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Non-Linear Eigenvalue Problems for Some Fourth Order Equations

I. MAXIMAL SOLUTIONS

by

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1. Introduction

This work was motivated by the paper [8] of F.ODEH and I. TADJBAKHSH who discussed two specific nonlinear eigenvalue problems which arise in the study of the equilibrium states of a thin rotating rod. They consider the nonlinear system

$$1.1)\qquad \begin{cases} u'' = \lambda \sin \theta , & 0 < t < 1 , \\ \theta'' = \lambda\, u \cos \theta , & 0 < t < 1 , \end{cases}$$

and the two sets of boundary conditions

A·) $\qquad u'(0) = \theta(0) = u(1) = \theta'(1) = 0 ,$

and

B·) $\qquad u'(0) = \theta'(0) = u(1) = \theta(1) = 0 .$

N. Bazley and B. Zwahlen [1] also studied equations (1.1) under the boundary conditions (A·).

These interesting papers employ a variety of methods to obtain information about the existence of solutions when $\lambda > \lambda_0$, the smallest positive eigenvalue of the linearized problem (linearized about zero). In particular, Odeh and Tadjbakhsh prove that there always is a nontrivial solution (in both cases) when $\lambda_0 < \lambda$. Moreover, they make the following conjecture: if $\lambda n < \lambda \leq \lambda n{+}1$ then there are (at least) n+1 distinct nontrivial solutions $(u_j(t), \theta_j(t))$ $j = 0,1,\ldots n$.

We became interested in these problems because the physical solution $(u(t), \theta(t))$ must satisfy (see the discussion on page 83 of [8])

1.2) $$|\theta(t)| < \frac{\Pi}{2} .$$

However, there is no discussion of the "size" of the solution obtained in [8] and [1].

In this report we formulate a general class of problems which include equations (1.1) and study the existence and uniqueness of "maximal solutions." While we are unable to prove that all solutions of equations (1.1) which satisfy the boundary conditions A or B, also satisfy (1.2), we are able to establish the existence of a maximal, "positive" solution which also satisfies condition (1.2).

The general problem is formulated in section 2. In section 3 we remind the reader of some basic facts about second order problems as developed in [2] and our previous work. Section 4 uses those results and an idea due to Picard [12] (in the second order case) to establish the existence of positive solutions. Section 5 is devoted to the unicity of such positive solutions and their role as "maximal" solutions. Because these positive solutions are maximal solutions and provide bounds on all solutions it is particularly relevant that our proof is a constructive proof. The basic existence proof is based on a nonlinear iteration which may be easily adapted to numerical computation. Finally in section 6 we establish the conjecture of Odeh and Tadjbakhsh for the boundary conditions B.

In part II of this work we turn to the application of fixed point theorems to prove the existence of other solutions. In particular the conjecture is established for the boundary conditions A.

2. The General Problem

Let

$$L_k[\varphi] \equiv (p_k(t)\varphi')' - c_k(t)\varphi(t) , \qquad k = 1,2$$

be two regular Sturm Liouville operators. That is,

$$\begin{cases} c_k(t) \in C[0,1], \quad c_k(t) \geq 0, & 0 \leq t \leq 1 \\ p_k(t) \in C'[0,1], \quad p_k(t) \geq p_o > 0 , & 0 \leq t \leq 1 \end{cases}$$

for some positive constant p_o.

Consider the nonlinear systems of ordinary* differential equations

$$\text{2.1)} \qquad \begin{cases} L_1[u] = \lambda\,\theta\, H_1(t,u,\theta) = \lambda F_1(t,u,\theta), & 0 < t < 1 , \\ L_2[\theta] = \lambda\, u\, H_2(t,u,\theta) = \lambda F_2(t,u,\theta), & 0 < t < 1 , \end{cases}$$

where the functions $u(t)$, $\theta(t)$ are required to satisfy the homogeneous boundary conditions

$$\text{2.2)} \qquad \begin{cases} A_o[u] \equiv a_o\, u(o) - b_o\, u'(o) = 0 , \\ A_1[u] \equiv a_1\, u(1) + b_1\, u'(1) = 0 , \\ B_o[\theta] \equiv \alpha_o\, \theta(0) - \beta_o \theta'(o) = 0 , \\ B_1[\theta] \equiv \alpha_1\, \theta(1) + \beta_1 \theta'(1) = 0 , \end{cases}$$

with

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*The operators $L_k[\varphi]$, $k = 1,2$ could equally well be two uniformly elliptic second order operators on a smooth domain $\Omega \subset \mathbb{R}^n$. However, the present treatment enables us to concentrate on the essential ideas and not get concerned with some technical "smoothness" questions.

$$2.3) \quad \begin{cases} \alpha_k, a_k, \beta_k, b_k \geq 0 \,, & k = 1,2 \\ a_k + b_k > 0 \,, \quad \alpha_k + \beta_k > 0 \,, \quad a_o + a_1 > 0, \; \alpha_o + \alpha_1 > 0 \,. & \end{cases}$$

For simplicity, we assume that the functions H_k (t,u,θ) are even, i.e.

$$2.4) \qquad H_k(t,u,\theta) = H_k(t, |u|, |\theta|), \quad k = 1,2 \,.$$

With this convention we see that $-\lambda$ is an eigenvalue with eigenfunction $(-u,\theta)$ whenever λ is an eigenvalue with eigenfunction (u,θ). Thus we may restrict our attention to the case where $\lambda > 0$.

Definition 2.1 The problem described by equations (2.1), (2.2) is called "normal" if

$$H_k(t,u,\theta) > 0 \,, \quad k = 1,2$$

for all $t \in [0,1]$ and all real u,θ.

Definition 2.2 The problem described by equations (2.1), (2.2) is called a "cut off" problem if there is a finite positive constant Θ such that

(i) $H_k(t,u,\theta) = H_k(t,u,\Theta), \quad \Theta \leq |\theta|, \; k = 1,2$

(ii) $H_k(t,u,\theta) > 0 \,, \quad |\theta| < \Theta \,, \; k = 1,2$

(iii) $H_2(t,u,\Theta) = 0 \quad \forall \; t \in [0,1]$ and all u .

