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CONVERGENCE OF A DISCRETIZATION FOR CONSTRAINED SPLINE FUNCTION PROBLEMS

by

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1. Introduction

In [Mangasarian - Schumaker (1969)], some generalizations of the basic ideas of spline functions were developed by considering certain minimization problems under a mixture of discrete and continuous inequality constraints, extending concepts in [Atteia (1968), Golumb-Jerome (1969), Jerome-Schumaker (1969), Ritter (1969)]. Sufficient and sometimes necessary conditions for a function to solve the minimization problem were presented via optional control techniques, but no computational methods were discussed. In the present paper we shall analyze the convergence of simple discretizations of the problem, such discretizations in many cases being finitely solvable by standard quadratic programming methods. Let us first define the problem.

Let m be a positive integer and let $1 < p < \infty$. For $i = 1,2,\dots,k$, let M_i be not identically zero linear differential operators on $[0,1]$ of degree less than m, and similarly for N_i, $i = 1,2,\dots,n$; we write

$$M_i x = \sum_{j=0}^{m-1} b_{ij}(t)\, x^{(j)}(t), \quad N_i x = \sum_{j=0}^{m-1} c_{ij}(t)\, x^{(j)}(t) .$$

We allow $k = 0$ and $n = 0$. Let L be a linear differential operator on $[0,1]$ of exact degree m,

$$Lx = \sum_{j=0}^{m} a_j(t)\, x^{(j)}(t) \;,\quad a_m(t) \neq 0 \text{ in } [0,1].$$

Let $W^{m,p}$ be the Sobolev space of real valued functions x on [0,1] such that $x^{(m-1)}$ is absolutely continuous and $x^{(m)} \in L^p(0,1)$. Then our minimization problem is to

$$\left.\begin{aligned} &\text{minimize} && f(x) \equiv \int_0^1 |Lx(t)|^p\, dt \\ &\text{over} && C = \{x;\ x \in W^{m,p},\ \alpha_i(t) \leq M_i x(t) \leq \beta_i(t) \text{ for } 0 \leq t \leq 1, \\ & && i = 1,\ldots,k,\ \gamma_i \leq N_i x(\xi_i) \leq \delta_i \text{ for } i = 1,\ldots,n\} \end{aligned}\right\} \quad (1.1)$$

where α_i and β_i are given functions, the γ_i and δ_i are given scalars, and the ξ_i are points in [0,1]. Some simple generalization is possible by allowing one-sided constraints or by allowing the N_i to be difference operators but we shall not consider this here. It is shown in [Mangasarian-Schumaker (1969)] that, if C is nonempty, if $a_j \in C^j[0,1]$ for $j = 0,\ldots m$, if $b_{ij} \in C[0,1]$ for $i = 1,\ldots,k$ and $j = 0,\ldots,m-1$, if $c_{ij} \in C[0,1]$ for $i = 1,\ldots,n$ and $j = 0,\ldots,m-1$, and if α_i and β_i lie in $M_i W^{m,p}$ for $i = 1,\ldots,k$, then there exists a solution x^* to the problem in Equation 1.1. <u>We shall assume the above hypotheses to hold throughout the ensuing discussion</u>. Since the solution x^* may hit the boundary of C at unknown points, perhaps countably many times, the computation of x^* is difficult. One obvious way to handle this is as follows.

Let $h > 0$ be some mesh size, say $h = \frac{1}{Q}$, and let $[0,1]$ be partitioned by $t_i = ih$, $i = 0, \ldots, \frac{1}{h} = Q$; we suppose that all the points ξ_i lie on this mesh for all h to be used, that is, $\frac{\xi_i}{h}$ is an integer (this assumption is not necessary but simplifies the notation). Our first discretization consists in merely replacing the continuous constraints by discrete ones, that is, we

$$\left.\begin{aligned} &\text{minimize} \quad f(x) = \int_0^1 |Lx(t)|^p dt \\ &\text{over} \quad C_1(h) \equiv \{x;\ x \in W^{m,p},\ \alpha_i(t_j) \leq M_i x(t_j) \leq \beta_i(t_j) \text{ for} \\ &\qquad j = 0, \ldots, Q,\ i = 1, \ldots, k,\ \gamma_i \leq N_i x(\xi_i) \leq \delta_i \text{ for} \\ &\qquad i = 1, \ldots, n\}\ . \end{aligned}\right\} \quad (1.2)$$

As analyzed in [Ritter (1969)], this problem can be solved in the common case of $p = 2$ in finitely many steps by minimizing a quadratic function of $2k(Q+1) + m + 2n$ variables subject to $2k(Q+1) + m + 2n$ linear inequality constraints. We shall prove the following (Section 2, Theorem 2.1): <u>All weak limit points (in the $W^{m,p}$ sense), at least one of which exists, of a sequence of solutions to the first discretization in Equation 1.2 must solve the original problem of Equation 1.1; if the solution to the original problem is unique, the approximating solutions converge to it $W^{m,p}$ weakly and in particular the function and the first m-1 derivatives converge uniformly</u>.

If one must take h very small to obtain a reasonable approximation to x^*, one might well be satisfied to have only approximate values of x^* at the grid points t_i rather than throughout $[0,1]$; if $x^*(t)$ were desired and $x^*(t_i)$ was accurately known, unconstrained interpolation could be used to generate a reasonable approximation to x^*. Thus we are lead to a second, more complete, discretization. If z is a function defined at least on the mesh points $t_i = ih$, $i = 0, \dots, \frac{1}{h} = Q$, let $D = D_h$ be the operator such that

$Dz(t_i) = \dfrac{z(t_{i+1}) - z(t_i)}{h}$, $i = 0, 1, \dots, Q-1$. We then have

$D^\ell z(t_i) = \dfrac{1}{h^\ell} \sum_{j=0}^{\ell} (-1)^{\ell-j} \binom{\ell}{j} z(t_{i+j})$ as a natural analogue of the ℓ - th derivative

of z. We therefore define

$$M_{i,h} z(t_j) = \sum_{\ell=0}^{m-1} b_{i\ell}(t_j)\, D^\ell z(t_j), \quad 0 \le j \le Q - m + 1,$$

$$N_{i,h} z(t_j) = \sum_{\ell=0}^{m-1} c_{i\ell}(t_j) D^\ell z(t_j), \quad 0 \le j \le Q - m + 1,$$

$$L_h z(t_j) = \sum_{\ell=0}^{m} a_\ell(t_j)\, D^\ell z(t_j), \quad 0 \le j \le Q - m .$$

