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Non-Linear Eigenvalue Problems for Some Fourth Order Equations

II. FIXED POINT METHODS

by

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1. Introduction

Let

1.1) $$L_k[\varphi] \equiv (p_k(t)\varphi')' - C_k(t)\,\varphi(t), \quad k = 1, 2,$$

be two regular Sturm Liouville operators defined on $[0, 1]$. That is

1.2) $$\begin{cases} p_k(t) \in C'[0,1], \quad C_k(t) \in C[0,1], \\ p_k(t) \geq p_0 > 0, \quad C_k(t) \geq 0. \end{cases}$$

Consider the nonlinear system of ordinary differential equations

1.3) $$\begin{cases} L_1[u] = \lambda\theta H_1(t, u, \theta), & 0 < t < 1, \\ L_2[\theta] = \lambda u H_2(t, u, \theta), & 0 < t < 1, \end{cases}$$

where the functions $u(t)$, $\theta(t)$ are required to satisfy the boundary conditions

1.3a) $$\begin{cases} A_0[u] \equiv a_0 u(0) - b_0 u'(0) = 0, \; A_1[u] \equiv a_1 u(1) + b_1 u'(1) = 0, \\ B_0[\theta] \equiv \alpha_0 \theta(0) - \beta_0 \theta'(C) = 0, \; B_1[\theta] \equiv \alpha_1 \theta(1) + \beta_1 \theta 1(1) = 0, \end{cases}$$

with

1.3b) $$\begin{cases} a_k, \alpha_k, b_k, \beta_k \geq 0, \quad k = 1, 2, \\ a_k + b_k > 0, \quad \alpha_k + \beta_k > 0, \quad a_0 + a_1 > 0, \quad \alpha_0 + \alpha_1 > 0. \end{cases}$$

The functions $H_k(t, u, \theta)$ are even, and positive, i.e.

1.3c) $$H_k(t, u, \theta) = H_k(t, |u|, |\theta|) > 0, \quad k = 1, 2 .$$

In a companion paper [12] we studied such problems under a set of assumptions which allowed the iterative construction of a "maximal", "positive", solution. In this report we apply the Schauder fixed-point theorem to obtain (under appropriate hypotheses) the existence of solutions having a specified number of zeros.

This work, and the work described in [12], was motivated by a problem studied by F. Odeh and I. Tadjbakhsh [10] and N. Bazley and B. Zwahlen [1]. These authors consider the nonlinear system

1.4) $$\begin{cases} u'' = \lambda \sin \theta , & 0 < t < 1, \\ \theta'' = \lambda u \cos \theta, & 0 < t < 1, \end{cases}$$

subject to the boundary conditions

A.) $$u'(0) = u(1) = 0, \qquad \theta(0) = \theta'(1) = 0 ,$$

or the boundary conditions.

B.) $$u'(0) = \theta'(0) = \theta(1) = u(1) = 0 .$$

Because of the physical interpretation of the function $(u(t), \theta(t))$ an important condition which was not imposed by these earlier authors is

1.4a) $$|\theta(t)| < \frac{\pi}{2} .$$

The case of the boundary conditions (B) has been discussed in [12]. Thus one of our major aims is to obtain "physical" solutions of (1.4) subject to the boundary conditions (A). This result will follow from the general results obtained here together with a simple construction based on the a priori estimates of [12].

The approach we use here is closely related to our work [11] on sublinear Hammerstein equations and is related to the work of Pimbley [13] and Wolkowisky [14]. Indeed, using the method of [11, lemma 7] these results go over to problem involving pairs of integral equations with oscillation kernels. Nevertheless, at this time, we limit ourselves to the case of differential equations.

In Section 2 we discuss some preliminary ideas relating these problems to the theory of "oscillation" kernels [4] and variational problems. In Section 3 we develop some basic facts of oscillation theory for fourth order problems. Section 4 is devoted to the basic existence theorem. In Section 5 we show how the results of [12] may be used to obtain additional existence theorems. In particular we obtain results which apply to the problem A of Odeh and Tadjbakhsh.

2. Preliminary Notions

In addition to the assumptions (1.3c), we assume that the functions $H_k(t, u, \theta)$ satisfy

P.1) $$H_k(t, u, \theta) \in C[0,1] \times C[-\infty, \infty] \times C[-\infty, \infty], \quad k = 1, 2.$$

P.2) $$0 < a \le H_k(t, u, \theta) \le b, \quad k = 1, 2$$

where a and b are positive constants.

Let

2.1) $$A \equiv \{(q_1(t), q_2(t)) \in C[0,1] \times C[0,1]; \; a \le q_k(t) \le b, \; k = 1, 2\}$$

2.2) $$B \equiv \{(q_1(t), q_2(t)) \in L^2(0,1) \times L^2(0,1); \; a \le q_k(t) \le b, \text{ a.e.}, \; k = 1, 2\}$$

For any pair $(q_1(t), q_2(t)) \in B$ we consider the linear eigenvalue problem

2.3) $$\begin{cases} L_1[u] = \lambda\theta q_1(t); & A_0[u] = A_1[u] = 0, \\ L_2[\theta] = \lambda u q_2(t); & B_0[\theta] = B_1[\theta] = 0. \end{cases}$$

Let $K_1(s,t)$, $K_2(s,t)$ be the Green's functions associated with the operators $-L_1[u]$ and $-L_2[\theta]$ subject to the appropriate homogeneous boundary conditions ($A_j[u] = 0$, $B_j[\theta] = 0$.). Then the equations (2.3) are equivalent to

2.4a) $$u(t) = -\lambda \int_0^1 K_1(t,x)\, q_1(x)\, \theta(x)\, dx,$$

2.4b) $$\theta(x) = -\lambda \int_0^1 K_2(x,y)\, q_2(y)\, u(y)\, dy.$$

Upon substitution, we see that this pair of integral equations is equivalent to either

$$\text{2.5a)}\quad \begin{cases} u(t) = \lambda^2 \int_0^1 G_1(t,s)\, u(s)\, dx , \\ G_1(t,s) = \int_0^1 K_1(t,x)\, q_1(x)\, K_2(x,s)\, q_2(s)\, dx , \end{cases}$$

or

$$\text{2.5b)}\quad \begin{cases} \theta(t) = \lambda^2 \int_0^1 G_2(t,s)\, \theta(s)\, ds , \\ G_2(t,s) = \int_0^1 K_2(t,x)\, K_1(x,s)\, q_1(s)\, q_2(x)\, dx . \end{cases}$$

The kernels $K_j(s,t)$, and therefore the kernels $G_j(s,t)$ (1.), are "oscillation kernels" in the sense of Gantmacher-Krein [4] and hence a great deal is known about their spectrum. In particular, consider equations (2.5a) or (2.5b). The spectrum consists of positive, simple eigenvalues

$$\text{2.6)}\quad 0 < \lambda_0^2 < \lambda_1^2 < \cdots < \lambda_k^2 < \cdots .$$

Moreover, the associated eigenfunctions $\varphi_k(t)$, $k = 0, 1, \cdots$ satisfy the "oscillation" condition. That is, in the open interval $(0, 1)$ $\varphi_k(t)$ has exactly k "nodal" zeros and no other zeros.

Thus, returning to our original problem, we see that the eigenvalues are real and occur in pairs $(\lambda_k, -\lambda_k)$. Indeed, if λ is an eigenvalue with associated eigenfunction $(u(t), \theta(t))$, then $-\lambda$ is an eigenvalue associated with $(-u(t), \theta(t))$.

(1.) The theory developed in [4] is restricted to the symmetric case $G(s,t) = G(t,s)$. However the results required here are valid in the general case. Gantmacher-Krein assert the validity of their results in the general case and cite references to the Russian literature. A discussion of the general case was given by S. Karlin in classroom lectures and will appear in his book [6].

Thus we may restrict ourselves to a consideration of the positive eigenvalues

2.7) $$0 < \lambda_1 < \lambda_2 \cdots$$

If $(u_k(t),\ \theta_k(t))$ is the eigenfunction associated with λ_k, then each function $u_k(t)$ or $\theta_k(t)$ has exactly k interior nodal zeros and no other zeros.

Another useful fact about oscillation kernels which is clearly related to the above remarks is the variation diminishing property. That is, for $f(t) \in C[0,1]$ let $Z(f)$ denote the number of interior nodal zeros of $f(t)$. Let $K(s,t)$ be an oscillation kernel and

2.8a) $$\varphi(s) = \int_0^1 K(s,t)\,\psi(t)\,dt\ ,$$

then

2.8b) $$Z(\varphi) \le Z(\psi)\quad .$$

The representations (2.5a), (2.5b) show that λ_j is a continuous function of $(q_1(t),\ q_2(t)) \in B$, (see [7 , page 213]).