A pair of functions $u(t),\theta(t)$ is called a solution to a cut off problem if and only if they satisfy equations (2.1), (2.2) and

$$|\theta(t)| < \Theta \,.$$

<u>Remark</u>: The problem of Odeh and Tadjbakhsh described by (1.1) is reduced to a cut off problem by setting

$$H_1(t,u,\theta) = \begin{cases} \dfrac{\sin \theta}{\theta}, & |\theta| \le \dfrac{\pi}{2}, \\[2ex] \dfrac{2}{\pi}, & |\theta| > \dfrac{\pi}{2}, \end{cases}$$

$$H_2(t,u,\theta) = \begin{cases} \cos \theta, & |\theta| \le \dfrac{\pi}{2}, \\[2ex] 0, & |\theta| > \dfrac{\pi}{2}. \end{cases}$$

We assume that $F_k(t,u,\theta) \varepsilon C'$ except possibly at $|\theta| =$ (") in the cut off case. We will consider the following hypotheses on the coefficients

H.1) The function $F_1(t,u,\theta)$ is monotone nondecreasing in θ and $F_2(t,u,\theta)$ is monotone nondecreasing in u. We write

2.5) $$\frac{\partial}{\partial \theta} F_1(t,u,\theta) \ge 0, \quad \frac{\partial}{\partial u} F_2(t,u,\theta) \ge 0,$$

even though this statement may not be true at $|\theta| =$ (") . Observe that these conditions may be rewritten as

2.5a) $$\begin{cases} H_1(t,u,\theta) + \theta \dfrac{\partial}{\partial \theta} H_1(t,u,\theta) \ge 0, \\[3ex] H_2(t,u,\theta) + u \dfrac{\partial}{\partial u} H_2(t,u,\theta) \ge 0. \end{cases}$$

H.2) There are two functions $G_1(t,\theta)$, $G_2(t,u)$ such that

$$0 \le H_1(t,u,\theta) \le G_1(t,\theta)$$

$$0 \le H_2(t,u,\theta) \le G_2(t,u)$$

for all $t \in [0,1]$ and all u, θ.

H.3) The functions $H_k(t,u,\theta)$ are monotone nonincreasing in $|u|, |\theta|$. That is,

2.6) $$u \frac{\partial}{\partial u} H_k(t,u,\theta) \leq 0, \ \theta \frac{\partial}{\partial \theta} H_k(t,u,\theta) \leq 0, \ k = 1,2.$$

However, the system (2.1), (2.2) should be genuinely "nonlinear." Hence, in addition to (2.6) we assume if $\bar{u}, \bar{\theta}$ are positive and C is a constant with $C > 1$, then

2.6a) $$H_k(t, C\bar{u}, C\bar{\theta}) < H_k(t, \bar{u}, \bar{\theta}), \quad k = 1,2 .$$

3. Second Order Problems - A REVIEW

Let $L[\varphi]$ be a regular Sturm Liouville operator and consider the nonlinear boundary value problem

3.1) $$\begin{cases} L[\varphi] = f(t,\varphi), & 0 < t < 1 \\ A_o[\varphi] = A_1[\varphi] = 0 \end{cases}$$

where the boundary operators $A_o[\varphi]$, $A_1[\varphi]$ are described by equation (2.2), (2.3). The function $f(t,\varphi)$ is continuous in (t,φ) and satisfies a Lipschitz condition in φ with Lipschitz constant γ.

Definition 3.1 Let $\varphi_1(t), \varphi_2(t) \in C^1[0,1]$. We say φ_1 dominates φ_2 if

3.2a) $$\varphi_2(t) < \varphi_1(t), \quad 0 < t < 1$$

3.2b) $$\varphi_1(0) = \varphi_2(0) \Longrightarrow \varphi_2'(0) < \varphi_1'(0),$$

3.2c) $$\varphi_1(1) = \varphi_2(1) \Longrightarrow \varphi_1'(1) < \varphi_2'(1).$$

If $\varphi_1(t)$ dominates $\varphi_2(t)$ we write

3.3) $$\varphi_2 \prec \varphi_1$$

The concept of domination* arises in the study of second order equations through the strong form of the maximum principle and Hopf's lemma [2]. Together these principles give the following assertion: If

3.4a) $$L[\varphi] \leq 0$$

then

3.4b) $$\varphi(t) \geq \text{Min } \{0, \varphi(0), \varphi(1)\} .$$

Moreover, if equality (in (3.4b)) occurs at any interior point, then

3.4c) $$\varphi(t) \equiv \text{constant}.$$

Furthermore, if $\varphi(0) \geq 0$ and

3.5a) $$\varphi(0) = \min \varphi(t), \quad 0 \leq t \leq 1,$$

then either (3.4c) holds or

3.5b) $$\varphi'(0) > 0 .$$

similarly, if $\varphi(1) \geq 0$ and $\varphi(t)$ assumes its minimum at $t = 1$, then either (3.4c) holds or

3.5c) $$\varphi'(1) < 0 .$$

These facts lead to the following basic lemma.

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*We will make essential use of this concept only in section 5 when we are concerned with uniqueness.

Lemma 3.1 If $L[\varphi] \leq 0$, $A_o[\varphi] = A_1[\varphi] = 0$, then either $\varphi(t) \equiv 0$ or

$$0 \prec \varphi(t).$$

The next lemmas collect some basic facts about solutions of the problem (3.1) as developed in [2] and [11]. In [11] we developed the basic ideas for the special case where

$$L \equiv \left(\frac{d}{dt}\right)^2, \qquad b_o = b_1 = 0.$$

However, using lemma 3.1 one may easily adapt the proofs to the general case.

Lemma 3.2 Let $f(t,\varphi)$ be bounded for all (t,φ). Suppose $a(t) \in C^2[0,1]$ satisfies

3.6) $$\begin{cases} L[a] \leq f(t,a), & L[a] \not\equiv f(t,a) \\ A_o[a] \geq 0, \quad A_1[a] \geq 0, \end{cases}$$

Then there is a function $u(t)$ which is a solution of equation (3.1) which satisfies

3.7) $$u(t) \prec a(t)$$

Moreover, if $z(t)$ is any other solution of equation (3.1) which satisfies

3.8a) $$z(t) \leq a(t),$$

then

3.8b) $$z(t) \leq u(t).$$

Finally, if $f_1(t,u) \geq f(t,u)$ for all u, then the solution u_1, of

$$L[u_1] = f_1(t,u_1), \quad A_o[u_1] = A_1[u_1] = 0$$

which is determined by this process satisfies

3.8c) $$u_1 \prec u.$$

Similarly, let $b(t) \in C^2[0,1]$ satisfies

3.9) $$\begin{cases} L[b] \geq F(t,b) , & L[b] \equiv f(t,b), \\ A_o[b] \leq 0 , \; A_1[b] \leq 0 . \end{cases}$$

Then, there is a function $v(t)$ which is a solution of equation (3.1) which satisfies

3.10) $$b(t) \prec v(t) .$$

Moreover, if $z(t)$ is any solution of equation (5.1) which satisfies

3.11a) $$b(t) \leq z(t).$$

then

3.11b) $$v(t) \leq z(t) .$$

Finally, if $f_2(t,u) \leq f(t,u)$ for all u, then the solution $v_2(t)$ of

$$L[v_2] = f_2(t,v_2), \; A_o[v_2] = A_1[v_2] = 0$$

which is determined by this process satisfies

3.11c) $$v \prec v_2.$$

Proof: Consider the iteration

$$L[z_{n+1}] - \gamma z_{n+1} = f(t,z_n) - \gamma z_n$$

with $z_o(t) = a(t)$ or $z_o(t) = b(t)$. The argument proceeds by induction as in [11] .

On the basis of this lemma we define two operations $U(a)$, $V(b)$ by

3.12) $$U(a) = u(t) , \qquad V(b) = v(t) .$$

Lemma 5.3 If $f(t, \varphi)$ is monotone nondecreasing in φ, then equation (3.1) has a unique solution.