Our second discretized problem is now to

$$\left.\begin{aligned}
&\text{minimize} \quad f_h(x_h) = h \sum_{j=0}^{Q-m} |L_h x_h(t_j)|^p \\
&\text{over} \quad C_2(h) \equiv \{x_h; x_h \equiv (x_h(0), x_h(t_1), \ldots, x_h(1))^T \in \mathbb{R}^{Q+1}, \\
&\qquad -\varepsilon_h + \alpha_i(t_j) \leq M_{i,h} x_h(t_j) \leq \beta_i(t_j) + \varepsilon_h \text{ for } j = 0, \ldots, \\
&\qquad Q - m + 1, i = 1, \ldots, k, \ -\varepsilon_h + \gamma_i \leq N_{i,h} x_h(\xi_i) \leq \delta_i + \varepsilon_h \\
&\qquad \text{for } i = 1, \ldots, n\}.
\end{aligned}\right\} \quad (1.3)$$

Here ε_h, which tends to zero, gives a small expansion of the constraint set as $h \to 0$. If some such expansion is not allowed, the set $C_2(h)$ can be empty [Daniel (1969b, 1970)]; as will be clear from the use made of the expansion by ε_h, constraints of the form $\alpha(t) \leq x(t) \leq \beta(t)$ or $\alpha \leq x(\xi) \leq \beta$ need not be expanded. This problem in the common case of $p = 2$ can be solved in finitely many steps since it involves a quadratic function of $Q + 1$ variables subject to $2k(Q - m + 2) + 2n$ linear inequalities. Since in general $Q = \frac{1}{h}$ is large, seeking only approximations to $x^*(t_i)$ reduces the difficulty considerably. Under slight additional hypotheses on the problem, we shall prove the following, roughly stated (Section 3, Theorem 3.1): All $W^{m,p}$ weak limit points (of certain "interpolations"), at least one of which exists, of a sequence of solutions to the discretization in Equation 1.3 must solve the original problem of Equation 1.1; if the solution to the original problem is unique, the ("interpolations" of the) approximate solutions converge to it $W^{m,p}$ - weakly and in particular the function values and the first $m - 1$ divided differences converge uniformly to x^* and its first $m - 1$ derivatives.

2. Analysis of the first, simpler, discretization.

We have yet to define a norm on the space $W^{m,p}$; two common norms, which are equivalent as is well known, are

$$\|x\|_o \equiv \{ \sum_{i=0}^{m} \int_0^1 |x^{(i)}(t)|^p dt\}^{\frac{1}{p}} ,$$

$$\|x\|_1 \equiv \{ \sum_{i=1}^{m} |x(\theta_i)|^p + \int_0^1 |x^{(m)}(t)|^p dt\}^{\frac{1}{p}}$$

for $0 \leq \theta_1 < \theta_2 < \ldots < \theta_m \leq 1$. For some positive a, A, we have $a\|x\|_o \leq \|x\|_1 \leq A\|x\|_o$ for all x in $W^{m,p}$, as is well known.

From the computational standpoint, serious difficulties arise if the original problem, Equation 1.1, admits solutions of arbitrarily large norm. For example, the the functions $x_n(t) = n$ form a minimizing sequence (in fact, they are all solutions) for $\int_0^1 |x^{(2)}(t)|^2 dt$ over the set of x satisfying $0 \leq x^{(1)}(t) \leq 1$ but has no convergent subsequence. In this situation our analysis to follow could not guarantee that the approximate solutions have limit points; to avoid this we must eliminate problems admitting solutions of arbitrarily large norm. We pause to see what this means. For $0 \leq \ell \leq k$, let $S_\ell \equiv \{x; x \in W^{m,p}, Lx \equiv 0, M_i x \equiv 0$ for $1 \leq i \leq \ell$, and $N_i x(\xi_i) = 0$ for $1 \leq i \leq n\}$. It is shown in [Mangasarian-Schumaker (1969)] that, if d_{k+1} is the dimension of S_k and if d_{k+1-j} is the dimension of $S_{k+1-j} - S_{k+2-j}$ for $1 \leq j \leq k$, then there exist points $\theta_{\ell,i}$ with

$0 \leq \theta_{\ell,1} < \theta_{\ell,2} < \ldots < \theta_{\ell,d_\ell} \leq 1$ for $1 \leq \ell \leq k+1$, with the points $\{\theta_{k+1,i}\}^{d_{k+1}}$, being completely arbitrary in $[0,1]$, such that

$$\left.\begin{aligned} \|x\| \equiv \{ \sum_{\ell=1}^{k} \sum_{j=1}^{d_\ell} |M_\ell x(\theta_{\ell j})|^p + \sum_{j=1}^{d_{k+1}} |x(\theta_{k+1,j})|^p + \sum_{i=1}^{n} |N_i x(\xi_i)|^p \\ + \int_0^1 |Lx(t)|^p dt\}^{\frac{1}{p}} \end{aligned}\right\} \quad (2.1)$$

defines a norm on $W^{m,p}$. We remark that if for some ℓ_o one has $M_{\ell_o} x \equiv x$, then one may take $d_{\ell_o} = m$, all other $d_\ell = 0$, eliminate the sum in N_i from the Equation 2.1, and take arbitrary distinct points for $\theta_{\ell_o,i}$ to define the norm. By the usual Sobolev inequalities it is simple to show that this norm is in fact equivalent to $||\cdot||_o$ and $||\cdot||_1$.

Lemma 2.1 $\|\cdot\|$ is equivalent to $\|\cdot\|_o$ and $\|\cdot\|_1$.

Proof: The existence of an A' such that $\|x\| \leq A'\|x\|_1$ for all x in $W^{m,p}$ is clear from the usual Sobolev inequalities; we ask whether or not an $a' > 0$ exists such that $\|x\| \geq a'\|x\|_1$. If not, we can find $x_n \in W^{m,p}$ such that $\|x_n\| \to 0$ but $\|x_n\|_1 = 1$; by the weak compactness of the sphere in $(W^{m,p}, \|\cdot\|_1)$, we may assume x_n converges $(W^{m,p}, \|\cdot\|_1)$ - weakly to some x in $W^{m,p}$. Since L is bounded from $(W^{m,p}, \|\cdot\|_1)$ into $L^p(0,1)$, since M_i and N_i are bounded from $(W^{m,p}, \|\cdot\|_1)$ into $C[0,1]$, and since $\int_0^1 |Lx_n(t)|^p dt$

tends to zero, we have $\|x\| = 0$ and therefore $x \equiv 0$. Since x_n converges uniformly to x and $\|x_n\|_1 = 1$, we have $\int_0^1 |x_n^{(m)}(t)|^p dt$ converging to 1. Then

$$\left\{\int_0^1 |Lx_n(t)|^p dt\right\}^{\frac{1}{p}} \geq \left\{\int_0^1 |a_m(t)x_n^{(m)}(t)|^p dt\right\}^{\frac{1}{p}} - \left\{\int_0^1 \left|\sum_{i=0}^{m-1} a_i(t)x_n^{(i)}(t)\right|^p dt\right\}^{\frac{1}{p}}$$

which then is bounded away from zero since $|a_m(t)| \geq \varepsilon > 0$ for some ε, since $\int_0^1 |x_n^{(m)}(t)|^p dt \to 1$, and since $x_n^{(i)}$ converges uniformly to $x^{(i)} \equiv 0$ for $0 \leq i \leq m-1$. This contradicts $\|x_n\| \to 0$. Q.E.D.

