$

Returning to the definition (6.15) we see that if (6.18) holds for all small δ , then it must be true that for $\frac{T}{2} - \varepsilon \leq t \leq \frac{T}{2} + \varepsilon$ $(D^{-2}v)(x,t)$ is a polynomial in t of degree not greater than 2 with coefficients which are functions of x lying in $\Delta(B)$ (since $D^{-2}v \ \varepsilon \ \Delta(B)$). Differentiating $D^{-2}v$ twice with respect to t in $\frac{T}{2} - \varepsilon \leq t \leq \frac{T}{2} + \varepsilon$ we see that there is a function $\tilde{v}(x)$ with $\tilde{v} \ \varepsilon \ \Delta(B)$ such that

$$v(x,t) \equiv \tilde{v}(x),\ x \ \varepsilon \Omega,\ \frac{T}{2} - \varepsilon \leq t \leq \frac{T}{2} + \varepsilon .$$

But if $v(x,t)$ is a generalized solution of $\frac{d^2v}{dt^2} + Bv = 0$ such that $v \ \varepsilon \ \Delta(B)$ and $v(x,t)$ is constant with respect to t then we must have

$$B\ v(x,t) \equiv B\ \tilde{v}\ (x) \equiv 0.$$

But the only elements $\tilde{v} \ \varepsilon \ \Delta(B)$ for which $B\tilde{v} = 0$ are of the form $\tilde{v} \equiv$ const. Therefore, for $\frac{T}{2} - \varepsilon \leq t \leq \frac{T}{2} + \varepsilon$

$$v(x,t) \equiv \tilde{v}(x) \equiv \text{const.},\ v_t(x,t) \equiv 0.$$

Since the energy associated with the generalized solution $v(x,t)$ is constant, we conclude that $v(x,t) \equiv \hat{v}(x) \equiv$ const., $v_t(x,T) \equiv \hat{v}_t(x) \equiv 0$ and the proof of Theorem 4 is complete.

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