Corollary: Suppose $f(t,\varphi) \leq 0$ and is monotone nondecreasing in φ for $\varphi \geq 0$. Then there exists a unique nonnegative solution $\varphi(t)$. Similarly, suppose $f(t,\varphi) \geq 0$ and is monotone nondecreasing in φ for $\varphi \leq 0$. Then there exists a unique nonpositive solution $\varphi(t)$.

Proof: We consider only the first case. We observe that if there is a solution $\varphi(t)$ of equation (3.1) it is nonnegative.

Let

$$f_o(t,\varphi) = \begin{cases} f(t,\varphi), & \varphi \geq 0 \\ f(t,0), & \varphi \leq 0 . \end{cases}$$

Then, $\varphi(t)$ is a solution of equation (3.1) if and only if $\varphi(t)$ is a solution of

$$L[\varphi] = f_o(t,\varphi(t)).$$

But, this equation has a unique solutions because $f_o(t,\varphi)$ is nondecreasing in φ.

Lemma 3.4 Suppose $f(t,\varphi) \leq 0$. Suppose there is a constant $k > 0$ such that

$$f(t,\varphi) = 0 \quad \text{for} \quad k \leq \varphi .$$

Let $\varphi(t)$ be a solution of equation (3.1). Then

$$0 \leq \varphi(t) \leq k$$

Proof: Suppose there is a point $t_o \in (0,1)$ such that

$$\varphi(t_o) > k.$$

Then there is an interval $[\rho, \delta]$ about t_o such that

3.13) $$\varphi(t) \geq k \quad \text{for} \quad t \in [\rho, \delta] \ .$$

Naturally, we take $[\rho, \delta]$ as large as possible.

<u>Case 1</u>: $\rho = 0, \ \delta = 1$. Then $L[\varphi] \equiv 0$ and the maximum principle asserts that $\varphi(t) \equiv 0 < k$.

<u>Case 2</u>: $\rho = 0, \ \delta < 1$. Then $\varphi(\delta) = k$ and $\varphi(0)$ is a maximum of $\varphi(t)$ for $t \in [0, \delta]$. If $\varphi(t)$ is not constant on this interval, we have

$$\varphi'(0) < 0 \ .$$

However, the boundary condition, $A_o[\varphi] = 0$, implies that $\varphi(0) = 0$ or $\varphi'(0) \cdot \varphi(0) > 0$. Since $\varphi(0) \geq k$ we have a contradiction.

<u>Case 3</u>: $\rho > 0, \ \delta = 1$. In this case we see that

$$\varphi'(1) > 0 \ , \quad \varphi(1) \geq k \ ,$$

but the boundary condition, $A_1[\varphi] = 0$, implies that

$$\varphi(1) = 0 \quad \text{or} \quad \varphi'(1)\, \varphi(1) < 0.$$

<u>Case 4</u>: $0 < \rho < \delta < 1$. Then $\varphi(\rho) = \varphi(\delta) = k$ and

$$L[\varphi] = 0 \ , \qquad \rho < t < \delta \ .$$

The maximum principle asserts that

$$\varphi(t) = k \ , \quad \rho \leq t \leq \delta \ .$$

4. "Positive" Solutions

We now return to the general problem (2.1).

Definition 4.1 A pair of functions $(u(t), \theta(t))$ will be called a "positive solution" of equation (2.1) if they are a solution and also satisfy

$$4.1) \qquad u \prec 0 \prec \theta$$

Note: If $(u(t), \theta(t))$ is a solution, so is $(-u(t), -\theta(t))$. Moreover, if either function, $u(t)$ or $\theta(t)$, is nonpositive (but not identically zero) the other function dominates the zero function.

Definition 4.2 A positive solution $(u(t), \theta(t))$ will be called a "maximal" solution if: whenever (w, Φ) is another nontrivial solution of equation (2.1), (2.2) (not necessarily positive) then

$$4.2) \qquad |\Phi(t)| \leq \theta(t) \qquad |w(t)| \leq |u(t)| = -u(t) .$$

Note: By the remarks above, (4.2) is equivalent to

$$4.2') \qquad \Phi(t) \leq \theta(t) , \qquad u(t) \leq w(t) .$$

Lemma 4.1 Suppose $(w(t), \Phi(t))$ is a nontrivial solution of equation (2.1),(2.2) and $\Phi(t) \geq 0$. Then (w, Φ) is a positive solution.

Proof: Apply lemma 3.1 and the remarks above.

Consider now the linear problem obtained by "linearizing" equations (2.1) about $(u, \theta) = (0,0)$. We obtain

$$4.3) \qquad \begin{cases} L_1[u] = \lambda\theta H_1(t,0,0) , & A_o[u] = A_1[u] = 0 , \\ L\ [\theta] = \lambda\theta H_2(t,0,0) , & B_o[u] = B_1[\theta] = 0 . \end{cases}$$

Let $K_1(s,t)$, $K_2(s,t)$ be the "Green's Functions" associated with the operators $-L_1[u]$ and $-L_2[\theta]$ respectively, subject to the appropriate homogeneous boundary conditions ($A_j[u] = B_j[\theta] = 0$, $j = 1,2$). Then the equations (6.3) are equivalent to

$$u(t) = -\lambda \int_0^1 K_1(t,x)\, H_1\, (x,0,0)\, \theta(x) dx \ ,$$

$$\theta(x) = -\lambda \int_0^1 K_2(x,y)\, H_2\, (y,0,0)\, u(y)\, dy \ .$$

On substitution, we obtain

4.4) $$\theta(t) = \lambda^2 \int_0^1 G(t,s)\theta(s) ds \ ,$$

with

4.4a) $$G(t,s) = \int_0^1 K_2(t,x) K_1(x,s) H_1\, (s,0,0) H_2\, (x,0,0) d\, x \ .$$

The kernel* $G(t,s)$ is a positive (nonnegative) kernel. Hence, the smallest eigenvalue λ_o^2 corresponds to an eigenfunction of constant sign (see [4], [5], [6], [7]). Thus, we may normalize the eigenfunction $(u_o(t), \theta_o(t))$ associated with the smallest positive eigenvalue $\lambda_o > 0$ so that

4.5) $$u_o < 0 < \theta_o \ .$$

Moreover, if $\lambda_o < \lambda$ we may scale $(u_o(t), \theta_o(t))$ so that (4.5) holds and

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*In fact, $G(t,s)$ is an oscillation kernel in the sense of Gantmacher-Krein [3]. However, we will not make use of this fact in this report.

4.6) $$R_k \equiv [1 - \frac{\lambda H_k(t, u_o, \theta_o)}{\lambda_o H_k(t,0,0)}] < 0, \; k = 1,2 .$$

A straightforward calculation now shows that

4.7) $$\begin{cases} L_1[u_o] \le \lambda F_1(t,u_o,\theta_o) , & A_o[u_o] = A_1[u_o] = 0 , \\ L_2[\theta_o] \ge \lambda F_2(t,u_o,\theta_o) , & B_o[\theta_o] = B_1[\theta_o] = 0 . \end{cases}$$

These inequalities, together with the mappings of lemma 3.2 enable us to construct an "increasing" sequence $(u_n(t), \theta_n(t))$.