Now for any x in C, the values $|M_\ell x(\theta_{\ell,i})|$, $|N_i x(\xi_i)|$, and $\int_0^1 |Lx(t)|^p dt$ (since $f(x) \leq f(x^*)$) are uniformly bounded. Thus C will be bounded if and only if the $\{|x(\theta_{k+1,j})|\}$ are uniformly bounded for $1 \leq j \leq d_{k+1}$. If $d_{k+1} = 0$ this is certainly true; if $d_{k+1} \neq 0$, there exists a nonzero function z in S_k and thus $x^* + \alpha z \in C$ for all scalars α and $f(x^* + \alpha z) = f(x^*)$. Therefore the original problem admits of an a priori bound on its solutions if and only if $d_{k+1} = 0$. <u>We hereafter assume that $d_{k+1} = 0$.</u> Computationally this may be accomplished simply by adding one continuous constraint $|x(t)| \leq E$ for some large E or m discrete constraints $|x(\theta_i)| \leq E$ and thus this is not a computationally significant restriction. Adding $|x(t)| \le E$ means that we may use as our norm the simple expression

$$\left\{ \sum_{i=1}^{m} |x(\theta_i)|^p + \int_0^1 |Lx(t)|^p dt \right\}^{\frac{1}{p}} . \tag{2.2}$$

We hereafter assume that the points $\theta_{\ell,i}$ of Equation 2.1 are mesh points for all h used. If some $M_{\ell\rho}\, x \equiv x$, then we need only assume that the m points θ_i of Equation 2.2 are mesh points for all h used.

Now let, for $Q = \frac{1}{h}$, x_Q^* be a solution to the simple discretized problem of Equation 1.2; $C_1(h)$ is not empty since, in particular, $x^* \in C_1(h)$ where x^* solves the original problem. Since $f(x_Q^*) \le f(x^*)$, since $|M_\ell x_Q^* (\theta_{\ell,j})| \le \max\{|\alpha_\ell(\theta_{\ell,j})|, |\beta_\ell(\theta_{\ell,j})|\}$, and since $|N_i x_Q^* (\xi_i)| \le \max\{|\delta_i|, |\gamma_i|\}$, we conclude that $\|x_Q^*\|$ is uniformly bounded. We note that x_Q^* is not necessarily in C; it is however "near" to C as the following more general lemma demonstrates.

Lemma 2.2 If $x \in W^{m,p}$ and $-\varepsilon + \alpha_i(t_j) \le M_i x(t_j) \le \beta_i(t_j) + \varepsilon$ for $i = 1, \ldots, k$ and $j = 0, \ldots, Q$, then $-(\varepsilon + \eta_i(h)) + \alpha_i(t) \le M_i x(t) \le \beta_i(t) + (\varepsilon + \eta_i(h))$ for $0 \le t \le 1$, where

$$\eta_i(h) = \max \left[\{\int_0^1 |\beta_i^{(1)}(t)|^p dt\}^{\frac{1}{p}}, \{\int_0^1 |\alpha_i^{(1)}(t)|^p dt\}^{\frac{1}{p}} \right] h^{\frac{1}{q}} + B_i \|x\|_o h^{\frac{1}{q}}$$

$$+ \|x\|_o w_i(h) \text{ where } \frac{1}{p} + \frac{1}{q} = 1, \ |b_{i\ell}(t)| \le B_i, \ |b_{i\ell}(t_1) - b_{i\ell}(t_2)| \le$$

$$\le w_i(h) \text{ if } |t_1 - t_2| \le h, \ \ell = 0, \ldots, m-1.$$

Proof: For the upper bound,

$$M_i x(t) - \beta_i(t) = M_i x(t) - M_i x(t_j) + M_i x(t_j) - \beta_i(t_j) + \beta_i(t_j) - \beta_i(t) .$$

Since

$$\beta_i \ \varepsilon \ M_i \ W^{m,p} \subset W^{1,p}, \ |\beta_i(t) - \beta_i(t_j)| \le \int_{t_j}^t |\beta_i^{(1)}(t)| dt$$

$$\le \{\int_0^1 |\beta_i^{(1)}(t)|^p dt\}^{\frac{1}{p}} |t - t_j|^{\frac{1}{q}}. \text{ For } |M_i x(t) - M_i x(t_j)| \le$$

$$\sum_{\ell=0}^{m-1} |b_{i\ell}(t) x^{(\ell)}(t) - b_{i\ell}(t_j) x^{(\ell)}(t_j)| \le B_i \|x\|_o |t - t_j|^{\frac{1}{q}} + \|x\|_o w_i(|t - t_j|)$$

arguing as for β_i. Thus letting t_j be such that $|t - t_j| \le h$, we have $M_i x(t) - \beta_i(t) \le \varepsilon + \left[\{\int_0^1 |\beta_i^{(1)}(t)|^p dt\}^{\frac{1}{p}} + B_i \|x\|_o \right] h^{\frac{1}{q}} + \|x\|_o w_i(h)$. Similarly for the lower bound. Q. E. D.

We note that if the $b_{i\ell}$ are Hölder continuous with exponent greater than or equal to $\frac{1}{q}$, then $|N_i(h)| \le F h^{\frac{1}{q}}$ for a constant F uniformly bounded whenever $\|x\|$ is bounded.

<u>Lemma 2.3</u> If functions $x_Q \in W^{m,p}$ satisfy $-\varepsilon_Q + \alpha_i(t) \leq M_i x_Q(t) \leq \beta_i(t) + \varepsilon_Q$ for $i = 1,\ldots,k$ and $-\varepsilon_Q + \gamma_i \leq N_i x_Q(\xi_i) \leq \delta_i + \varepsilon_Q$ for $i = 1,\ldots,n$, if $\lim_{Q\to\infty} \varepsilon_Q = 0$ and if x_Q converges $W^{m,p}$ weakly to x, then $x \in C$.