In the special case where $L_1 \equiv L_2$, $A_j \equiv B_j$ the linear eigenvalue problem (2.3) is essentially self adjoint and we know even more about the spectrum. The eigenvalue λ_k are given by the variational characterization of Courant [2], Weyl, Ritz etc. That is,

2.9a) $$\lambda_0^2 = \operatorname*{Max}_{u \not\equiv 0} \frac{\int_0^1 [q_1(t)]^{-1}\,(L_1[u])^2\,dt}{\int_0^1 q_2(t)\,(u(t))^2\,dt}$$

and for $j \geq 1$,

2.9b) $$\lambda_j^2 = \operatorname*{Min}_{S_{j-1}} \operatorname*{Max}_{u \in S_{j-1}^{\perp}} \frac{\int_0^1 [q_1(t)]^{-1} (L_1[u])^2 \, dt}{\int_0^1 q_2(t) \, (u(t))^2 \, dt}$$

where S_k denotes an arbitrary k dimensional subspace of $W_2^2(0,1)$ whose elements satisfy the boundary conditions

2.9c) $$A_0[u] = A_1[u] = 0$$

and $S_k^{\perp}$ denotes the orthogonal complement of S_k in $L_2[(0,1), q_2 \, dt]$, i.e., $\varphi(t) \in S_k^{\perp}$ if (2.9c) holds and

2.9d) $$\int_0^1 q_2(t) \, \varphi(t) \, u(t) \, dt = 0$$

for every $u(t) \in S_k$.

From this basic fact we obtain the following lemma.

<u>Lemma 2.1</u>. Let $(q_1(t;\sigma), q_2(t;\sigma)) \in A$, $0 \leq \sigma \leq \infty$ be a one parameter family of pairs of functions which is continuous in B as a function of σ. Let $\lambda_j(\sigma)$ denote the j'th positive eigenvalue of

2.10) $$\begin{cases} L_1[u] = \lambda \theta q_1(t;\sigma), & A_0[u] = A_1[u] = 0, \\ L_1[\theta] = \lambda u q_2(t;\sigma), & A_0[\theta] = A_1[\theta] = 0. \end{cases}$$

Suppose $\sigma_1 < \sigma_2$ implies

2.11) $$q_j(t,\sigma_1) \le q_j(t,\sigma_2), \quad q_1(t,\sigma_1)\, q_2(t,\sigma_1) \not\equiv q_1(t,\sigma_2)\, q_2(t,\sigma_2) .$$

Then the eigenvalue $\lambda_j(\sigma)$ is a continuous function of σ and

2.12) $$\lambda_j(\sigma_1) > \lambda_j(\sigma_2) .$$

Moreover, for each j, there exist two positive constants Λ_j and M_j such that

2.13) $$0 < \Lambda_j \le \lambda_j(q_1, q_2) \le M_j$$

for all $(q_1, q_2) \in B$.

3. Linear Problems - Oscillation Theory

In this section we develop some further properties of system (2.3). Our fundamental tool is an extension of some basic results of W. Leighton and Z. Nehari [9] .

Let $\lambda > 0$ be a fixed constant and let $(q_1(t),\ q_2(t)) \in A$ and consider the linear differential equation

$$
(3.1)\qquad \begin{cases} L_1[u] = \lambda\theta q_1(t)\,, & 0 < t < 1 \\ L_2[\theta] = \lambda u q_2(t)\,, & 0 < t < 1 \end{cases}
$$

Lemma 3.1. Let $(u(t),\ \theta(t))$ be a solution of equation (3.1) and let $a \in [0, 1)$. If $u(a)$, $u'(a)$, $\theta(a)$, $\theta'(a)$ are nonnegative (but not all zero), then the functions $u(x)$, $u'(x)$, $\theta(x)$, $\theta'(x)$ are all positive for $a < x \le 1$.

Proof: In the case where

$$
L_1[u] = L_2[u] \equiv u''
$$

this result is lemma 2.1 of [9]. In the general case we use the representations (Volterra integral equations)

$$
(3.2a)\qquad \begin{cases} \theta(s) = \theta(a) + p_2(a)\ \theta'(a) \displaystyle\int_a^s \frac{dx}{p_2(x)} + \lambda \int_a^s \frac{dx}{p_2(x)} \int_a^x u(t)\, q_2(t)\, dt \\ \qquad + \displaystyle\int_a^s \frac{dx}{p_2(x)} \int_a^x C_2(t)\ \theta(t)\ dt\,, \end{cases}
$$

$$
3.2b) \quad \begin{cases} u(s) = u(a) + p_1(a)\, u'(a) \displaystyle\int_a^s \frac{dx}{p_1(x)} + \lambda \int_a^s \frac{dx}{p_1(x)} \int_a^x \theta(t) q_1(t)\, dt \\ \quad + \displaystyle\int_a^s \frac{dx}{p_1(x)} \int_a^x C_1(t)\, u(t)\, dt \, . \end{cases}
$$

Case 1. $\theta(a) + \theta'(a) > 0$. There is an interval $(a, a+\delta)$ in which $\theta(t)$ is positive. Let us assume $\theta(t)$ is known and use equation (3.2b) to obtain $u(t)$ in this interval. Since we are dealing with a Volterra integral equation we may use Picard iterations with $u_0(t) \equiv u(a)$. A straightforward induction shows that $u_n(t)$ is positive on $(a, a+\delta)$ and hence

$$
u(t) \geq 0, \quad a < t < a + \delta \, .
$$

Using this result in equation (3.2a) we see that

$$
\theta(a + \delta) > 0 \, .
$$

Hence $\theta(t)$, $u(t)$ are (strictly) positive for $t \in (a, 1]$. Using the representations (3.1a) and (3.1b) we see that $\theta'(t)$ and $u'(t)$ are also positive for $t \in (a, 1]$.

Case 2. $u(0) + u'(0) > 0$. A similar argument (reversing the roles of u and θ) completes the proof in this case.

Lemma 3.2. Let $(u(t), \theta(t))$ be a solution of equation (3.1) and let $a \in (0, 1]$. Suppose $u(a) \geq 0$, $\theta(a) \geq 0$ while $u'(a) \leq 0$, $\theta'(a) \leq 0$ (but not all zero). Then for $t \in [0, a)$ we have

$$
u(t) > 0, \quad \theta(t) > 0, \quad u'(t) < 0, \quad \theta'(t) < 0 \, .
$$

Proof: As in the proof of Lemma 2.2 of [9], we let $s = 1 - t$ and apply Lemma 3.1.

Lemma 3.3. Let $(u(t), \theta(t))$ be a nontrivial solution of equation (3.1) and let $a \in (0, 1)$. Suppose either

$$u(a) = u'(a) = 0$$

or

$$\theta(a) = \theta'(a) = 0 \ .$$

Then, in (at least) one of the two intervals $[0, a)$, $(a, 1]$ all four functions $u(t)$, $u'(t)$, $\theta(t)$, $\theta'(t)$ are different from zero.

Proof: This result follows from the two preceding lemmas exactly as in [9].

These rather elementary results are the basis of some interesting theorems on the "continuity" of the spectrum of equation (2.3) which are stronger than the results mentioned earlier ([7 , page 213]).

Lemma 3.4. Suppose there is a sequence $(q_1^{(k)}(t), q_2^{(k)}(t)) \in A$, $(k = 1, 2, \cdots)$ and functions $(\bar{q}_1(t), \bar{q}_2(t)) \in B$, such that

$$q_j^{(k)} \rightharpoonup \bar{q}_j(t) \text{ weakly in } L_2[0,1] \text{ as } k \to \infty, \quad j = 1, 2. \tag{3.3}$$

Let $(u_n^{(k)}(t), \theta_n^{(k)}(t))$ and $\lambda \sigma_n^k$ be the n'th eigenfunction and n'th positive eigenvalue of

$$3.4)\quad \begin{cases} L_1[u_n^{(k)}] = \lambda\sigma_n^{(k)}\,\theta_n^{(k)}\,q_1^k(t), & A_0[u_n^{(k)}] = A_1[u_n^{(k)}] = 0\,, \\ L_2[\theta_n^{(k)}] = \lambda\sigma_n^{(k)}\,u_n^{(k)}\,q_2^k(t), & B_0[\theta_n^{(k)}] = B_1[\theta_n^{(k)}] = 0\,. \end{cases}$$

Suppose there is a positive constant $\overline{\sigma}$ such that

$$3.4a)\quad \sigma_n^{(k)} \rightarrow \overline{\sigma} \quad \text{as} \quad k \rightarrow \infty\,.$$

Finally, suppose there are functions $\overline{u}(t), \overline{\theta}(t) \in C^1(0,1]$ such that

$$3.4b)\quad \begin{cases} u_n^{(k)}(t) \rightarrow \overline{u}(t) & \text{in } C^1[0,1] \text{ as } k \rightarrow \infty\,, \\ \theta_n^{(k)}(t) \rightarrow \overline{\theta}(t) & \text{in } C_1[0,1] \text{ as } k \rightarrow \infty\,. \end{cases}$$

Then the functions $(\overline{u}(t), \overline{\theta}(t))$ are the n'th eigenfunction associated with the n'th (positive) eigenvalue $\lambda\overline{\sigma}$ of

$$3.5)\quad \begin{cases} L_1[\overline{u}] = \lambda\overline{\sigma}\,\overline{\theta}\,q_1, \text{ a, e;} & A_0[\overline{u}] = A_1[\overline{u}] = 0\,, \\ L_2[\theta] = \lambda\overline{\sigma}\,\overline{u}\,\overline{q}_2, \text{ a, e;} & B_0[\theta] = B_0[\overline{\theta}] = 0\,. \end{cases}$$

<u>Proof</u>: While the functions $(\overline{q}_1(t), \overline{q}_2(t))$ need not be continuous, they belong to B. Moreover, the functions $\overline{u}(t), \overline{\theta}(t)$ are weak solutions of equations (3.5). Hence, strong solutions. Thus, as in the development of equations (2.5a), (2.5b) we see that $\overline{u}(t)$ and $\overline{\theta}(t)$ are separately eigenfunctions of a linear integral equation whose kernel is an oscillation kernel. Thus each has only a finite number of interior zeros in $(0,1)$ and each such interior zero is a nodal zero. Let N be the number of interior zeros.