Lemma 4.2 Let H.1 and H.2 hold. Suppose $(u_{n-1}(t), \theta_{n-1}(t))$ satisfy

4.8a) $$\begin{cases} L_1[u_{n-1}] \le \lambda F_1(t,u_{n-1}(t), \theta_{n-1}(t)) , \\ L_2[\theta_{n-1}] \ge \lambda F_2(t,u_{n-1}(t), \theta_{n-1}(t)) , \end{cases}$$

and

4.8b) $$\begin{cases} A_o[u_{n-1}] \ge 0 , & A_1[u_{n-1}] \ge 0 , \\ B_o[\theta_{n-1}] \le 0 , & B_1[\theta_{n-1}] \le 0 . \end{cases}$$

Let $u_n(t)$ be the solution of the nonlinear equation

4.9) $$L_1[u_n] = \lambda F_1(t,u_n,\theta_{n-1}(t)), \; A_o[u_n] = A_1[u_n] = 0$$

determined by lemma 3.2. That is

4.10) $$u_n = U(u_{n-1}) .$$

Then, unless $u_{n-1}(t)$ satisfies equation (4.9) and $u_n(t) \equiv u_{n-1}(t)$, we have

4.11a) $$u_n < u_{n-1},$$

and

4.11b) $$L_2[\theta_{n-1}] \geq \lambda F_2(t, u_{n-1}, \theta_{n-1}) \geq \lambda F_\theta(t, u_n(t), \theta_{n-1}) .$$

Thus, we may choose $\theta_n(t)$ as the solution of

4.12) $$L_2[\theta_n] = \lambda F_2(t, u_n(t), \theta_n), \; B_o[\theta_n] = B_1[\theta_n] = 0 ,$$

determined by lemma 3.2. That is

4.13) $$\theta_n = V(\theta_{n-1}) .$$

Then, unless θ_{n-1} satisfies equation (4.12) and $\theta_{n-1}(t) \equiv \theta_n(t)$, we have

4.14a) $$\theta_{n-1} \prec \theta_n$$

and

4.14b) $$L_1[u_n] = \lambda F_1(t, u_n, \theta_{n-1}) \leq \lambda F_1(t, u_n, \theta_n) .$$

In either case

4.15) $$u_n \leq u_{n-1}, \; \theta_{n-1} \leq \theta_n ,$$

and equations (4.8a), (4.8b) hold with n-1 replaced by n.

Proof: The condition H.2 permits us to apply lemma 3.2 while (4.11b) and (4.14b) follow from H.1.

Corollary 1. Suppose $0 < \lambda_o < \lambda$. Then we may choose $(u_o(t), \theta_o(t))$ as the solutions of the linear eigenvalue problem (4.3) associated with λ_o which also satisfy (4.5),(4.6) and (4.7). Thus we generate a sequence $(u_n(t), \theta_n(t))$, with

4.16) $$u_n(t) \leq u_{n-1}(t) \prec 0 \prec \theta_{n-1}(t) \leq \theta_n(t) .$$

The functions $u_n(t)$, $\theta_n(t)$ will satisfy equation (4.9) and (4.12) respectively. Moreover, either $u_n(t) \equiv u_{n-1}(t)$, or

4.17a) $$u_n \prec u_{n-1}.$$

And, either $\theta_n(t) \equiv \theta_{n-1}(t)$, or

4.17b) $$\theta_{n-1} \prec \theta_n.$$

Corollary 2. Let H.3 hold also. Then the functions $u_n(t)$, $\theta_n(t)$ are the unique solutions of equations (4.9) and (4.12) respectively.

Proof: Apply the corollary to lemma 3.3.

Corollary 3. If we are dealing with a cut off problem and we further "scale" $(u_o(t), \theta_o(t))$ so that $\theta_o(t) < \Theta$, then

$$\theta_n(t) \le \Theta \quad , \quad n = 1,2,\ldots.$$

Proof: Apply lemma 3.4.

We obtain our next result from the same argument.

Lemma 4.3 Let H.1 and H.2 hold. Suppose $(w_{n-1}(t), \Phi_{n-1}(t))$ satisfy

4.18a) $$\begin{cases} L_1[w_{n-1}] \ge \lambda F_1(t, w_{n-1}(t), \Phi_{n-1}(t)), \\ L_2[\Phi_{n-1}] \le \lambda F_2(t, w_{n-1}(t), \Phi_{n-1}(t)), \end{cases}$$

and

4.18b) $$\begin{cases} A_o[w_{n-1}] \le 0, \quad A_1[w_{n-1}] \le 0 \\ B_o[\Phi_{n-1}] \ge 0, \quad B_1[\Phi_{n-1}] \ge 0. \end{cases}$$

Let $w_n(t)$ be the solution of the nonlinear equation

4.19) $$L_1[w_n] = \lambda F_1(t, w_n, \Phi_{n-1}(t)) , \quad A_o[w_n] = A_1[w_n] = 0$$

determined by lemma 3.2. That is

$$w_n = V(w_{n-1}) .$$

Then, unless $w_n(t) \equiv w_{n-1}(t)$ and $w_{n-1}(t)$ satisfies (4.19),

4.20a) $$w_{n-1} < w_n$$

and

4.20b) $$L_2[\Phi_{n-1}] \leq \lambda F_2(t, w_n, \Phi_{n-1}) .$$

Thus, we may choose $\Phi_n(t)$ as the solution of

4.21) $$L_2[\Phi_n] = \lambda F_2(t, w_n(t), \Phi_n) , \quad B_o[\Phi_n] = B_1[\Phi_n] = 0 ,$$

determined by lemma 3.2. That is

$$\Phi_n = U(\Phi_{n-1}) .$$

Then, unless $\Phi_n(t) \equiv \Phi_{n-1}(t)$ and Φ_{n-1} satisfies (4.21)

4.22a) $$\Phi_n < \Phi_{n-1} ,$$

and

4.22b) $$L_1[w_n] = \lambda F_1(t, w_n, \Phi_{n-1}) \geq \lambda F_1(t, w_n, \Phi_n) .$$

In either case

$$w_{n-1} \leq w_n , \quad \Phi_n \leq \Phi_{n-1}$$

and equations (4.18a), (4.18b) hold with $n-1$ replaced by n.