<u>Proof</u>: This is obvious since $x_Q^{(j)}(t)$ converges to $x^{(j)}(t)$ for each t in $[0,1]$ for $0 \leq j \leq m-1$, and the constraints involve derivatives of order at most $m-1$. Q. E. D.

<u>Lemma 2.4</u> As $\varepsilon_Q \geq 0$ tends to zero, the minimum of f over the set $C\varepsilon_Q = \{x; x \in W^{m,p}, -\varepsilon_Q + \alpha_i(t) \leq M_i x(t) \leq \beta_i(t) + \varepsilon_Q$ for $1 \leq i \leq k$ and $0 \leq t \leq 1$, $-\varepsilon_Q + \gamma_i \leq N_i x(\xi_i) \leq \delta_i + \varepsilon_Q$ for $1 \leq i \leq n\}$ converges to the minimum of f over C.

<u>Proof:</u> Clearly each set C_{ε_Q} is weakly closed, and for the minimization problem over C_{ε_Q} we may restrict ourselves to those x satisfying $f(x) \leq f(x^*)$ since $x^* \in C_{\varepsilon_Q}$, where x^* minimizes f over C. Since, for all x in this set, $\|x\|$ is uniformly bounded, the weakly lower semicontinuous functional f attains its minimum over the weakly compact set C_{ε_Q} at some point x_Q. Since $\|x_Q\|$

is uniformly bounded, we may assume that x_Q converges weakly to some x, which must be in C by Lemma 2. . Thus $f(x) \leq \liminf_{Q \to \infty} f(x_Q)$ $\leq \lim\sup f(x_Q) \leq f(x^*)$ since $f(x_Q) \leq f(x^*)$ for all Q. Thus $\lim_{Q \to \infty} f(x_Q)$ $= f(x^*)$. Q. E. D.

We can now prove our discretization result for the simpler discretization.

Theorem 2.1 Let the general assumptions of Section 1 hold and let the fixed points $\theta_{\ell,i}$ defining the norm $\|\cdot\|$ in Equation 2.1 (or θ_i in Equation 2.2) be mesh points in our discretization for all h. Let $x_Q^* \in W^{m,p}$ solve the problem of Equation 1.2, that is minimize $f(x)$ over $C_1(h)$. Then $f(x_Q^*)$ converges to $f(x^*)$ and all $W^{m,p}$ weak limit points, at least one of which exists, of $\{x_Q^*\}$, minimize f over C; if x^* minimizing f over C is unique, then x_Q^* converges weakly to x^*, that is, $x_Q^{*(i)}$ converges uniformly to $x^{*(i)}$ for $0 \leq i \leq m-1$ and $x_Q^{*(m)}$ converges L^p - weakly to $x^{*(m)}$.

Proof: Arguing as in Lemma 2.4 we see that x_Q^* always exists, $f(x_Q^*) \leq f(x^*)$, and there exists a constant E such that $\|x_Q^*\| \leq E$. Thus, by Lemma 2.2, there exist functions $\eta_i(h)$ for $1 \leq i \leq k$ tending to zero with h, and such that $x_Q^* \in C_Q \equiv \{x;\ x \in W^{m,p},\ -\eta_i(h) + \alpha_i(t) \leq M_i x(t) \leq \beta_i(t) + \eta_i(h)$

for $1 \leq i \leq k$, $\gamma_i \leq N_i x(\xi_i) \leq \delta_i$ for $1 \leq i \leq n\}$. By Lemma 2.4, $\zeta_Q = \min_C f - \min_{C_Q} f$

tends to zero. We write

$$f(x^*) = \min_C f = \min_{C_Q} f + \zeta_Q \leq f(x_Q^*) + \zeta_Q \leq f(x^*) + \zeta_Q \tag{2.3}$$

This implies, since $\zeta_Q \to 0$, that $f(x_Q^*) \to f(x^*)$. Since $\{x_Q^*\}$ is bounded, it has weak limit points. For any such weak limit point x' with $x_{Q'}^*$, weakly converging to x', we have $x' \varepsilon C$ by Lemma 2.3 and thus

$$f(x^*) \leq f(x') \leq \liminf_{Q' \to \infty} f(x_{Q'}^*) = \lim_{Q \to \infty} f(x_Q^*) = f(x^*)$$

which says that $f(x') = f(x^*)$, that is, x' minimizes f over C. The remainder follows from the definition of convergence in $W^{m,p}$. Q. E. D.

We have not been able to estimate the rate of convergence as a function of h.

3. Analysis of the more complete discretization.

We shall here use the norm $\|\cdot\|$ defined in Equation 2.1 via points $\theta_{\ell,i}$ (or θ_i in Equation 2.2) and we shall assume that the $\theta_{\ell,i}$ are mesh points for all h used. We shall analyze our complete discretization, the relationship between the problems in Equation 1.1 and Equation 1.3, by the general discretization analysis of [Daniel (1969a, 1969b, 1970)]; for completeness our arguments are self-contained.

We wish to use roughly the arguments of Theorem 2.1 in this case also. If x_h^* minimizes f_h over $C_2(h)$, we unfortunately cannot talk about $f(x_h^*)$ or $f_h(x^*)$ as in Equation 2.3 in the proof of Theorem 2.1 since these make no sense in our new situation. Instead, with x^* we shall associate a point $y_h = r_h x^* \varepsilon C_2(h)$ by a "discretization" or "restriction" mapping r_h such that $|f_h(r_h x^*) - f(x^*)|$ converges to zero with h. Similarly, with x_h^* we shall associate a $z_h = p_h x_h^* \varepsilon W^{m,p}$ "converging into C" by an "interpolation" or "prolongation" mapping p_h and such that $|f(p_h x_h^*) - f_h(x_h^*)|$ converges to zero with h. We can then imitate the proof of Theorem 2.1 by replacing Equation 2.3 by roughly $f(x^*) = \min_{C_Q} f + \zeta_Q \leq f(p_h x_h^*) + \zeta_Q = f_h(x_h^*) + \zeta_Q + [f(p_h x_h^*) - f_h(x_h^*)] \leq f_h(r_h x^*) + \zeta_Q + [f(p_h x_h^*) - f_h(x_h^*)] = f(x^*) + \zeta_Q + [f(p_h x_h^*) - f_h(x_h^*)] + [f_h(r_h x^*) - f(x^*)]$.