Because of the $C^1[0,1]$ convergence there is a k_0 such that $k \geq k_0$ implies that $u_n^{(k)}(t)$ has **at least** N interior nodal zeros. Since each $u_n^{(k)}(t)$ has exactly n interior nodal zeros we have

3.6) $$N \leq n.$$

Let $n \geq 1$ and let

$$0 < \xi_1^{(k)} < \xi_2^{(k)} < \cdots < \xi_n^{(k)} < 1$$

be the n interior zeros of $u_n^{(k)}(t)$. There is a subsequence (k') and a set of values

$$0 \leq \bar{\xi}_1 \leq \bar{\xi}_2 \leq \cdots \leq \bar{\xi}_n \leq 1$$

such that

3.7) $$\xi_j^{(k')} \to \bar{\xi}_j \quad \text{as} \quad k' \to \infty .$$

If $N < n$ then either there is a pair

3.8a) $$0 < \bar{\xi}_j = \bar{\xi}_{j+1} < 1$$

or

3.8b) $$\bar{\xi}_1 = 0 ,$$

or

3.8c) $$\bar{\xi}_n = 1 .$$

However, if (3.8a) occurs then $\bar{u}(t)$ has a double zero at $\bar{\xi}_j$. If

$$\bar{\theta}(\bar{\xi}_j)\ \bar{\theta}'(\bar{\xi}_j) \geq 0$$

then (because of the linearity) we may take

$$\bar{\theta}(\bar{\xi}_j) \geq 0, \quad \bar{\theta}'(\bar{\xi}_j) \geq 0$$

and then Lemma 3.1 contradicts the boundary conditions at $t = 1$. On the other hand, if

$$\bar{\theta}(\bar{\xi}_j) \geq 0, \quad \bar{\theta}'(\bar{\xi}_j) \leq 0$$

then the boundary conditions at $t = 0$ and Lemma 3.2 lead to a contradiction.

If (3.8b) occurs as a result of $\xi_1^{(k')} \to 0$ then (because of the boundary condition at $t = 0$) $\bar{u}(t)$ has a double zero at $t = 0$. However, we also have

$$\bar{\theta}(0)\, \bar{\theta}'(0) \geq 0 .$$

Hence, the boundary conditions at $t = 1$ and Lemma 3.1 lead to a contradiction.

A similar argument disposes of the case (3.8c).

This result leads us to consider another basic assumption.

P.3) For every fixed n there are constants A_n, B_n such that λ_n, the $n^{'th}$ positive eigenvalue of the linear eigenvalue problem (2.3), satisfies

3.9) $$0 < A_n \leq \lambda_n \leq B_n$$

for all $(q_1(t), q_2(t)) \in A$.

Theorem 3.1. Suppose P.3 holds. Let $(q_1^{(k)}(t), q_2^{(k)}(t)) \in A$ for $k = 1, 2, \cdots$. Suppose there are two functions $(\bar{q}_1(t), q_2(t))$ such that

3.10) $q_j^{(k)}(t) \longrightarrow \bar{q}_j(t)$ weakly in $L_2(0, 1)$, $k \to \infty$, $j = 1, 2$.

The convexity of A implies that $(\bar{q}_1, \bar{q}_2) \in B$.

Let $\lambda_n^{(k)}$ and $(u_n^{(k)}(t), \theta_n^{(k)}(t))$ be the n'th (positive) eigenvalue and the corresponding eigenfunction of equations (2.3) with $q_j(t)$ replaced by $q_j^{(k)}(t)$. Let $(u_n^{(k)}(t), \theta_n^{(k)}(t))$ be normalized so that

$$\max (\| \theta_n^{(k)} \|_\infty, \| u_n^{(k)} \|_\infty) = 1$$

where

$$\| f \|_\infty = \max \{ |f(t)|, \quad 0 \le t \le 1 \}.$$

Let $\bar{\lambda}_n$ and $(\bar{u}(t), \bar{\theta}(t))$ be the n'th (positive) eigenvalue and corresponding eigenfunction of equations (2.3) with $q_j(t)$ replaced by $\bar{q}_j(t)$. Then

*) $$\begin{cases} \lambda_n^{(k)} \to \bar{\lambda}, \\ u_n^{(k)}(t) \to \bar{u}(t) \quad \text{in} \quad C'[0,1], \\ \theta_n^{(k)}(t) \to \bar{\theta}(t) \quad \text{in} \quad C'[0,1]. \end{cases}$$

<u>Proof</u>: There is a subsequence (k') and a constant μ and two functions $U(t)$, $\Theta(t)$ such that

$$\lambda_n^{(k')} \to \mu$$

$$u_n^{(k)}(t) \to U(t) \quad \text{in} \quad C'[0,1],$$

$$\theta_n^{(k)}(t) \to \Theta(t) \quad \text{in} \quad C'[0,1].$$

On applying Lemma 3.3 we see that

$$\mu = \bar{\lambda}$$

$$U(t) = \bar{u}(t), \quad \Theta(t) = \bar{\theta}(t) .$$

A straightforward argument based on the uniqueness of the quantities $\bar{\lambda}$, $\bar{u}(t)$, $\bar{\theta}(t)$ shows that the entire sequence converges.

Having established this result, one is naturally led to the question: when does P.3 hold?? Clearly, lemma 2.1, the variational characterization of $\bar{\lambda}_n$ given by equation (2.9b), asserts that P.3 holds in the symmetrizable case. It seems reasonable to conjecture that P.3 always holds. However, we have not established this assertion. On the other hand, the methods of this section may be used to establish this fact for certain cases. These results are presented in an appendix. We note that the case (A) of Odeh and Tadjbakhsh is included.

4. The Basic Existence Theorem

In this section we return to our original nonlinear problem (1.3), (1.3a). We assume that P.1, P.2, and P.3 hold.

Let

4.1)
$$\begin{cases} h_k(t) = H_k(t, 0, 0), & k = 1, 2. \\ g_k(t) = H_k(t, \infty, \infty), & k = 1, 2. \end{cases}$$

Let λ_j, $j = 0, 1, \ldots$ denote the positive eigenvalues of the linear eigenvalue problem

4.2a)
$$\begin{cases} L_1[v] = \lambda\phi\, h_1(t), & A_0[v] = A_1[v] = 0, \\ L_2[\phi] = \lambda v\, h_2(t), & B_0[\phi] = B_1[\phi] = 0. \end{cases}$$

Let μ_j, $j = 0, 1, \ldots$ denote the positive eigenvalues of the linear eigenvalue problem

4.2b)
$$\begin{cases} L_1[w] = \mu\psi g_1(t), & A_0[w] = A_1[w] = 0, \\ L_2[\psi] = \mu w g_2(t), & B_0[\psi] = B_1[\psi] = 0. \end{cases}$$

Naturally, we assume

4.2c)
$$\lambda_j < \lambda_{j+1}; \quad \mu_j < \mu_{j+1}.$$

Let $\lambda > 0$ be fixed. Let $(q_1(t), q_2(t)) \in A$.

Let

(4.3)
$$\begin{cases} \sigma_n = \sigma_n(q_1, q_2) \\ U_n(t) = U_n(t; q_1, q_2) \\ \Theta_n(t) = \Theta_n(t; q_1, q_2) \end{cases}$$

denote the n'th positive eigenvalue and eigenfunction respectively of the linear eigenvalue problem

(4.4a)
$$\begin{cases} L_1[U_n] = \lambda\sigma_n \Theta_n q_1(t), \quad A_0[U_n] = A_1[U_n] = 0\,, \\ L_2[\Theta_n] = \lambda\sigma_n U_n q_2(t), \quad B_0[\Theta_n] = B_1[\Theta_n] = 0\,, \end{cases}$$

normalized so that

4.4b)
$$\max\{\|U_n\|_\infty, \|\Theta_n\|_\infty\} = 1\ .$$

Note: Since the eigenvalues are all simple, this normalization determines (U_n, Θ_n) up to sign.

Remark: Each function $U_n(t)$, $\Theta_n(t)$ has exactly n nodal zeros in $(0,1)$ and no other interior zeros.