Theorem 4.1 Suppose $\lambda_o < \lambda$, H.1 and H.2 hold. Suppose $(u_o(t), \theta_o(t))$ are the eigenfunctions of the linear eigenvalue problem (4.3) which also satisfy (4.5), (4.6), (4.7). Suppose there exists a pair of functions (w, Φ) such that

4.23a) $$w \prec u_o \prec 0 \prec \theta_o \prec \Phi ,$$

4.23b) $$L_1[w] \geq \lambda F_1(t,w,\Phi) , \quad A_o[w] \leq 0, \quad A_1[w] \leq 0 ,$$

4.23c) $$L_2[\Phi] \leq \lambda F_2(t,w,\Phi), \quad B_o[\Phi] \geq 0 , \quad B_1[\Phi] \geq 0 .$$

Then, there exists a positive solution $(u(t), \theta(t))$ of equations (2.1), (2.2). Moreover, either (w,Φ) is a solution or

4.24) $$w \prec u \prec u_o \prec 0 \prec \theta_o \prec \theta \prec \Phi .$$

Proof: Let $(u_n(t), \theta_n(t))$ be the monotone sequence generated by lemma 4.2 with (u_o, θ_o) chosen as above. Let (w_n, Φ_n) be the monotone sequence generated by lemma 4.3 with $w_o \equiv w$, $\Phi_o \equiv \Phi$. We shall prove

4.25a) $$w_o \leq u_n , \quad \theta_n \leq \Phi_o ,$$

4.25b) $$w_n \leq u_o , \quad \theta_o \leq \Phi_n .$$

Then the theorem will follow from standard estimates and the Ascoli-Arzela lemma. Indeed, each pair of sequences (u_n, θ_n), (w_n, Φ_n) will converge to a solution pair (u, θ) and $(\hat{w}, \hat{\Phi})$ respectively. Thus, there may be two solutions.

The proof follows by induction. By (4.23a) we have (4.25a), (4.25b) for $n = 0$. Suppose

$$w_o \leq u_{n-1}, \quad \theta_{n-1} \leq \Phi_o .$$

Then

$$L_1[u_{n-1}] \leq \lambda F_1(t, u_{n-1}, \theta_{n-1}) \leq \lambda F_1(t, u_{n-1}, \Phi_o).$$

Using lemma 4.2 we construct $u_n(t)$ which satisfies (4.9) and using lemma 3.2 we construct a function $\hat{v}(t)$ which satisfies

$$L_1[\hat{v}] = \lambda F_1(t, \hat{v}, \Phi_o) \ , \ A_o[\hat{v}] = A_1[\hat{v}] = 0$$

and

$$w_o = w \leq \hat{v} \leq u_n \ .$$

Thus, we establish (4.25a) for all n. A similar argument establishes (4.25b) and completes the proof.

<u>Theorem 4.2</u> Let $\lambda_o < \lambda$, H.1 and H.2 hold. Suppose we have a cut off problem. Then there is a maximal solution $(u(t), \theta(t))$.

<u>Proof</u>: Let

$$W = \text{MAX} \ \{\lambda F_1(t, w, \textcircled{''}); \ 0 \leq t \leq 1, \ |w| < \infty\} \ ,$$

and let $w(t)$ be the solution of

$$\begin{cases} L_1[w] = W \geq \lambda F_1(t, v, \textcircled{''}) \ , \ \forall \ v(t) \\ A_o[w] = 0 \ , \quad A_1[w] = 0 \ . \end{cases}$$

Let $\Phi(t) \equiv \textcircled{''}$. Then

$$\begin{cases} L_2[\Phi] = -C_2(t) \ \textcircled{''} \leq \lambda F_2(t, w, \textcircled{''}) = 0 \\ B_o[\Phi] \geq 0 \ , \quad B_1[\Phi] \geq 0 \ . \end{cases}$$

Thus, the pair (w, Φ) satisfy, the conditions of theorem 6.1 and there is a positive solution $(\hat{w}(t), \hat{\Phi}(t))$ which is the limit of $(w_n(t), \Phi_n(t))$.

Let $(v(t), \Psi(t))$ be any other solution. Then, because (v, Ψ) is a solution to the cut off problem, we have

4.26) $$|\Psi(t)| \leq \Theta .$$

Hence

$$\lambda F_1(t, v(t), \Psi(t)) \leq \lambda F_1(t, v(t), \Theta) \leq W .$$

Therefore

$$w(t) \leq v(t) .$$

And, of course, $(-v(t), -\Psi(t))$ is also a solution so that

4.27) $$|v(t)| \leq -w(t) = |w(t)| .$$

An induction, based on lemma 3.2 and lemma 4.3 shows that

$$w_n \leq v(t), \quad \Psi(t) \leq \Phi_n(t) .$$

The theorem follows at once.

Returning to the normal (non cutoff) problems, we seek conditions which will guarantee the existence of a pair $(w(t), \Phi(t))$ satisfying (4.23a), (4.23b) and (4.23c). Clearly, the conditions H.1 and H.2 are not sufficient because these conditions include the linear case.

<u>Theorem 4.3</u> Let $\lambda_o < \lambda$. Let H.1 and H.2 hold. Let $K_1(s,t)$, $K_2(s,t)$ be the Green's functions of $-L_1[u]$, $-L_2[\theta]$ respectively which were discussed earlier.

Suppose there are four positive constants M, U_o, Θ_o, α with $0 < \alpha < 1$, such that

4.28a) $$\begin{cases} H_k(t,0,0) \geq H_k(t,u(t),\theta(t)), & k = 1,2 \\ K_j(t,s)\, H_j(t,u(s),\theta(s)) \leq M & j = 1,2, \end{cases}$$

4.28b) $$\lambda^2 \int_0^1 K_1(s,t)\, K_2(x,s)\, H_1(t, u(t), \theta(t))\, H_2(s, \hat{u}(s), \hat{\theta}(s))\, ds \leq \alpha ,$$

for all functions $u(x)$, $\theta(x)$, $\hat{u}(x)$, $\hat{\theta}(x)$ which satisfy

4.28c) $$\begin{cases} U_o \leq |u(x)|, \; |\hat{u}(x)| , & 0 \leq x \leq 1, \\ \Theta_o \leq |\theta(x)|, \; |\hat{\theta}(x)| , & 0 \leq x \leq 1 . \end{cases}$$

Then there exists a pair (w, Φ) with $w(t) \leq -U_o < \Theta_o \leq \Phi(t)$ which satisfy (4.23a), (4.23b) and (4.23c). Finally, there exists a positive solutions $(u(t), \theta(t))$

Proof: Consider the inhomogeneous, nonlinear equation

4.29) $$\begin{cases} L_1[v] = \lambda\Psi \; H_1(t, v - U_o, \Psi + \Theta_o) + \lambda\, \Theta_o\, H_1(t,0,0) \\ L_2[\Psi] = \lambda\, v \; H_2(t, v - U_o, \Psi + \Theta_o) - \lambda\, U_o\, H_2(t,0,0), \\ A_o[v] = A_1[v] = B_o[\Psi] = B_1[\Psi] = 0 . \end{cases}$$

We shall show that there exists a "positive" solution, i.e. a solution (v, Ψ) with

4.30) $$v(t) \leq 0 \leq \Psi(t) .$$

Let

$$
4.31)\quad
\begin{cases}
g_1(s) = \lambda\, \Theta_o \displaystyle\int_0^1 K_1(s,t)\, H_1(t,0,0)\, dt\ , \\
g_2(s) = \lambda\, U_o \displaystyle\int_0^1 K_2(s,t)\, H_2(t,0,0)\, dt\ , \\
K_o = (\lambda M \, \|g_1\|_\infty + \|g_2\|_\infty)/(1-\alpha) \\
K_1 = \lambda K_o \cdot M + \|g_1\|_\infty\ .
\end{cases}
$$