Having outlined our approach and the reasons for constructing certain mappings p_h and r_h we now proceed with the technical details. We define the restriction r_h in the obvious manner. Let $y_h = r_h x^*$ be the discrete mesh function (that is, defined at points $t_i = ih$ only) defined by $y_h(t_i) \equiv x^*(t_i)$.

We need to develop some tools for using divided differences. For any ℓ with $0 \leq \ell \leq m$, by Peano's theorem we can write

$$D^\ell x^*(t) = \frac{1}{(\ell-1)!} \int_0^1 D_t^\ell (t - \tau)_t^{\ell-1} \; x^{*(\ell)}(t)\, dt$$

where the D_t indicates differences with respect to t and where

$$(t-\tau)_+^{\ell-1} = \begin{cases}(t-\tau)^{\ell-1} & \text{for } t-\tau \geq 0\\ 0 & \text{for } t-\tau \leq 0\end{cases}$$

as usual. $K_\ell(\tau - t) \equiv \frac{1}{(\ell-1)!} D_t^\ell (t-\tau)_+^{\ell-1}$ is a "basic spline" [Curry-Schoenberg (1966)], that is

$$K_\ell(s) \equiv \frac{1}{h^\ell(\ell-1)!} \sum_{i=0}^{\ell} (-1)^{\ell-i} \binom{\ell}{i} (ih-s)_+^{\ell-1}$$

vanishes identically for $s \geq \ell h$ and $s \leq 0$, is strictly positive for s in $(0,\ell h)$, and lies in $C^{\ell-2}(-\infty,\infty)$. Thus $D^\ell t^\ell \equiv \ell ! = \int_0^{\ell h} K_\ell(s)ds$, and we find

$$\int_0^{\ell h} K_\ell(s)ds = 1.$$

Also

$$K_\ell(s) = \frac{1}{h^\ell(\ell-1)!} \sum_{i=0}^{\ell} (-1)^{\ell-i} \binom{\ell}{i} (i - \frac{s}{h})_+^{\ell-1}\ h^{\ell-1} \leq \frac{G}{h}$$

for some fixed G since we have $0 \leq \frac{s}{h} \leq \ell$.

We now show that $r_h x^* \varepsilon C_2(h)$ for large enough ε_h.

<u>Lemma 3.1</u> $-\varepsilon_h + \alpha_i(t_j) \leq M_{i,h} y_h(t_j) \leq \beta_i(t_j) + \varepsilon_h$ for $0 \leq j \leq Q-m+1$ and $1 \leq i \leq k$, and $-\varepsilon_h + \gamma_i \leq N_{i,h} y_h(\xi_i) \leq \delta_i + \varepsilon_h$ for $1 \leq i \leq n$ where

$\varepsilon_h = Gh^{\frac{1}{q}}$ and $G \geq \|x^*\|_o \sum_{\ell=0}^{m-1} \|b_{i\ell}\|_\infty \ell^{\frac{1}{q}}$ for $1 \leq i \leq k$, and

$G \geq \|x^*\|_o \sum_{\ell=0}^{m-1} \|c_{i\ \ell}\|_\infty \ell^{\frac{1}{q}}$ for $1 \leq i \leq h$.

Proof: For $0 \leq \ell \leq m-1$, $|D^\ell y_h(t_j) - x^{*(\ell)}(t_j)| = |D^\ell x^*(t_j) - x^{*(\ell)}(t_j)|$

$$= \left|\int_{t_j}^{t_j + \ell h} K_\ell(\tau - t_j)[x^{*(\ell)}(\tau) - x^{*(\ell)}(t_j)]dt\right| \leq \sup_{t_j \leq \tau \leq t_j + \ell h} |x^{*(\ell)}(\tau) - x^{*(\ell)}(t_j)|.$$

Now $|x^{*(\ell)}(\tau) - x^{*(\ell)}(t_j)| \leq \int_{t_j}^{\tau} |x^{*(\ell+1)}(s)|ds \leq \|x^*\|_o |\tau - t_j|^{\frac{1}{q}} \leq \|x^*\|_o |\ell h|^{\frac{1}{q}}$.

More generally, $|M_{i,h}y_h(t_j) - M_i x^*(t_j)| = |M_{i,h}x^*(t_j) - M_i x^*(t_j)| \leq$

$\sum_{\ell=0}^{m-1} |b_{i\ell}(t_j)\| D^\ell x^*(t_j) - x^{*(\ell)}(t_j)| \leq \sum_{l=0}^{m-1} |b_{i\ell}(t_j)| \|x^*\|_o |\ell h|^{\frac{1}{q}}$ and similarly

for N_i and $N_{i,h}$. The lemma follows since $\alpha_i(t_j) \leq M_i x^*(t_j) \leq \beta_i(t_j)$ and

similarly for N_i. Q. E. D.

As our last step in treating r_h, we show that $|f_h(r_h x^*) - f(x^*)|$ converges to zero.

Lemma 3.2 $\lim_{h \to 0} |f_h(r_h x^*) - f(x^*)| = 0$.

Proof: Since the m-times continuously differentiable functions are dense in $W^{m,p}$, we can find such a function z arbitrarily near x^* and such that $|f(x^*) - f(z)|$ is arbitrarily small. Since, for $0 \le \ell \le m$,

$$|D^\ell x^*(t_j) - D^\ell z(t_j)| \le \int_{t_j}^{t_j+\ell h} K_\ell(\tau - t_j)\,|x^{*(\ell)}(\tau) - z^{(\ell)}(\tau)|\,dt$$

$$\le \left\{\int_{t_j}^{t_j+\ell h} K_\ell(\tau - t_j)dt\right\}^{\frac{1}{q}} \left\{\int_{t_j}^{t_j+\ell h} K_\ell(\tau - t_j)\,|x^{*(\ell)}(\tau) - z^{(\ell)}(\tau)|^p dt\right\}^{\frac{1}{p}}$$

$$\le \left\{G\ell \int_0^1 |x^{*(\ell)}(\tau) - z^{(\ell)}(\tau)|^p dt\right\}^{\frac{1}{p}}$$ which is arbitrarily small, and

since the functions a_ℓ are bounded, it is also clear that $|f_h(r_h x^*) - f_h(r_h z)|$ can be made arbitrarily small independent of h by choosing z near x^*. Thus we are through if, after fixing z, we can show that $|f(z) - f_h(r_h z)|$ tends to zero. By using the triangle inequality we immediately find