Given $(q_1, q_2) \in A$, and hence (U_n, Θ_n), let $\alpha \in (0,\infty)$ and let

4.5a)
$$\begin{cases} \rho_n = \rho_n(q_1, q_2, \alpha) \\ V_n(t) = V_n(t, q_1, q_2, \alpha) \\ \Psi_n(t) = \Psi_n(t, q_1, q_2, \alpha) \end{cases}$$

denote the n'th positive eigenvalue and eigenfunction respectively of the linear eigenvalue problem

4.5b)
$$\begin{cases} L_1[V_n] = \lambda \rho_n \Psi_n H_1(t, \alpha U_n(t), \alpha \Theta_n(t)), \\ L_2[\Psi_n] = \lambda \rho_n V_n H_2(t, \alpha U_n(t), \alpha \Theta_n(t)), \\ A_0[V_n] = A_1[V_n] = B_0[\Psi_n] = B_1[\Psi_n] = 0, \end{cases}$$

normalized so that

4.5c)
$$\max \{\|V_n\|_\infty \; \|\Psi_n\|_\infty\} = 1 .$$

Note: Because the functions $H_k(t, u, \theta)$ are even, the functions $H_k(t, \alpha U_n(t), \alpha \Theta_n(t))$ are well defined.

Lemma 4.1: The quantities $U_n(t; q_1, q_2)$, $\Theta_n(t; q_1, q_2)$, $H_1(t, \alpha U_n, \alpha \Theta_n)$, $H_2(t, \alpha U_n, \alpha \Theta_n)$, $\rho_n(q_1, q_2, \alpha)$, $V_n(t, q_1, q_2, \alpha)$, $\Psi_n(t, q_1, q_2, \alpha)$ are continuous functions of (q_1, q_2, α) in the following sense. If

$$\alpha_n^{(k)} \to \bar{\alpha} < \infty \quad \text{as } k \to \infty$$

$$q_j^{(k)}(t) \rightharpoonup \bar{q}_j(t) \in B, \quad \text{weakly in } L_2(0,1) \text{ as } k \to \infty$$

then

4.6a)
$$\begin{cases} U_n^{(k)}(t) = U_n(t; q_1^{(k)}, q_2^{k}) \Longrightarrow U_n(t; \bar{q}_1, \bar{q}_2) \text{ uniformly on } [0,1] \\ \Theta_n^{(k)}(t) = \Theta_n(t; q_1^{(k)}, q_2^{(k)}) \Longrightarrow \Theta_n(t; \bar{q}_1, \bar{q}_2) \text{ uniformly on } [0,1] \end{cases}$$

4.6b)
$$H_j(t, \alpha^{(k)} U_n^{(k)}, \alpha^{(k)} \Theta_n^{(k)}) \Longrightarrow H_j(t, \bar{\alpha} U_n, \bar{\alpha} \Theta_n) \text{ uniformly on } [0,1]$$

$$\rho_n(q_1^{(k)}, q_2^{(k)}, \alpha^{(k)}) \to \rho(\bar{q}_1, \bar{q}_2, \bar{\alpha})$$

4.6c)
$$\begin{cases} V_n(t;q_1^{(k)}, q_2^{(k)}, \alpha^{(k)}) \Longrightarrow V_n(t, \overline{q}_1, \overline{q}_2, \overline{\alpha}) \text{ uniformly on } [0,1], \\ \Psi_n(t;q_1^{(k)}, q_2^{(k)}, \alpha^{(k)}) \Longrightarrow \Psi_n(t, \overline{q}_1, \overline{q}_2, \overline{\alpha}) \text{ uniformly on } [0,1]. \end{cases}$$

Proof: Apply theorem 3.1.

Lemma 4.2. Suppose

4.7)
$$\lambda_n < \lambda < \mu_n$$

Let $(q_1(t), q_2(t)) \in A$. Then there is at least one value of $\alpha \in (0,\infty)$ such that

$$\rho_n(q_1, q_2, \alpha) = 1$$

Proof: By Lemma 4.1, for fixed $(q_1, q_2) \in A$, $\rho_n(q_1, q_2; \alpha)$ is a continuous function of α. The lemma follows from the observation that

4.8a)
$$\lim_{\alpha \to 0} \rho_n(q_1, q_2, \alpha) = \lambda_n/\lambda < 1$$

and

4.8b)
$$\lim_{\alpha \to \infty} \rho_n(q_1, q_2, \alpha) = \mu_n/\lambda > 1$$

Lemma 4.3: Let (4.7) hold. There is a positive constant $\alpha_1 > 0$ such that, for all $(q_1(t), q_2(t)) \in A$ and all $\alpha \in (0,\alpha_1)$ we have

4.9)
$$\rho_n(q_1, q_2, \alpha) < 1 .$$

Proof: Assume the lemma is false. Using the continuity of $\rho_n(q_1, q_2, \alpha)$ and condition (4.8a), we may assume that there is a sequence $(q_1^{(k)}(t); q_2^{(k)}(t)) \in A$ and a sequence $\alpha^{(k)} \in (0,\infty)$ such that

4.10a) $$\alpha^{(k)} \to 0 \quad \text{as} \quad k \to \infty$$

4.10b) $$\rho_n(q_1^{(k)}, q_2^{(k)}, \alpha^{(k)}) = 1 \quad \text{for all} \quad k = 1, 2, \ldots$$

However, we may extract a subsequence (k') and a pair of functions $(\bar{q}_1(t), \bar{q}_2(t)) \in B$ so that

4.11) $$q_j^{(k')}(t) \longrightarrow \bar{q}_j(t), \quad \text{weakly in } L_2(0,1), \; u = 1, 2.$$

Applying Lemma 4.1 or Theorem 3.1 we see that

$$\lim_{k' \to \infty} \rho_n(q_1^{(k')}, q_2^{(k')}, \alpha^{(k')}) = \frac{\lambda_n}{\lambda} < 1$$

which contradicts equation (4.10b).

<u>Lemma 4.4</u>: Let (4.7) hold. There is a finite positive constant α_2, such that, for all $(q_1(t), q_2(t)) \in A$ and all $\alpha \in (\alpha_2, \infty)$ we have

4.12) $$\rho_n(q_1, q_2, \alpha) > 1 .$$

<u>Proof:</u> Assume the lemma is false. Using Lemma 4.1 and condition (4.8b) we may assume that there is a sequence $(q_1^{(k)}(t), q_2^{(k)}(t)) \in A$ and a sequence $\alpha^{(k)} > 0$ such that

4.13a) $$\alpha^{(k)} \to \infty \quad \text{as} \quad k \to \infty$$

4.13b) $$\rho_n(q_1^{(k)}, q_2^{(k)}, \alpha^k) = 1 \; , \quad k = 1, 2, \ldots \; .$$

Using Lemma 4.1, and extracting enough subsequences we may also assume that there are two functions $(\bar{q}_1(t), \bar{q}_2(t)) \in B$ so that

$$\begin{cases} q_j^{(k)}(t) \longrightarrow \bar{q}_j(t) \quad \text{weakly in } L_2(0,1), \quad j = 1, 2, \\ U_n^{(k)}(t) = U_n(t; q_1^{(k)}, q_2^k) \Longrightarrow U_n(t, \bar{q}_1, \bar{q}_2) \quad \text{uniformly}, \\ \Theta_n^{(k)}(t) = U_n(t; q_1^{(k)}, q_2^{(k)}) \Longrightarrow \Theta_n(t, \bar{q}_1, \bar{q}_2) \quad \text{uniformly}. \end{cases}$$

Then

$$H_j(t; \alpha^{(k)} U_n^{(k)}(t), \alpha^{(k)} \Theta_n^k(t)) \to H_j(t, \infty, \infty) = g_j(t), \quad j = 1, 2$$

uniformly on all closed intervals not containing the zeros (at most 2n) of

$$U_n(t; \bar{q}_1, \bar{q}_2) \Theta_n(t; \bar{q}_1, \bar{q}_2) .$$

Thus, this convergence is $L_2(0,1)$ convergence and

$$\rho_n(q_1^{(k)}, q_2^{(k)}, \alpha^{(k)}) \to {}^{\mu_n}/_{\lambda} > 1$$

which contradicts (4.13b).

<u>Theorem 4.1:</u> Let inequality (4.7) hold. Let A_n, B_n be the constants of (3.9) describing P.3. Let

4.14a) $$\alpha_0 = \frac{\lambda}{B_n} \alpha_1 ,$$

4.14b) $$\alpha_3 = \frac{\lambda}{A_n} \alpha_2 .$$

Then

4.15) $$\alpha_0 < \alpha_1 < \alpha_2 < \alpha_3 .$$

Also, let F be a mapping defined on

$$S \equiv A \times [\alpha_0, \alpha_3]$$

by

4.16) $$F(q_1, q_2, \alpha) = \{H_1(t, \alpha U_n, \alpha \Theta_n), H_2(t, \alpha U_n, \alpha \Theta_n), \frac{\alpha}{\rho_n(q_1, q_2, \alpha)}\} .$$

Then F has a fixed-point $(\bar{q}_1, \bar{q}_2, \bar{\alpha})$. Finally let

$$\bar{\alpha}\, U_n(t, \bar{q}_1, \bar{q}_2) = u(t) \quad ,$$

$$\bar{\alpha}\, \Theta_n(t, \bar{q}_1, \bar{q}_2) = \theta(t) \ .$$

Then $(u(t), \theta(t))$ is a solution of equations (1.3), (1.3a) and each function $u(t)$ or $\theta(t)$ has exactly n interior nodal zeros in $(0, 1)$.