Let S be the convex set

$$
4.32)\qquad S = \{ (\bar{v}(t), \bar{\Psi}(t)) \ \varepsilon\ C[0,1]\ ;\ -K_1 \le \bar{v}(t) \le 0 \le \bar{\Psi}(t) \le K_o \}
$$

Let $(\bar{v}(t), \bar{\Psi}(t)) \ \varepsilon\ S$ and let $V(t)$, $\Psi(t)$ be the solutions of the linear equations

$$
4.33)\quad
\begin{cases}
L_1[V] = \lambda \bar{\Psi}\, H_1(t, \bar{v} - U_o, \bar{\Psi} + \Theta_o) + \lambda\, \Theta_o\, H_1(t,0,0)\ , \\
L_2[\Psi] = \lambda V H_2(t, V-U_o, \bar{\Psi} + \Theta_o) - \lambda\, U_o\, H_2(t,0,0)\ . \\
A_o[V] = A_1[V] = B_o[\Psi] = B_1[\Psi] = 0
\end{cases}
$$

Using the integral representations of the solution, we have

$$
4.34a)\qquad V(s) = -\lambda \int_0^1 K_1(s,t)\, \bar{\Psi}(t)\, H_1(t, v(t) - U_o, \bar{\Psi}(t) + \Theta_o)\, dt - g_1(s)
$$

and

$$
4.34b)\qquad \Psi(x) = \int_0^1 G(x,t) \bar{\Psi}(t) dt + \lambda \int_0^1 K_2(x,s)\, H_2(s, V-U_o, \bar{\Psi} + \Theta_o)\, g_1(s)\, ds + g_2(x),
$$

where

4.34c) $G(x,t) = \lambda^2 \int_0^1 K_1(s,t)K_2(x,s)H_1(t,\bar{v}(t) - V_o, \bar{\Psi}(t) + \Theta_o)H_2(s, V(s)-V_o, \bar{\Psi}(s) + \Theta_o)\, ds$

From (4.34a) and (4.28a) we see that

4.35a) $-K_1 \leq V(t) \leq 0$.

From (4.34b), (4.28a) and (4.28b) we see that

4.35b) $0 \leq \alpha K_o + \lambda M \cdot \|g_1\|_\infty + \|g_2\|_\infty = K_o$.

Thus, equations (4.33) provide a mapping of S into S . Standard estimates show that this is a compact continuous mapping. Thus, there is a fixed point, i.e. a solution of equations (4.29) which satisfy (4.30).

Let

4.36) $w = v - U_o \leq -U_o, \quad \Phi = \Psi + \Theta_o \geq \Theta_o$.

Then

$$L_1[w] = \lambda \Phi H_1(t,w,\Phi) + \lambda \Theta_o[H_1(t,0,0) - H_1(t,w,\Phi)] + C_1 U_o ,$$

$$L_2[\Phi] = \lambda w H_2(t,w,\Phi) - \lambda U_o[H_2(t,0,0) - H_2(t,w,\Phi)] - C_2 \Theta_o .$$

Thus

4.37a) $$\begin{cases} L_1[w] \geq \lambda F_1(t,w,\Phi) \\ A_o[w] = -A_o U_1 \leq 0, \ A_1[w] = -A_1 U_1 \leq 0 . \end{cases}$$

and

4.37b) $$\begin{cases} L_2[\Phi] \le \lambda F_2(t,w,\Phi) \\ B_o[\Phi] = \alpha_o \, \Theta_o \ge 0 \,, \quad B_1[\Phi] = \alpha_1 \, \Theta_o \ge 0 \,. \end{cases}$$

The theorem now follows from Theorem 4.1.

5. Uniqueness of Positive Solutions - Existence of Maximal Solutions

In this section we strengthen the hypothesis on the functions $H_k(t,u,\theta)$ $(k = 1,2)$ and study the unicity of the positive solution.

Lemma 5.1 Let H.1 and H.3 hold. Then, of course H.2 holds as well. Suppose

$$\lambda_o < \lambda$$

and there are two distinct positive solutions (v_1, Ψ_1), (v_2, Ψ_2).

Then, there are two positive solutions (u, θ) and (v, Ψ) which satisfy

5.1) $$u \prec v \prec 0 \prec \Psi \prec \theta \,.$$

Proof: Let $(u_o(t), \theta_o(t))$ be the eigenfunctions of the linear eigenvalue problem (4.3) which also satisfy (4.5), (4.6), (4.7) and

5.2) $$v_k \prec u_o \prec 0 \prec \theta_o \prec \Psi_k \,, \; k = 1,2.$$

Since both pairs (v_k, Ψ_k) satisfy the conditions (4.23a), (4.23b), (4.23c) we may apply Theorem 4.1 to obtain a positive solution (v, Ψ) which satisfies

$$v_k \le v \prec 0 \prec \Psi \le \Psi_k \qquad k = 1,2 \,.$$

Suppose

5.3) $$v_1(t) \not\equiv v(t), \quad \Psi_1(t) \not\equiv \Psi(t) .$$

Then

5.4) $$\begin{cases} L_2[\Psi] = \lambda F_2(t,v,\Psi) \geq \lambda F_2(t,v_1,\Psi) , \\ B_o[\Psi] = B_1[\Psi] = 0 . \end{cases}$$

By lemma 3.2 there is a function $a(t)$ which satisfies

5.5a) $$L_2[a] = \lambda F_2(t,v_1,a), \quad B_o[a] = B_1[a] = 0$$

and, either $\Psi(t) \equiv a(t)$ or

5.5b) $$\Psi \prec a .$$

But, since $v_1(t) \leq 0$ and H.3 holds, the corollary to lemma 3.3 asserts that the solution of (5.5a) is unique. Hence

$$a(t) \equiv \Psi_1(t) .$$

Thus, using (5.3), we have

$$0 \prec \Psi \prec \Psi_1 .$$

A similar argument shows that

$$v_1 \prec v .$$

On the other hand, if (5.3) does not hold we apply the same argument to (v_2, Ψ_2).

<u>Lemma 5.2</u> Suppose H.1 and H.3 hold and there are two positive solutions of equations (2.1), (2.7) which satisfy (5.1) Let α be any constant such that

(5.6a) $$0 < \alpha < 1 ,$$

(5.6b) $$\alpha\theta \leq \Psi .$$

Then

5.7) $$v \prec \alpha u .$$

Similarly, if

5.8) $$v \leq \alpha u ,$$

then

5.9) $$\alpha\theta \prec \Psi .$$

Proof: Using (2.6a) we see that

$$L_1[\alpha u] = \lambda \alpha \theta H_1(t,u,\theta) < \lambda \alpha \theta H_1(t,\alpha u,\alpha\theta) = \lambda F_1(t,\alpha u,\alpha\theta) .$$

Using (5.6b) we have

$$L_1[\alpha u] < \lambda F_1(t,\alpha u, \Psi), \quad A_o[\alpha u] = A_1[\alpha u] = 0 .$$

Using lemma (3.2), there is a function $w(t)$ which satisfies

5.10) $$\begin{cases} L_1[w] = \lambda F_1(t,w,\Psi) , \quad A_o[w] = A_1[w] = 0 . \\ w \prec \alpha u . \end{cases}$$

However, because H.3 holds, the corollary to lemma 3.3 implies that $w(t) \equiv v(t)$ and the lemma is proven in the first case. The other case follows by a completely similar argument.