$$|f_h(r_h z)^{\frac{1}{p}} - f(z)^{\frac{1}{p}}|^p \le \sum_{i=0}^{Q-m} \int_{ih}^{ih+h} \left| \sum_{\ell=0}^{m-1} \left[a_\ell(t_i)\, D^\ell z(t_i) - a_\ell(t) z^{(\ell)}(t)\right] \right|^p dt$$

$$+ \int_{1-mh}^{1} \left| \sum_{\ell=0}^{m-1} a_\ell(t)\, z^{(\ell)}(t) \right|^p dt$$

the latter term of which clearly tends to zero with h for fixed z . For the former term, since $z^{(\ell)}$ is continuous, we have $D^{\ell} z(t_i) = z^{(\ell)}(\lambda_i)$ for some λ_i in $(t_i, t_i + \ell h)$. Then the former term equals

$$\sum_{i=0}^{Q-m} \int_{ih}^{ih+h} \left| \sum_{\ell=0}^{m-1} \left[a_\ell(t_i)\, z^{(\ell)}(\lambda_i) - a_\ell(t)\, z^{(\ell)}(t) \right] \right|^p dt .$$

Since a_ℓ and $z^{(\ell)}$ are both continuous, the term under the integral sign for this fixed z is bounded by some function $w(h)$ tending to zero with h and thus the whole expression is bounded by $\sum_{i=0}^{Q-m} \int_{ih}^{ih+h} w(h)dt \leq w(h)$. Q.E.D.

Remark. The preceding lemma is of some independent interest. As a special case, it says that $h \sum_{i=0}^{Q-m} |D^m x(t_i)|^p$ converges to $\int_0^1 |x^{(m)}(t)|^p dt$ for all x in $W^{m,p}$; since the sum looks something like a Riemann sum for the integral, it is interesting that convergence can be proved. This is vital for the work in this paper since [Mangasarian-Schumaker (1969)] did not give broad necessary continuity conditions for x^*; we know of examples in which $x^{*(m)}$ has countably infinitely many finite jumps although the constraining functions are very smooth.

Next we must consider a mapping p_h of x_h^* into $p_h x_h^* = z_h \in W^{m,p}$ near C with $|f(p_h x_h^*) - f_h(x_h^*)|$ converging to zero. Let v_h be an m-vector function on the mesh points, $v_h = (v_{h,o}, \ldots, v_{h,m-1})^T$, solving

$$\left.\begin{aligned}
&v_{h,o}(t_{j+1}) = v_{h,o}(t_j) + h\, v_{h,1}(t_j)\\
&\quad\vdots\\
&v_{h,m-2}(t_{j+1}) = v_{h,m-2}(t_j) + h\, v_{h,m-1}(t_j)\\
&v_{h,m-1}(t_{j+1}) = v_{h,m-1}(t_j) + h\left[-\sum_{i=0}^{m-1} a_i(t_j) v_{h,i}(t_j) + L_h x_h^*(t_j)\right]\\
&\qquad \text{for } 0 \le j \le Q-m, \text{ with } v_{h,i}(o) = D^i x_h^*(o) \text{ for } 0 \le i \le m-1.
\end{aligned}\right\} \qquad (2.4)$$

For convenience we have assumed $a_m(t) \equiv 1$ without loss of generality. Clearly then we have that

$$v_{h,i}(t_j) = D^i x_h^*(t_j) \text{ for } 0 \le i \le m-1 \text{ and } 0 \le j \le Q-m.$$

Consider the m- vector function V_h on $[0,1]$, $V_h = (V_{h,0}, \ldots, V_{h,m-1})^T$, solving the system of differential equations

$$\left.\begin{aligned}
&V_{h,0}^{(1)} = V_{h,1}\\
&\quad\vdots\\
&V_{h,m-2}^{(1)} = V_{h,m-1}\\
&V_{h,m-1}^{(1)} = -\sum_{i=0}^{m-1} a_i V_{h,i} + u_h\\
&\text{with } V_{h,i}(0) = v_{h,i}(0) \text{ for } i = 0,\ldots,m-1, \text{ and } u_h(t) = L_h x_h^*(t_i) \text{ for}\\
&t_i \le t < t_{i+1} \text{ and } 0 \le i \le Q-m,\ u_h(t) = 0 \text{ for } t \ge 1-(m-1)h .
\end{aligned}\right\} \qquad (2.5)$$

We see immediately that the v_h is obtained by applying Euler's method to solve the system in Equation 2.5 which, for convenience, we write as

$$V_h^{(1)} = A\,V_h + e\,u_h,\ V_h(0) = v_h(0) \tag{2.6}$$

where A is the obvious matrix and $e = (0,0,\ldots,0,1)^T$.

We now define $z_h = p_h x_h^* \equiv V_{h,0}$. We notice that $z_h = V_{h,0}$ solves the equation $L z_h = u_h$ and thus

$$\int_0^1 |L z_h(t)|^p dt = \int_0^1 |u_h(t)|^p dt = h \sum_{i=0}^{Q-m} |L_h x_h^*(t_i)|^p, \text{ that is,}$$

$$f(p_h x_h^*) = f_h(x_h^*)\ . \tag{2.7}$$

Thus we have accomplished the goal of making $|f(p_h x_h^*) - f_h(x_h^*)|$ tend to zero; we now check to see if $z_h = p_h x_h^*$ is "near" C by relating V_h to v_h.

Now we write

$$V_h(t_{j+1}) = V_h(t_j) + \int_{t_j}^{t_{j+1}} [A(t)V(t) + e\,u_h(t)]dt$$

and

$$v_h(t_{j+1}) = v_h(t_j) + \int_{t_j}^{t_{j+1}} [A(t_j)v_h(t_j) + e\,u_h(t)]dt.$$

Letting $e_h(t_j) = V_h(t_j) - v_h(t_j)$ and arguing in the usual way we find, writing

$\|e_h(t_j)\|_\infty = \max\limits_{0 \le i \le m-1} |e_{h,i}(t_j)|$ and $F = \max\limits_{0 \le t \le 1} \|A(t)\|_\infty$,

$$\|e_h(t_j)\|_\infty \le \frac{\exp(F) - 1}{(1+\frac{1}{q})F} h^{\frac{1}{q}} \int_0^1 |(A(t)\ V_h(t))^{(1)}|^p dt \quad .$$

Thus v_h and V_h will be uniformly close if $AV_h \varepsilon W^{1,p}$ and is uniformly bounded in $W^{1,p}$. If $A^{(1)} \varepsilon C[0,1]$, that is $A \varepsilon C^1[0,1]$, then since $(AV_h)^{(1)} = A^{(1)} V_h + A^2 V_h + A\ e\ u_h$ and $\|A^{(1)}\|_\infty$, $\|A\|_\infty$, and $\int_0^1 |u_h(t)|^p dt$ are uniformly bounded, $A V_h$ will be uniformly bounded in $W^{1,p}$ if V_h is uniformly bounded in $L^p(0,1)$; this finally is clearly true if $V_h(0)$ is uniformly bounded in $\mathbb{R}^m$.