<u>Proof</u>: The inequalities (4.15) follow immediately from the inequality (4.7). By Lemma 4.1, the mapping is continuous. Clearly, S is convex. Moreover, standard estimates, together with the continuity of $H_j(t, u, \theta)$ show that F is compact. Finally we will show that F maps S into S. Clearly,

$$(H_1(t, \alpha U_n, \alpha \Theta_n), \ H_2(t, \alpha U_n, \alpha \Theta_n)) \in A.$$

Thus we need only show that

4.17) $$\alpha_0 \le \frac{\alpha}{\rho_n(q_1, q_2, \alpha)} \le \alpha_3 \quad .$$

If $\alpha \in [\alpha_0, \alpha_1]$, then from (4.9) we see that

4.18a) $$\frac{\alpha}{\rho_n(q_1, q_2, \alpha)} \ge \alpha_0 \ .$$

From (3.9) we have, $\alpha \in [\alpha_0, \alpha_2]$ implies

4.18b) $$\frac{\alpha}{\rho_n(q_1, q_2, \alpha)} \leq \frac{\alpha_2 \lambda}{A_n} = \alpha_3 .$$

If $\alpha \in [\alpha_1, \alpha_3]$ then (3.9) implies that

4.18c) $$\alpha_0 = \frac{\alpha_1 \lambda}{B_n} \leq \frac{\alpha}{\rho_n(q_1, q_2, \alpha)} .$$

Finally, if $\alpha \in [\alpha_2, \alpha_1]$, then (4.12) implies that

4.18d) $$\frac{\alpha}{\rho_n(q_1, q_2, \alpha)} \leq \alpha_3 .$$

The inequalities (4.18a), (4.18b), (4.18c), (4.18d) show that F maps S into S.

The Schauder fixed-point theorem [3] asserts the existence of a fixed point, $(\bar{q}_1(t), \bar{q}_2(t), \bar{\alpha})$. Then

4.19) $$\rho_n(\bar{q}_1, \bar{q}_2, \alpha) = 1 .$$

Moreover, using equations (4.4a) and (4.4b) together with the fact that

$$H_j(t, \bar{\alpha} U_n, \bar{\alpha} \Theta_n) = \bar{q}_j(t), \quad j = 1, 2$$

we see that

$$U_n(t) = V_n(t), \ \Theta_n(t) = \Psi_n(t)$$

and the functions $u(t), \theta(t)$ satisfy equations (1.3),(1.3a) .

5. Modified Problems

In many cases of interest Theorem 4.1 cannot be applied directly. For example, condition P.2 does not hold in the case of "cut-off" problems discussed in [12]. In general, condition P.2 does not hold when

$$\lim_{\substack{|u|\to\infty \\ |\theta|\to\infty}} H_1(t,u,\theta)\ H_2(t,u,\theta) = 0 .$$

However, it often happens that one may modify the functions $H_k(t, u, \theta)$ for large u, θ without changing the set of solutions, and the modified problem satisfies all the hypotheses of Theorem 4.1. We turn our attention to a special class of such problems.

Following [12] we assume

H.1) There is a constant $\Theta > 0$ (which may be $+\infty$) and in the region $R \equiv \{(t, u, \theta);\ 0 \le t \le 1,\ |u| < \infty,\ |\theta| < \Theta\}$ the function

$$F_1(t, u, \theta) = \theta\, H_1(t, u, \theta)$$

is monotone nondecreasing in θ while the function

$$F_2(t, u, \theta) = uH_2(t, u, \theta)$$

is monotone nondecreasing in u. We write

$$\text{5.1)} \quad \begin{cases} \dfrac{\partial}{\partial\theta} F_1(t, u, \theta) = H_1(t, u, \theta) + \theta \dfrac{\partial}{\partial\theta} H_1(t, u, \theta) \ge 0 , \\[2ex] \dfrac{\partial}{\partial u} F_2(t, u, \theta) = H_2(t, u, \theta) + u \dfrac{\partial}{\partial u} H_2(t, u, \theta) \ge 0 . \end{cases}$$

H.3) The functions $H_k(t, u, \theta)$ are monotone nonincreasing in $|u|$, $|\theta|$. That is

5.2) $$u \frac{\partial}{\partial u} H_k(t, u, \theta) \le 0, \quad \theta \frac{\partial}{\partial \theta} H_k(t, u, \theta) \le 0, \quad k = 1, 2 .$$

However, the system (1.3), (1.3a) should be genuinely "nonlinear". Hence, in addition to (5.2) we assume if $\bar{u}$, $\bar{\theta}$ are positive and C is a constant with $C > 1$, then

5.2a) $$H_k(t, C\bar{u}, C\bar{\theta}) < H_k(t, \bar{u}, \bar{\theta}), \quad k = 1, 2, \quad C\bar{\theta} < \Theta .$$

H.4) There are four positive constants M, U_0, Θ_0, α with $0 < \alpha < 1$, $0 < \Theta_0 < \Theta$ such that

5.3a) $$K_j(t, s)\, H_j(t, u(s), \theta(s)) \le M, \quad j = 1, 2$$

5.3b) $$\lambda^2 \int_0^1 K_1(s, t)\, K_2(x, s)\, H_1(t, u(t), \theta(t))\, H_2(s, \hat{u}(s), \hat{\theta}(s))\, ds \le \alpha$$

for all functions $u(x)$, $\theta(x)$, $\hat{u}(x)$, $\hat{\theta}(x)$ which satisfy

5.3c) $$\begin{cases} U_0 \le |u(x)|, \ |\hat{u}(x)|, & 0 \le x \le 1 , \\ \Theta_0 \le |\theta(x)|, \ |\hat{\theta}(x)| \le \Theta, & 0 \le x \le 1 . \end{cases}$$

<u>Lemma 5.1.</u> Let $\lambda_0 < \lambda$, Let H.1, H.3 and H.4 hold. Then there exists a unique maximal, positive solution $(u(t), \theta(t))$ of equations (1.3), (1.3a) which satisfies

5.4) $$|\theta(t)| < \Theta .$$

That is, $(u(t), \theta(t))$ satisfy (5.4), equations (1.3), (1.3a) and

5.5) $$u(t) < 0 < \theta(t), \qquad 0 < t < 1.$$

Moreover, if $(v(t), \phi(t))$ is any other nontrivial solution

5.6) $$\begin{cases} |v(t)| \le -u(t) = |u(t)|, \\ |\phi(t)| \le \theta(t) = |\theta(t)| . \end{cases}$$

Finally, if $\phi(t)$ has no interior zeros, then

$$\phi(t) = \pm\theta(t).$$

<u>Proof</u>: See Theorem 4.3 and Theorem 5.3 of [12].

<u>Theorem 5.1.</u> Let

5.7) $$\lambda_n < \lambda$$

and H.1, H.3, H.4 hold. Let $(u(t), \theta(t))$ be the maximal, positive solution whose existence is asserted in Lemma 5.1. Suppose there are two positive constants $U_1, \Theta_1 \ge \Theta_0$ such that

5.8) $$\begin{cases} |u(t)| \le U_1, \quad 0 \le t \le 1, \\ \theta(t) \le \Theta_1 < \Theta, \quad 0 \le t \le 1 . \end{cases}$$

Suppose also that there are two functions $G_k(t, u, \theta)$, $k = 1, 2$ such that

5.9) $$H_k(t, u, \theta) = G_k(t, u, \theta), \quad |u| \le U_1, \quad |\theta| \le \Theta_1$$

and H.1, H.3 hold for all (u, θ) when $G_k(t, u, \theta)$ is substituted for

$H_k(t, u, \theta)$. In that case H.4 also holds. Finally, suppose that P.1, P.2, P.3 hold when $G_k(t, u, \theta)$ is substituted for $H_k(t, u, \theta)$. Then there exist at least n distinct solutions $(u_j(t), \theta_j(t))$ $j = 0, 1, \ldots, n$, of equation (1.3), (1.3a). The function $(u_j(t), \theta_j(t))$ are characterized by the fact that each functions has exactly j interior nodal zeros in (0, 1). Of course, $(u_0(t), \theta_0(t)) = (u(t), \theta(t))$.

Proof: Consider equations (1.3), (1.3a) with $H_k(t, u, \theta)$ replaced by $G_k(t, u, \theta)$. Let $\{\mu_j\}$ be the eigenvalues of equation (4.2b) when

$$g_k(t) = G_k(t, \infty, \infty).$$

The integral representation (2.5b) and (5.3b) together with (5.7) imply

$$\lambda_j \le \lambda_n < \lambda < \mu_0 \le \mu_j \quad , \qquad j = 0, 1, \ldots, n.$$

Thus, Theorem 4.1 asserts the existance of a solution $(u_j(t), \theta_j(t))$ of this modified problem which is characterized by the fact that

$$Z(u_j) = Z(\theta_j) = j, \qquad j = 0, 1, \ldots, n \quad .$$

However, using Lemma 5.1 we see that all of these functions satisfy

$$|u_j(t)| \le |u(t)| \le U_1 \quad ,$$

$$|\theta_j(t)| \le \theta(t) \le \Theta_1 \quad .$$

Using (5.9), we see that $(u_j(t), \theta_j(t))$ is also a solution of the original problem.

Having obtained this result, we are naturally led to ask: "When can we construct the functions $G_k(t, u, \theta)$?"