Theorem 5.1 Let H.1 and H.3 hold. Let

$$\lambda_o < \lambda .$$

Then there is at most one positive solution of equations (2.1), (2.2) .

Proof: Suppose there are two positive solutions. By lemma 5.1 we may assume that there are two positive solutions (u,θ), (v,Ψ) which satisfy (5.1).

There is a positive number $\alpha < 1$ such that

5.11) $$\alpha\theta \leq \Psi ,$$

but

5.12) $$\alpha\theta \not\prec \Psi .$$

To see this we merely observe that for β small enough, $\beta\theta \prec \Psi$.

We may let β increase until either $\beta\theta(t_o) = \Psi(t_o)$ for some interior point t_o, or $\beta\theta'(0) = \Psi'(0)$ or $\beta\theta'(1) = \Psi'(1)$.

Then, using lemma 5.2, and (5.11),

$$v \prec \alpha u .$$

In particular, $v \leq \alpha u$. Using lemma 5.2 again,

$$\alpha\theta \prec \Psi$$

which contradicts (5.12).

Remark: The above uniqueness theorem applies to the cut off case as well as the normal case. The fact that $0 < \alpha < 1$ implies that we have been in the region

$$|\theta| \leq \Theta .$$

Theorem 5.2: Suppose H.1 and H.3 hold and

$$\lambda_o < \lambda .$$

Suppose also that the hypothesis of theorem 4.3 holds. Then the positive solution constructed in theorem 4.3 is also a maximal solution.

Proof: Let (v, Ψ) be any solution. Let

$$v_1 = \max |v(t)| , \Psi_1 = \max |\Psi(t)| .$$

Let

$$U_2 = U_o + v_1, \quad \Theta_2 = \Theta_o + \Psi_1 .$$

Then, following the construction of theorem 4.3 we may construct a pair (w, Φ) such that

$$w(t) \leq v(t) \ , \ \Psi(t) \leq \Phi(t) \ ,$$

and (4.23b), (4.23c) hold. A simple induction similar to the basic proof of Theorem 4.2 shows that the iteration (w_n, Φ_n) constructed in the proof of theorem 4.3 satisfy

$$w_n(t) \leq v(t), \quad \Psi(t) = \Phi_n(t) \ .$$

Thus the functions$(w_n(t), \ \Phi_n(t)\ )$ converge to a positive solution $(\hat{u}(t), \hat{\theta}(t)\ )$ which also satisfies

$$\hat{u}(t) \leq v(t), \ \Psi(t) \leq \hat{\theta}(t) \ .$$

However, there is only one positive solution and the theorem follows at once.

A very similar argument shows that the maximal solution is monotone in λ.

Theorem 5.3 Let $(u(t,\lambda), \theta(t,\lambda))$ denote the maximal solution of equations (2.1) (2.2). Assume H.1, H.3 hold, $\lambda_o < \lambda$ and the hypothesis of theorem 4.3 hold. Then

$$u(t,\lambda + \delta) \leq u(t,\lambda) \leq \theta(t,\lambda) \leq \theta(t,\lambda + \delta) \ .$$

Proof: Let

$$v_1 = \max |u(t,\lambda)| \ , \ \Psi_1 = \max |\theta(t,\lambda)|$$

and

$$U_2 = U_o + v_1, \ \Theta_2 = \Theta_o + \Psi_1 \ .$$

As in theorem 4.3 we construct a pair (w, Φ) so that (4.23b), (4.23c) hold and

$$w(t) \le u(t,\lambda), \ \theta(t,\lambda) \le \Phi(t) .$$

Consider equations (2.1), (2.2) with λ replaced by $\lambda + \delta$. Since

$$L_1[u(t,\lambda)] = \lambda F_1(t,u,\theta) \le (\lambda + \delta) F_1(t,u,\theta)$$

$$L\ [\theta(t,\lambda)] = \lambda F\ (t,u,\theta) \ge (\lambda + \delta) F\ (t,u,\theta)$$

we may use the induction of lemma 4.2 to produce a sequence which increases and, as in the proof of theorem 4.1, we have

$$w(t) \le u_n(t) \le u(t,\lambda) \le \theta(t,\lambda) \le \theta_n(t) \le \Phi(t) .$$

Thus the sequence $(u_n(t), \theta_n(t))$ will converge to the unique positive solution and the theorem is proven.

6. Other Solutions - Special Cases

Let us now consider the very special case where equation (2.1) take the form

6.1) $$\begin{cases} u'' = \lambda \theta H_1(u,\theta) . \\ \theta'' = \lambda u H\ (u,\theta) . \end{cases}$$

subject to the boundary conditions (B) (ot Odeh and Tadjbakhsh) or the boundary condition

(S) $$u(0) = u(1) = 0 , \quad \theta(0) = \theta(1) = 0 .$$

Let

6.2a) $$P = H_1(0,0) H_2(0,0) ,$$

6.2b) $$J = H_2(0,0)/H_1(0,0) .$$

Consider the linear eigenvalue problem

6.3) $$u'' = \lambda \theta H_1(0,0), \qquad \theta'' = \lambda u H\ (0,0).$$

In the case of the boundary conditions (S·) the eigenvalues are

6.4S) $$\lambda_j = \pm \frac{(\pi j)^2}{\sqrt{P}} ,$$

while the eigenfunctions are given by ($\lambda_j > 0$)

6.5S) $$\begin{cases} u_j(t) = A \sin \pi j t \\ \theta_j(t) = -\sqrt{J}\ u_j(t) = -\sqrt{J}\ (A \sin \pi j t) . \end{cases}$$

In the case of the boundary conditions (B·) we have

6.4B) $$\lambda_j = \pm \left(\frac{2j+1}{2}\pi\right)^2 / \sqrt{P}$$

and, for $\lambda_j > 0$,

6.5B) $$\begin{cases} u_j(t) = A \cos \frac{2j+1}{2}\pi t \\ \theta_j(t) = -\sqrt{J}\ u_j(t) = -\sqrt{J}\ (A \cos \frac{2j+1}{2}\pi t) . \end{cases}$$

We must also consider the differential equation (6.3) on other intervals. For this reason we introduce the following notation. Let