<u>Lemma 3.3</u> If the coefficients a_i defining the operator L are in $C^1[0,1]$, then there exists a constant K such that $\|v_h(t_j) - V_h(t_j)\|_\infty \le K h^{\frac{1}{q}}$ for $0 \le j \le Q-m$, $\frac{1}{p} + \frac{1}{q} = 1$. Also $\|V_{h,o}\| = \|p_h x_h^*\|$ is uniformly bounded.

<u>Proof</u>: Because of the preceding arguments, we need only show that $V_h(0) = v_h(0)$ is uniformly bounded in $\mathbb{R}^m$. Because of Equation 2.4, we can write $v_h(t_j)$ via

$$v_h(t_j) = h \sum_{i=0}^{j-1} [I + hA(t_{j-1})]\ldots[I + hA(t_{i+1})]u_h(t_i) + [I + hA(t_{j-1})]\ldots[I + hA(t_o)]\ v_h(0). \tag{2.8}$$

Consider the term in this expression involving the sum; this, call it w_h, solves Equation 2.4 with $w_h(0) = 0$ and is therefore within $O(h^{\frac{1}{q}})$ of the solution W_h to Equation 2.6 with $W_h(0) = 0$ by our preceding arguments. Note that by applying the operators $M_{\ell,h}$ at $\theta_{\ell,r}$ and $N_{r,h}$ at ξ_r to the first components of the vectors on both sides of Equation 2.8, we immediately see that $v_h(0)$ solves a certain system of linear equations. Applying one of this operators to the first component of the left hand side yields merely that operator applied to x_h^*, and these values are uniformly bounded at the $\theta_{\ell,r}$ and ξ_r. Applying an operator D^ℓ for $0 \leq \ell \leq m-1$ to $w_{h,0}$ merely gives $w_{h,\ell}$ which is uniformly close to $W_{h,\ell} = W_{h,0}^{(\ell)}$ which is uniformly bounded; it then follows that application of one of the operators $M_{\ell,h}$ or $N_{r,h}$ to $w_{h,0}$ gives uniformly bounded values.

Thus we have found that $v_h(0)$ solves a linear system with right hand side uniformly bounded in $\mathbb{R}^m$. A typical row in the matrix B_h of this system consists of, say, $M_{\ell,h}$ applied at $\theta_{\ell,r}$ to the components of the first row of the matrix function whose value at t_j is

$$[I + hA(t_{j-1})]\ldots[I + hA(t_o)] \ .$$

Arguing as we have done above it is easy to show that such an expression converges uniformly to the row (the collection of which forms a matrix B) consisting of the application of M_ℓ at $\theta_{\ell,r}$ to the components of the first row of the matrix function whose value at t is

$$\exp[\int_0^t A(\tau)d\tau] .$$

A matrix B of such rows however must be of full rank since by assumption there are no nonzero functions $x \varepsilon W^{m,p}$ such that $\|x\| = 0$. If we only apply those operators at those points which in the limit give an $m \times m$ nonsingular matrix, as we can always do since rank $(B) = m$, then for small h the matrices multiplying $v_h(0)$ are uniformly nonsingular and therefore the $v_h(0) = V_h(0)$ are uniformly bounded in $\mathbb{R}^m$. Since $V_h(0)$ is uniformly bounded it follows that $\|V_{h,o}\|$ is also. Q. E. D.

We can now prove convergence for the more complex discretization.

<u>Theorem 3.1</u> Let the general assumptions of Section 1 hold, and let the fixed points $\theta_{\ell,i}$ defining the norm $\|\cdot\|$ in Equation 2.1 (or the θ_i in Equation 2.2) be mesh points in our discretization for all h. Let the problem in Equation 1.1 not admit solutions of arbitrarily large $W^{m,p}$ - norm, for example, some $M_\ell x \equiv x$. Let $x_h^* \varepsilon \mathbb{R}^{Q+1}$ solve the problem in Equation 1.3, that is minimize $f_h(x_h)$ over $C_2(h)$, where $\varepsilon_h \geq Gh^{\frac{1}{q}}$ and G is defined in Lemma 3.1; one may thus take $h^{\frac{1}{q}} = 0(\varepsilon_h)$ for small h. Suppose the functions a_i defining L lie in $C^1[0,1]$. Let $p_h x_h^* = z_h$ solve

$$Lz_h = \begin{cases} L_h x_h^*(t_i) & \text{for } t_i \le t < t_{i+1},\ 0 \le i \le Q-m \\ 0 & \text{for } t \ge 1-(m-1)h\ . \end{cases}$$

Then $f_h(x_h^*)$ converges to $f(x^*)$ and all $W^{m,p}$ weak limit points, at least one of which exists, of $\{p_h x_h^*\}$, minimize f over C; if x^* minimizing f over C is unique, then $p_h x_h^*$ converges weakly to x^*. If (some subsequence of) $p_h x_h^*$ converges weakly to a point x, then x_h^* and its first $m-1$ difference approximations $D^\ell x_h^*$ evaluated at the points $t_i = ih$, $0 \le i \le Q-\ell$, converge uniformly to x and its first $m-1$ derivatives at the points t_i.