<u>Theorem 5.2.</u> Suppose $H_k(t, u, \theta)$ satisfy H.1, H.3 and H.4 for

5.10) $$0 \le u \le U_1 , \quad 0 \le \theta \le \Theta_1 < \Theta ,$$

where U_1 and Θ_1 satisfy (5.8). Suppose there exists an $\varepsilon_0 > 0$ such that, when $t \in [0, 1]$ and (5.10) holds,

5.11) $$\begin{cases} \dfrac{\partial}{\partial u} H_k(t, u, \theta) + \varepsilon_0 \dfrac{\partial}{\partial u} \dfrac{\partial}{\partial \theta} H_k(t, u, \theta) \le 0, & k = 1, 2 , \\[2ex] \dfrac{\partial}{\partial \theta} H_k(t, u, \theta) + \varepsilon_0 \dfrac{\partial}{\partial u} \dfrac{\partial}{\partial \theta} H_k(t, u, \theta) \le 0, & k = 1, 2 . \end{cases}$$

Let

5.12a) $$\gamma_1(t, \theta) = \frac{\frac{\partial}{\partial \theta} H_1(t, U_1, \Theta_1)}{H_1(t, u, \Theta_1)} ,$$

and

5.12b) $$\gamma_2(t, \theta) = \frac{\frac{\partial}{\partial u} H_2(t, U_1, \theta)}{H_2(t, U_1, \theta)} .$$

Suppose

5.13a) $$\rho_1 \equiv \min_{0 \le u \le U_1} \{1 + \Theta_1 \gamma_1(t, u)\} > 0$$

and

5.13b) $$\rho_2 \equiv \min_{0 \le \theta \le \Theta_1} \{1 + U_1 \gamma_2(t, \theta)\} > 0 .$$

Then, one may construct the functions $G_k(t, u, \theta)$ having the properties specified in Theorem 5.1.

Proof: We proceed in two stages, first we obtain a function $q_1(t, u, \theta)$ which satisfies

5.14) $$H_1(t, u, \theta) = q_1(t, u, \theta), \quad |\theta| \le \Theta_1 ,$$

and has the desired properties in the strip

5.15) $$S \equiv \{(t, u, \theta); \; 0 \le t \le 1, \quad 0 \le |u| \le U_1, \quad |\theta| < \infty\} .$$

Let r be a positive constant so large that

5.16a) $$\varepsilon_0^{-1} \le 2\sqrt{r}$$

and

5.16b) $$\max_{0 \le u \le U_1} |\gamma_1(t, u)| \le \frac{1}{10} \rho_1 \sqrt{r} .$$

Let m be a positive constant which satisfies

5.17a) $$0 < m < \frac{1}{20} ,$$

5.17b) $$m \Theta_1 \sqrt{2r} \le \frac{1}{10} \rho_1 .$$

Let

5.18) $$E(t, u, \theta) \equiv \exp\{\frac{1}{m} \gamma_1(t, u)(|\theta| - \Theta_1) - r(|\theta| - \Theta_1)^2\} ,$$

and set

5.19) $$q_1(t, u, \theta) = \begin{cases} H_1(t, u, \theta), & |\theta| \le \Theta_1 \\ H_1(t, u, \Theta_1)\ \{(1-m) + m\ E(t, u, \theta)\}, & \Theta_1 < |\theta|\ . \end{cases}$$

Then, for $\Theta_1 < \theta$

5.20) $$\frac{\partial}{\partial\theta} q_1(t, u, \theta) = H_1(t, u, \Theta_1)\{\gamma_1(t, u) - 2rm(\theta - \Theta_1)\}\ E(t, u, \theta) < 0\ .$$

Moreover, as a function of θ, $q_1(t, u, \theta) \in C^1$. For $\Theta_1 < \theta$ we have

5.21a) $$q_1(t, u, \theta) + \theta \frac{\partial}{\partial\theta} q_1(t, u, \theta) = H_1(t, u, \Theta_1)[(1-m) + A(t, u, \theta)E(t, u, \theta)]$$

where

5.21b) $$A(t, u, \theta) = \{[m + \Theta_1\gamma_1(t, u)] - 2mr\Theta_1(\theta - \Theta_1) + \gamma_1(t, u)(\theta - \Theta_1) - 2mr(\theta - \Theta_1)^2$$

If $|u| \le U_1$, then

5.22a) $$m + \Theta_1\gamma(t, u) \ge \rho + m - 1$$

5.22b) $$-\frac{1}{10}\ \rho \le -2m\ \Theta_1\ r(\theta - \Theta_1)\ E(t, u, \theta) \le 0,$$

5.22c) $$-\frac{1}{10}\ \rho \le \gamma_1(t, u)\ E(t, u, \theta)\ (\theta - \Theta_1) \le 0,$$

5.22d) $$-\frac{1}{10}\ \rho \le -2mr\ (\theta - \Theta_1)^2\ E(t, u, \theta) \le 0.$$

Hence,

$$\frac{7}{10}\rho + m - 1 \le A(t, u, \theta)\ E(t, u, \theta)\ ,$$

and

5.23) $$0 \le \frac{7}{10}\ \rho \le (1-m) + A(t, u, \theta)\ E(t, u, \theta).$$

Therefore, in S, the function $\theta_1(t, u, \theta)$ is strictly monotone in θ .

For $\Theta_1 < \theta$, $0 \le u$, consider

5.24) $$\frac{\partial}{\partial u} q_1(t, n, \theta) \le \frac{\partial}{\partial u} H_1(t, u, \Theta_1) \{(1-m) + m\ E(t, u, \theta)\} + \{\frac{\partial}{\partial \theta}\ \frac{\partial}{\partial u}\ H_1(t, u, \Theta_1)\}\ (\theta - \Theta_1)\ E(t, u, \theta) .$$

Since

$$(\theta - \Theta_1)\ E(t, u, \theta) \le \frac{1}{2\sqrt{r}} < \varepsilon_0(1-m)$$

we obtain

5.25) $$u\ \frac{\partial}{\partial u} q_1(t, u, \theta) \le 0, \quad |u| \le U_1, \quad |\theta| < \infty .$$

Finally for $\Theta_1 < \theta$ we have

$$\frac{\partial}{\partial u}\ \frac{\partial}{\partial \theta}\ q_1(t, u, \theta) = -2mr(\theta - \Theta_1)\ \frac{\partial}{\partial u}\ H_1(t, u, \Theta_1) +$$

$$A\ \frac{\partial}{\partial u}\ \frac{\partial}{\partial \theta}\ H_1(t, u, \Theta_1) + B\ \frac{\partial H_1}{\partial \theta}(t, u, \Theta_1)\ \frac{\partial H_1}{\partial u}\ (t, u, \Theta_1)$$

where A and B are bounded in S . Thus, using (5.20) and (5.11) we obtain an $\varepsilon_1 > 0$ such that, for $0 \le u \le U_1$, $0 \le \theta \le \infty$,

5.26) $$\frac{\partial q_1}{\partial \theta}(t, u, \theta) + \varepsilon_1 \frac{\partial^2}{\partial u \partial \theta}\ q_1(t, u, \theta) \le 0 .$$

Using a similar construction we may extend $q_1(t, u, \theta)$ to a function

$G_1(t, u, \theta)$ of the form

5.27) $$G_1(t, u, \theta) = \begin{cases} q_1(t, u, \theta) , & |u| \leq U_1 \\ q_1(t, u, \theta)\{(1-n) + n\widetilde{E}(t, u, \theta)\}, & U_1 < |u| . \end{cases}$$

Clearly, this function has the desired properties.

A similar construction yields $G_2(t, u, \theta)$ and the theorem is proven.

Corollary: In the case of problem A of Odeh and Tadjbakhsh [10], let

$$\lambda_n < \lambda .$$

Then there exist at least n distinct solutions $(u_j(t), \theta_j(t))$ $j = 0, 1, \ldots, n$ with

$$Z(u_j) = Z(\theta_j) = j .$$

Proof: See [12] for the formulation of this problem as a "cut-off" problem.

Remark: Clearly there are other constructions which yield the functions $G_k(t, u, \theta)$. In fact, my colleagues Ben Noble and Robert Turner have suggested other forms for slightly different conditions.

Appendix

This appendix is denoted to establishing P.3 in two cases of special interest. The basic tools are lemmas 3.1, 3.2 and 3.3.

We will be concerned with three problems. In all three cases we take

$$L_1[\varphi] \equiv L_2[\varphi] \equiv L[\varphi] \; .$$

The difficulties will arise from the boundary condition. Let $(q_1(t), q_2(t)) \in A$ and consider the differential equations

A.1)
$$\begin{cases} L[u] = \lambda\theta q_1 \, , \\ L[\theta] = \lambda u q_2 \, . \end{cases}$$

Problem S

$$u(0) = u(1) = \theta(0) = \theta(1) = 0.$$

Note: This is a symmetrizable problem and the remarks above apply.

Problem N

$$u(0) = u(1) = 0, \qquad \theta(0) = \theta'(1) = 0.$$

Problem A

$$u'(0) = u(1) = 0, \qquad \theta(0) = \theta'(1) = 0.$$

Note: These boundary conditions A are the boundary conditions A of Odeh and Tadjbakhsh [10]. The eigenvalues of these problems will be denoted by $\lambda_k(S)$, $\lambda_k(N)$, $\lambda_k(A)$ respectively.