5.6) $$\lambda_k(m, S)$$

be the k'th positive eigenvalue of the differential equations (6.3) on an interval of length m subject to the boundary conditions (S·). For example, consider the equation (6.3) on the interval (a, a + m) subject to the boundary conditions

$$u(a) = u(a+m) = \theta(a) = \theta(a+m) = 0 .$$

Then, the k'th positive eigenvalue is denoted by (6.6). Similarly, let

6.7) $$\lambda_k(m,B)$$

be the k'th eigenvalue of the differential equations (6.3) on an interval of length m subject to the boundary conditions (B). For example, $\lambda_k(m, B)$ denotes the k'th eigenvalue of equation (6.3) on the interval (a, a + m) subject to the boundary condition

$$u'(a) = \theta'(a) = u(a+m) = \theta(a+m) = 0 .$$

A straight forward calculation shows that

6.8) $$\begin{cases} \lambda_o(\frac{1}{k}, S) = \lambda_{k-1}(1,S), \quad k = 1, \ldots \\ \lambda_o(2m, S) = \lambda_o(m, B) \\ \lambda_o(\frac{2}{2k+1}, S) = \lambda_o(\frac{1}{2k+1}, B) = \lambda_k(1,B), \; k = 0,1,\ldots \end{cases}$$

These facts lead immediately to the following results

Lemma 6.1 Let H.1 and H.2 be satisfied. Let

$$\lambda_o(m,B) < \lambda$$

and suppose that $H_1(u,\theta)\, H_2(u,\theta)$ gets small enough for large (u,θ) that one knows that there is a positive solution $(u(t,m), \theta(t,m))$ of equations (6.1) subject to the boundary conditions B on an interval of length m, say (a, a + m) .

Then there is a positive solution $(U(t,m), \Theta(t,m))$ of equations (6.1) subject to the boundary condition S on the interval (a, a + 2m) . Moreover,

$$6.9) \quad \begin{cases} U'(a) = -U'(a+2m) < 0 \\ \Theta'(a) = -\Theta'(a+2m) > 0 \\ u'(a+m) = \Theta'(a+m) = 0 \end{cases}$$

Proof: Let

$$6.10a) \quad U(t) = \begin{cases} u(2a+m-t), & a \le t \le a+m \\ u(t-m), & a+m \le t \le a+2m \end{cases}$$

$$6.10b) \quad \Theta(t) = \begin{cases} \theta(2a+m-t), & a \le t \le a+m \\ \theta(t-m), & a+m \le t \le a+2m \end{cases}$$

A direct computation verifies that these functions have the desired properties

Theorem 6.1 Let H.1 and H.2 be satisfied. Let $k \ge 0$ and assume that

$$\lambda_o\left(\frac{1}{k+1}, S\right) = \lambda_k(1,S) < \lambda .$$

Suppose $H_1(u,\theta) H_2(u,\theta)$ gets small enough for large (u,θ) that one may apply lemma 6.1 to assert the existence of the function $U(t, \frac{1}{k+1})$, $\Theta(y, \frac{1}{k+1})$ of the previous lemma.

Then there is a solution $(U_k(t), \Theta_k(t))$ of equation (6.1) which satisfies the boundary conditions (S·). Moreover,

$$6.11) \quad U_k\left(\frac{\ell}{k+1}\right) = \Theta_n\left(\frac{\ell}{k+1}\right) = 0.$$

and these are the only zeros of $U_k(t)\Theta_k(t)$.

Proof: Let $a = 0$ and $U(t)$, $\Theta(t)$ be the functions whose existence is assumed by lemma 6.1. Let

6.12)
$$\begin{cases} U_k(t) \equiv (-1)^{\ell} \; U(t - \frac{\ell}{k+1}), & \frac{\ell}{k+1} \le t \le \frac{\ell+1}{k+1}, \; \ell = 0,1..k \\ \Theta_k(t) \equiv (-1)^{\ell} \; \Theta(t - \frac{\ell}{k+1}), & \frac{\ell}{k+1} \le t \le \frac{\ell+1}{k+1}, \; \ell = 0,1..k . \end{cases}$$

A direct computation verifies that $(U_k(t), \Theta_k(t))$ is the desired solution.

Theorem 6.2 Let H.1 and H.2 be satisfied. Let

$$\lambda_o(\frac{2}{2k+1}, S) = \lambda_o(\frac{1}{2k+1}, B) = \lambda_k(1,B) < \lambda .$$

Suppose that $H_1(u,\theta)$ $H_2(u,\theta)$ gets small enough for large (u,θ) that we may apply lemma 6.1 to assert the existence of the functions $U(t, \frac{2}{2k+1})$, $\Theta(t, \frac{2}{2k+1})$ of lemma 6.1.

Then there is a solution $(u_k(t), \theta_k(t))$ of equations (6.1) which satisfies the boundary conditions (B). Moreover

$$u_k(\frac{2\ell+1}{2k+1}) = \theta_k(\frac{2\ell+1}{2K+1}) = 0 , \quad \ell = 0,1,\ldots k.$$

and these are the only zeros of $u_k(t)$ $\theta_k(t)$ in $[0,1]$.

Proof: Let $a = -\frac{1}{2k+1}$. Let

6.14)
$$\begin{cases} u_k(t) = (-1)^{\ell} \; U(t - \frac{2(\ell+1)}{2k+1}), & \frac{2\ell+1}{2k+1} \le t \le \frac{2\ell+3}{2k+1}, \; \ell = -1,0,1,2\ldots k-1 \\ \theta_k(t) = (-1)^{\ell} \; \Theta(t - \frac{2(\ell+1)}{2k+1}), & \frac{2\ell+1}{2k+1} \le t \le \frac{2\ell+3}{2k+1}, \; \ell = -1,0,1,\ldots k+1 \end{cases}$$

Once more, a direct computation verifies that these functions have the desired features.

Remark: In the cut off case, we are assured of the existence of the necessary positive solutions. Thus, in particular, in the case of equations (1.1) subject to the boundary condition B, if

$$\lambda_k < \lambda \leq \lambda_{k+1}$$

there are at least (k+1) distinct nontrivial solutions $(u_j(t), \theta_j(t))$, $j = 0, 1 \ldots k$. The pair $(u_j(t), \theta_j(t))$ is characterized by the fact that each function has exactly j interior modal zeros and no other zeros.

Remark: This method of "patching together" positive solutions is clearly of limited applicability. Nevertheless, it is an interesting direct consequence of this theory of positive solutions.

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13 ABSTRACT

A constructive, nonlinear iterative method is developed for the construction of a "positive" solution $(u(t),\theta(t))$ of nonlinear fourth order ordinary differential equation of the form $u'' = \lambda\theta H_1(t,u,\theta)$, $\theta'' = \lambda u H_\theta(t,u,\theta)$. A solution $(u(t),\theta(t))$ is "positive" if $u(t) \leq 0 \leq \theta(t)$. Under appropriate hypothesis, these solutions are "maximal" in the sense that; if (ω,Φ) is any other solution, then $u \leq \omega$, $\Phi \leq \theta$. Thus, bounds on (u,θ) are a priori bounds on all solutions. Uniqueness is discussed. In special cases these positive solutions may be patched together to give other solutions.

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