Proof: By Lemma 3.1 and the hypothesis on ε_h, $r_h x^* \varepsilon C_2(h)$ so $C_2(h)$ is not empty. Since $C_2(h)$ is not empty, x_h^* exists. By Lemmas 3.3 and 2.2 and the facts that $v_{h,o} = x_h^* \varepsilon C_2(h)$, and $\|p_h x_h^*\|$ is uniformly bounded, there exist functions $\eta_i(h)$ tending to zero with h and such that $p_h x_h^* \varepsilon C_h \equiv$

$$\{x; x \varepsilon W^{m,p},\ -\eta_i(h) - \varepsilon_h - Kh^{\frac{1}{q}} + \alpha_i(t) \le M_i x(t) \le \beta_i(t) + \eta_i(h) + \varepsilon_h + Kh^{\frac{1}{q}}$$

$$\text{for } 1 \le i \le k,\ -\varepsilon_h - Kh^{\frac{1}{q}} + \gamma_i \le N_i x(\xi_i) \le \delta_i + \varepsilon_h + Kh^{\frac{1}{q}} \text{ for } 1 \le i \le n.\}.$$

By Lemma 2.4, $\zeta_h \equiv \min_C f - \min_{C_h} f$ tends to zero. We write

$$f(x^*) = \min_C f = \min_{C_h} f + \zeta_h \le f(p_h x_h^*) + \zeta_h = f_h(x_h^*) + \zeta_h,$$

the last equality following from the construction of p_h. Thus we have

$$f(x^*) \le f(p_h x_h^*) + \zeta_h = f_h(x_h^*) + \zeta_h \le f_h(r_h x^*) + \zeta_h$$

$$\le f(x^*) + \zeta_h + [f_h(r_h x^*) - f(x^*)].$$

From Lemma 3.2 and this inequality we conclude that $f(x^*) = \lim_{h\to 0} f(p_h x_h^*)$ $= \lim_{h \to 0} f_h(x_h^*)$. Since, from Lemma 3.3, $p_h x_h^*$ is bounded, it has weak limit points; for any such weak limit point x' we have $x' \in C$ by Lemma 2.3 and thus

$$f(x^*) \le f(x') \le \lim_{h' \to 0} \inf f(p_h x_h^*) = f(x^*)$$

which says that x' minimizes f over C. If $p_h x_h^*$ converges to some x weakly in $W^{m,p}$, then $p_h x_h^*$ and its first $m-1$ derivatives converge uniformly to x and its first $m-1$ derivatives. By Lemma 3.3, the numbers $v_{h,\ell}(t_j) = D^\ell v_{h,0}(t_j) = D^\ell x_h^*(t_j)$ are uniformly close to $V_{h,\ell}(t_j) = V_{h,0}^{(\ell)}(t_j) = p_h x_h^{*(\ell)}(t_j)$ for $0 \le \ell \le m-1$. Q.E.D.

We have not been able to estimate the rate of convergence as a function of h.

4. An elementary example.

Consider the example in [Mangasarian-Schumaker (1969)] with $m = 1$, $Lx \equiv x^{(1)}$, $k = 1$, $M_1 x \equiv x$, $\alpha_1(t) = t - t^2$, $\beta_1(t) = t$, $n = 1$, $N_1 x(\xi_1) = x(1)$, $\delta_1 = \gamma_1 = c \in [0,1]$, that is,

$$\text{minimize} \quad \int_0^1 |x^{(1)}(t)|^2 dt$$

$$\text{over} \qquad C = \{x; x \in W^{1,2},\ t - t^2 \le x(t) \le t,\ x(1) = c\}.$$

The unique solution to this problem is

$$x^*(t) = \begin{cases} t - t^2 & \text{for } 0 \le t \le 1 - \sqrt{c} \\ (2\sqrt{c} - 1)t + (1 + c - 2\sqrt{c}) & \text{for } 1 - \sqrt{c} \le t \le 1 \end{cases}$$

as pictured in Figure 4.1.

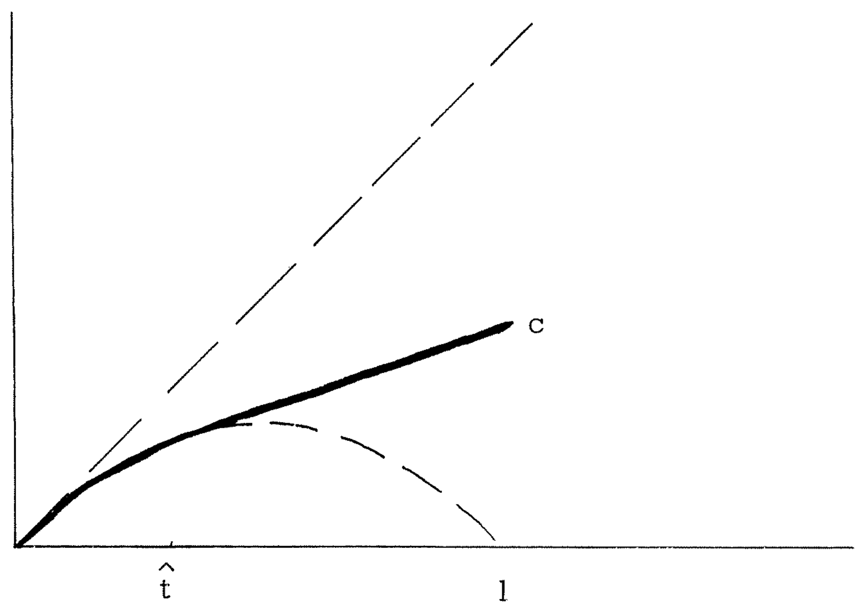

Figure 4.1

Let $\hat{t} = 1 - \sqrt{c}$, the point at which x^* leaves the lower curve.

If we use the simple discretization and merely discretize the constraints at $t_i = ih$, $0 \le i \le Q = \frac{1}{h}$, then if $\hat{t}_i = \max\ \{t_i; t_i \le \hat{t}\}$ then x_Q^* is just the piecewise linear interpolant of $t - t^2$ at t_i for $t \leq \hat{t}_i$ and is the linear interpolant between $\hat{t}_i - \hat{t}_i^2$ and c for $\hat{t}_i \leq t \leq 1$. For small enough h, the solution for the complete discretization is also unique; such discrete variational splines are studied in [Mangasarian-Schumaker (1970)].

If we define, for $c > \frac{1}{2}$, the numbers

$$\alpha_h = \min\{t_i;\ t_i \geq \sqrt{2\varepsilon_h}\},\ \beta_h = \max\{t_i;\ t_i \leq 1 - \sqrt{c}\},$$

the unique solution x_h^* for the complete discretization with $\varepsilon_h = h^{\frac{1}{2} - \delta}$, $\delta \in (0, \frac{1}{2})$ is

$$x_h^*(t_i) = \begin{cases} \varepsilon_h + t_i \left[\dfrac{\alpha_h - \alpha_h^2 - 2\varepsilon_h}{\alpha_h}\right] & \text{if } 0 \leq t_i \leq \alpha_h \\ t_i - t_i^2 - \varepsilon_h & \text{if } \alpha_h \leq t_i \leq \beta_h \\ \beta_h - \beta_h^2 - \varepsilon_h + (t_i - \beta_h)\left[\dfrac{c - \beta_h + \beta_h^2}{1 - \beta_h}\right] & \text{if } \beta_h \leq t_i \leq 1. \end{cases}$$

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