We now turn our attention to a basic boundary value problem.

Lemma A.1. Let λ be a fixed positive constant. Let $(q_1(t), q_2(t)) \in A$. There exists a unique pair $(u(t), \theta(t))$ which satisfies equation (A.1) and also satisfies the boundary conditions

$$\text{A.2)} \qquad \begin{cases} u(1) = u(0) = 0, \\ \theta(0) = 0, \quad \theta'(0) = 1. \end{cases}$$

Moreover, if $t_0 \in (0, 1)$ and $u(t_0) = 0$, then $u'(t_0) \neq 0$. Similarly, if $t_0 \in (0, 1)$ and $\theta(t_0) = 0$, then $\theta'(t_0) \neq 0$.

Proof: Let $u_j(t)$, $j = 0, 1, 2, 3$ be the basic solutions of equations (A.1) which satisfy the initial conditions

$$\text{A.3)} \quad \begin{cases} \left(\frac{d}{dt}\right)^k u_j(0) = \delta_{kj}; & k = 0, 1; \; j = 0, 1, 2, 3 \\ \left(\frac{d}{dt}\right)^k \left[\frac{1}{\lambda q_1} L[u_j](0)\right] = \delta_{k+2, j}; & k = 0, 1; \; j = 0, 1, 2, 3. \end{cases}$$

These functions exist. The existence of u_0, u_1 is clear when we view equations (A.1) as a fourth order equation for $u(t)$. The existence of u_2, u_3 is clear when we view equations (A.1) as a fourth order equation for $\theta(t)$. Moreover, they are linearly independent. A direct computation shows that

$$u(t) = -\left[\frac{u_3(1)}{u_1(1)}\right] u_1(t) + u_3(t)$$

is a solution. And, $\theta(t)$ is obtained from the differential equation. Suppose there were two solutions, say $u(t)$ and $v(t)$. Then $(u(t) - v(t)) = w(t)$ is a solution (of the fourth order equation in u) satisfying

$$w(0) = \frac{1}{\lambda q_1(0)} L[w](0) = \left(\frac{1}{\lambda q_1} L[w]\right)'(0) = 0.$$

Then, using Lemma 3.1 we would have

$$w'(0) \neq 0, \quad w'(0)\, w(1) > 0 .$$

But,

$$w(1) = 0.$$

The concluding remark of the lemma follows from Lemma 3.3.

Let

$$r(t) \equiv \frac{1}{\lambda q_1(t)} ,$$

$$P(t) \equiv \lambda q_2(t) \ .$$

For the remainder of this section we let $r(t)$ and $P(t)$ be continuous functions of a parameter σ. That is, the coefficients of equation (A.1) are

$$\lambda q_1(t, \sigma) = [r(t, \sigma)]^{-1} \text{ and } \lambda q_2(t, \sigma) = P(t, \sigma).$$

Let $u(t, \sigma)$, $\theta(t, \sigma)$ denote the solution of equation (A.1) which also satisfies the boundary conditions (A.2). With this notation we obtain a corollary to the preceding lemma.

Corollary: The functions $u(t, \sigma)$, $u'(t, \sigma)$, $\theta(t, \sigma)$ and $\theta'(t, \sigma)$ are continuous functions of σ.

Proof: The functions $u_j(t, \sigma)$ satisfying (3.13) (for each σ) are continuous in σ. This follows from general theorems for $u_0(t)$, $u_2(t)$. For $u_1(t)$ and $u_3(t)$ the continuity follows from the representations (3.2a), (3.2b). Also, those representations establish the continuity of $u'(t, \sigma)$, $\theta(t, \sigma)$, $\theta'(t, \sigma)$.

Following Section 2, let $Z(u, \sigma)$ denote the number of interior zeros of $u(t, \sigma)$ while $Z(\theta, \sigma)$ denotes the number of interior zeros of $\theta(t, \sigma)$. Because $K_1(s, t)$ is an oscillation kernel,

A.4) $$Z(u, \sigma) \le Z(\theta, \sigma)$$

Lemma A.2. For every σ_0 there is an $\varepsilon = \varepsilon(\sigma_0) > 0$ such that

A.5) $$|\sigma-\sigma_0| < \varepsilon \implies Z(\theta, \sigma) \geq Z(\theta, \sigma_0) .$$

Proof: Let $M = Z(\theta,\sigma_0)$. Let $\xi_0 = 0$ and for $j = 1, 2, \ldots, M$ let ξ_j denote the ordered interior zeros of $\theta(t, \sigma_0)$, i.e.

$$0 < \xi_j < \xi_{j+1} < 1, \quad \theta(\xi_j, \sigma_0) = 0 .$$

By Rolle's theorem there is a point η_j with

$$\xi_j < \eta_j < \xi_{j+1}, \quad j = 0, 1, \ldots M - 1$$

such that

$$\theta'(\eta_j,\sigma_0) = 0.$$

Let

$$\eta_M = \frac{1}{2}(\xi_M + 1) .$$

Let

$$\rho = \operatorname{Min} |\theta(\eta_j, \sigma_0)| > 0, \quad j = 0, 1, 2, \cdots M.$$

There is an $\varepsilon > 0$ such that $|\sigma - \sigma_0| < \varepsilon$ implies

$$|\theta(\eta_j, \sigma) - \theta(\eta_j, \sigma_0)| < \frac{1}{2} \rho .$$

Thus, there exist $M + 1$ points at which the continuous function $\theta(t, \sigma)$ alternates in sign. Hence, $\theta(t, \sigma)$ has at least M zeros.

Lemma A.3. If $\theta(1, \sigma_0) \neq 0$, there exists an $\varepsilon = \varepsilon(\sigma_0) > 0$ such that $|\sigma - \sigma_0| < \varepsilon$ implies that

A.6) $$Z(\theta, \sigma) = Z(\theta, \sigma_0) .$$

Proof: Suppose not. Then there is a sequence $\sigma_n \to \sigma_0$ such that

$$Z(\theta, \sigma_n) > Z(\theta, \sigma_0).$$

Let $\xi_j(\sigma_n)$ denote the zeros of $\theta(t, \sigma_n)$ as in the above lemma. Consider the vectors in $\mathbb{R}^{M+1}$

$$\xi^{(n)} = (\xi_1(\sigma_n), \xi_2(\sigma_n), \ldots, \xi_{M+1}(\sigma_n)), \quad n = 1, 2, \ldots .$$

There is a subsequence $\xi^{(n')}$ which converges to a limit vector $\xi = (\bar{\xi}_1, \bar{\xi}_2, \ldots, \bar{\xi}_{M+1})$. We observe that

$$0 \le \bar{\xi}_j \le \bar{\xi}_{j+1} \le 1$$

and

$$\theta(\bar{\xi}_j, \sigma_0) = 0 .$$

But $\theta(t, \sigma_0)$ has only M interior zeros. Hence, one of the following must occur.

Case 1. $\bar{\xi}_1 = 0$. But then the point $\eta_1(\sigma_n)$ at which $\theta'(\eta_1(\sigma_n), \sigma_n) = 0$ must also converge to zero. Hence

$$\theta'(0, \sigma_0) = 0.$$

But, of course, $\theta'(0, \sigma_0) = 1$.

Case 2. $\bar{\xi}_{n+1} = 1$. But then

$$\theta(1, \sigma_0) = 0$$

contrary to our assumption.

Case 3. There is a j such that

$$0 < \bar{\xi}_j = \bar{\xi}_{j+1} < 1 .$$

But then $\eta_j(\sigma_n) \to \bar{\xi}_j$ and

$$\theta'(\bar{\xi}_j, \sigma_0) = \theta(\bar{\xi}_j, \sigma_0) = 0$$

which is impossible.

Definition: A value σ will be called a "k-value" for the problem A (the problem N, the problem S) if

A.7) $$1 = \lambda_k(A, \sigma), \quad (1 = \lambda_k(N, \sigma), \quad 1 = \lambda_k(S, \sigma)).$$

We shall let A_k, N_k, S_k denote the set of all k values. That is

A.8) $$\begin{cases} A_k \equiv \{\sigma \mid 1 = \lambda_k(A, \sigma)\} , \\ N_k \equiv \{\sigma \mid 1 = \lambda_k(N, \sigma)\} , \\ S_k \equiv \{\sigma \mid 1 = \lambda_k(S, \sigma)\} . \end{cases}$$

We observe that S_k contains at most one element.

For the remainder of this section we assume that

$$\frac{\partial r(t, \sigma)}{\partial \sigma} \leq 0, \quad \frac{\partial P(t, \sigma)}{\partial \sigma} \geq 0$$

and, $\sigma_1 < \sigma_2$ implies that

$$P(t, \sigma_1) \not\equiv P(t, \sigma_2)$$

Lemma A.4. Suppose

$$1 = \lambda_k(S, \sigma_0), \quad 1 = \lambda_{k+1}(S, \sigma_2).$$

Then

A.9) $$\sigma_0 < \sigma_2 .$$

The set N_{k+1} is not empty; and, if $\sigma \in N_{k+1}$ then

A.10) $$\sigma_0 < \sigma < \sigma_2 .$$

Proof: The inequality (A.9) follows from Lemma 2.1. Let $(u(t, \sigma_0), \theta(t, \sigma_0))$ be the solution of equation (A.1) which satisfies the boundary conditions (A.2). Then $(u(t, \sigma_0), \theta(t, \sigma_0))$ is (except for scaler multiples) the k^{th} eigenfunction of the problem S. To see this, let $v(t)$ be a k^{th} eigenfunction of problem S. We need only verify that $\psi'(0) = r(0)\, L_1[v]'(0) \neq 0$. If $\psi'(0) = 0$ then Lemma 3.1 implies that $v(1) \neq 0$. However, $v(1) = 0$. We observe that because $K_1(S, t)$ and $K_2(S, t)$ are both oscillation kernels, we have

$$Z(u, \sigma_0) = Z(\theta, \sigma_0)$$

$$Z(u, \sigma_2) = Z(\theta, \sigma_2) .$$

Let σ increase from σ_0 to σ_2. Since all zeros of $\theta(t, \sigma)$ are nodal zeros we see that

$$\operatorname{Sgn} \theta'(1, \sigma_0) = (-1)^{k+1}$$

$$\operatorname{Sgn} \theta'(1, \sigma_2) = (-1)^{k+2} .$$

Thus, there must be at least one value of $\sigma \in (\sigma_0, \sigma_2)$ such that

A.11) $$\theta'(1, \sigma) = 0.$$

Now there is an $\varepsilon_0 = \varepsilon(\sigma_0, \sigma_2) > 0$ such that

A.12a) $$Z(\theta, \sigma_0) \leq Z(\theta, \sigma), \quad |\sigma - \sigma_0| < \varepsilon_0 ,$$

and

A.12b) $$Z(\theta, \sigma_2) \leq Z(\theta, \sigma), \quad |\sigma - \sigma_2| < \varepsilon_0 .$$

Moreover, for every point $\hat{\sigma}$ in the closed interval $[\sigma_0 + \varepsilon_0/2, \ \sigma_2 - \varepsilon_0/2]$ there is an $\varepsilon = \varepsilon(\hat{\sigma})$ such that

$$Z(\theta, \sigma) = Z(\theta, \hat{\sigma}), \quad |\sigma - \hat{\sigma}| < \varepsilon(\hat{\sigma}).$$

Thus, we may apply the Heine-Borel theorem to conclude that

$$Z(\theta, \sigma) \equiv \text{constant}, \quad \sigma_0 + \varepsilon_0/2 \leq \sigma \leq \sigma_2 - \varepsilon_0/2.$$

Thus, on letting $\varepsilon_0 \to 0$ we see that

$$Z(\theta, \sigma) \equiv \text{constant}, \quad \sigma_0 < \sigma < \sigma_2 .$$

This fact, combined with the inequalities (A.12a), (A.12b) and the fact that

$$Z(\theta, \sigma_2) = Z(\theta, \sigma_0) + 1$$

implies that

$$Z(\theta, \sigma) = k + 1 , \quad \sigma_0 < \sigma \leq \sigma_2 .$$

Thus

$$N_{k+1} \neq \phi .$$

Suppose there are values $\sigma \in N_{k+1}$ which do not lie in the interval (σ_0, σ_2).

Case 1: There is a value $\hat{\sigma} \in N_{k+1}$ and $\hat{\sigma} < \sigma_0$. Let $u(t, \hat{\sigma})$ be the solution of equation (A.1) which satisfies the boundary condition (A.2). Then as before $u(t, \hat{\sigma})$ must be the $(k+1)^{st}$ eigenfunction of problem N (except for scalar multiples).

Let σ increase from $\hat{\sigma}$ to σ_0. The argument given above shows that

$$k = Z(\theta, \sigma_0) \geq Z(\theta, \hat{\sigma}) = k + 1 .$$

This is impossible.

Case 2: There is a value $\hat{\sigma} \in N_{k+1}$ and $\hat{\sigma} > \sigma_2$. But again, the argument given above shows that

$$k + 1 = Z(\theta, \hat{\sigma}) > Z(\theta, \sigma_2) = k + 1 .$$

Thus, the lemma is proven.

Corollary: For any fixed value of σ

A.13) $$\lambda_k(S, \sigma) < \lambda_{k+1}(N, \sigma) < \lambda_{k+1}(S, \sigma) .$$

Proof: Let σ be fixed, and let $r(t, \sigma, \sigma') \equiv r(t, \sigma')$ while

$$P(t, \sigma; \sigma') = (\sigma')^2 P(t, \sigma) .$$

Applying the above ideas to $r(t, \sigma, \sigma')$, $P(t, \sigma, \sigma')$ as functions of σ' we obtain (in an obvious notation)

$$\lambda_k(S, \sigma, \sigma'_0) = \lambda_{k+1}(S, \sigma, \sigma'_2) = \lambda_{k+1}(N, \sigma, \sigma'_1)$$

for some $\sigma'_1 \in (\sigma'_0, \sigma'_2)$. But then

$$(\sigma'_0)^2 = \lambda_k(S, \sigma) < (\sigma'_1)^2 = \lambda_{k+1}(N, \sigma) < (\sigma'_2)^2 = \lambda_{k+1}(S, \sigma) .$$

We wish to obtain similar results for the k values of problem A and the eigenvalues related to problem A. Hence we consider another special problem.

Lemma A.5. There is a unique function pair $(v(t, \sigma), \psi(t, \sigma))$ which satisfies equation (A.1), (under the identification $v(t, \sigma) = u$, $\psi(t, \sigma) = \theta$) and the boundary conditions

$$v(1) = 0 , \quad v'(1) = 1 ,$$

$$\psi(0) = \psi'(1) = 0 .$$

Moreover, if $t_0 \in (0, 1)$ and $v(t_0) = 0$ then $v'(t_0) \neq 0$. Similarily, if $t_0 \in (0, 1)$ and $\psi(t_0) = 0$ then $v'(t_0) \neq 0$.

Proof: Let $v_j(t)$, $j = 0, 1, 2, 3$ be the basic solutions of equation (A.1) which satisfy the initial (terminal) conditions

$$\text{A.14)} \quad \begin{cases} \left(\frac{d}{dt}\right)^k v_j(1) = \delta_{k,j} , & k = 0, 1; \quad j = 0, 1, 2, 3 . \\ \left(\frac{d}{dt}\right)^k (r L_1[v_j])(1) = \delta_{k+2, j}, & k = 0, 1; \quad j = 0, 1, 2, 3 . \end{cases}$$

Then, a computation gives $v(t, \sigma)$ in the form

$$v(t, \sigma) = v_1(t, \sigma) + Mv_2(t, \sigma) .$$

The rest of the lemma follows exactly as the proof of Lemma A.1 .

Corollary: The functions $v(t, \sigma)$, $v'(t, \sigma)$, $\psi(t, \sigma)$, $\psi'(t, \sigma)$ are all continuous functions of σ .

Let $Z(v, \sigma)$, $Z(\psi, \sigma)$ denote the number of interior zeros of $v(t, \sigma)$ and $\psi(t, \sigma)$ respectively. Then, as before, because we are dealing with oscillation kernels

$$Z(\psi, \sigma) \leq Z(v, \sigma) .$$

Lemma A.6. For every σ_0 there is an $\varepsilon = \varepsilon(\sigma_0) > 0$ such that $|\sigma - \sigma_0| < \varepsilon$ implies that

$$Z(v, \sigma) \geq Z(v, \sigma_0) .$$

Proof: The proof of this lemma is exactly the same as the proof of Lemma A.2.

Lemma A.7. For every σ_0 for which $v(0, \sigma_0) \neq 0$ there is an $\varepsilon = \varepsilon(\sigma_0) > 0$ such that $|\sigma - \sigma_0| < \varepsilon$ implies that

$$Z(v, \sigma) = Z(v, \sigma_0)$$

Proof: The proof of this lemma is exactly the same as the proof of Lemma A.3.

Lemma A.8. Let

$$\sigma_1 \equiv \sup \{\sigma : \sigma \in N_k\} ,$$

$$\sigma_3 \equiv \inf \{\sigma : \sigma \in N_{k+1}\} .$$

We assume $-\infty < \sigma_3, \ \sigma_1 < \infty$.

Then

$$\sigma_1 < \sigma_3 ,$$

$$A_{k+1} \neq \phi ,$$

and, if $\sigma \in A_{k+1}$, then

$$\sigma_1 < \sigma < \sigma_3 .$$

The proof of this lemma is exactly the same as the proof of Lemma A.4 .

Corollary: For every fixed value of σ

A.15) $$\lambda_k(N, \sigma) < \lambda_{k+1}(A, \sigma) < \lambda_{k+1}(N, \sigma) < \lambda_{k+1}(S, \sigma) .$$

Proof: The proof of the corollary follows the same argument as the proof of the corollary to Lemma A.4.

Theorem A.1. For the special cases of problem A and problem N the hypothesis P.3 holds.

Proof: Since P.3 holds for problem S, the upper bound on λ_n follows from the inequalities (A.13), (A.15). The lower bound follows from an elementary argument based on the Krein-Rutman theory [8]. See [5] also.

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