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A NOTE ON THE EIGENVALUE PROBLEM FOR SUBLINEAR HAMMERSTEIN OPERATORS

by

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Technical Report #74

October 1969

This research is supported by the Office of Naval Research under Contract No. N-0014-67-A-0128-004 and by the U.S. Atomic Energy Commission.

1. INTRODUCTION

Consider the Hammerstein equation

(1.) $$x(s) = \lambda \int_0^1 K(s,t)\, f(t, x(t))\, dt$$

where $K(s,t)$ is continuous for $0 \le s,\ t \le 1$ and "positive," i.e.,

(1.1) $$K(s,t) > 0, \quad 0 < s < 1, \quad 0 < t < 1 .$$

The function $f(t,x)$ is assumed to have the form

(1.2) $$f(t,x) = xH(t,x)$$

where

(1.2a) $$H(t,x) = H(t,-x) > 0, \quad 0 \le t \le 1, \quad |x| < \infty$$

(1.2b) $$f_x(t,x) = H(t,x) + x\,H_x(t,x) > 0$$

(1.2c) $$x\,H_x(t,x) < 0, \quad x \neq 0.$$

Many authors have studied such problems. However, the work of George Pimbley [6], [7] is of particular interest for us.

A related problem which is basic for the study of equation (1.) is the linearized problem (linearized about zero)

(1.3) $$h(s) = \lambda \int_0^1 K(s,t)\, H(t,0)\, h(t)\, dt.$$

It is well known [3], [4] that there is a unique positive number $\lambda_0 > 0$ which is the smallest eigenvalue of equation (1.3). We may normalize the associated eigenfunction $h_0(t)$ to be positive for $0 < t < 1$.

Moreover, if λ is any other eigenvalue and $\varphi(t)$ is the associated eigenfunction then either

(1.3a) $|\lambda| > \lambda_0$ and $\exists\, t_0 \in (0, 1)$ such that $\varphi(t_0) = 0$

or

(1.3b) $\lambda = \lambda_0$ and $\varphi(t) = ch_0(t)$

for some constant c.

We will assume that the positive eigenvalue λ_0 is strictly monotone in $H(t, 0)$.

Definition: Let $g_1(t),\ g_2(t) \in C[0, 1]$ satisfy

(1.4) $0 < c_0 \leq g_1(t) \leq g_2(t), \qquad 0 \leq t \leq 1$.

Let $\mu(g_1),\ \mu(g_2)$ be the minimal positive eigenvalues of the linear eigenvalue problems

(1.4a) $Q(s) = \mu(g_k) \int_0^1 K(s, t)\, g_k(t)\, Q(t)\, dt\,, \quad K = 1, 2$.

We will say the kernel $k(s, t)$ satisfies condition M if $g_1(t) \not\equiv g_2(t)$ implies that

(1.4b) $\mu(g_2) < \mu(g_1)$.

Remark: We know of no examples where this condition is violated. Nevertheless, since it is this particular fact which is used, it seemed worthwhile to identify it.

Remark: When $K(t,s) = K(s,t)$ we see that $K(s,t)$ satisfies condition M from the variational characterization (see [1])

$$\mu(g) = \operatorname*{Min}_{\varphi(t)\not\equiv 0} \frac{\int_0^1 \varphi^2(t)\,dt}{\left|\int_0^1 \int_0^1 K(s,t)\,[g(s)g(t)]^{\frac{1}{2}}\,\varphi(s)\,\varphi(t)\,dt\,ds\right|} .$$

In this note we prove an existence theorem for "positive" solutions of equation (1.) which also enables us to establish a priori bounds on <u>all</u> solutions of equation (1.). A basic result is

<u>Theorem I:</u> Let $K(s,t)$ satisfy condition M . Let λ be a fixed constant satisfying

(1.5a) $$\lambda_0 < \lambda \quad .$$

Suppose there are two positive constants y_0, α with $0 < \alpha < 1$ such that

(1.5b) $$\lambda \int_0^1 K(s,t)\,H(t,y_0)\,dt \leq \alpha < 1 \ .$$

Then there exists a positive solution of equation (1.), say $\overline{y}(t)$. Moreover, if $v(t)$ is any solution of equation (1.) then

(1.6) $$v(t) \leq \overline{y}(t) \leq y_0 \ .$$

Finally, if $v(t)$ is any nontrivial, nonnegative solution of equation (1.), then

(1.7) $$v(t) \equiv \overline{y}(t)$$

Remark: Since $-v(t)$ is a solution whenever $v(t)$ is a solution, the inequality (1.6) implies

$$|v(t)| \leq \overline{y}(t) .$$

Using this estimate and the basic idea of Pimbley [7] and Wolkowisky [8] we then prove

Theorem II: Suppose $K(s,t)$ is an oscillation kernel. Suppose (1.5a), (1.5b) hold. Suppose

$$(1.8) \qquad \lambda_n < \lambda$$

$$(1.9) \qquad \lambda H(t, y_0) < \lambda_0 H(t, 0)$$

hold, where λ_n is the n^{th} smallest eigenvalue of equation (1.3). Then, there exist (at least) $n+1$ nontrivial solutions of equation (1.) $x_j(t)$, $j = 0, 1 \cdots n$. The function $x_j(t)$ has exactly j interior nodal zeros and no other interior zeros.

Theorem II is essentially Pimbley's theorem of [7]. Unfortunately there is a gap in the proof of lemma 5 of [7]. Our contribution is basically the use of the a priori estimate of Theorem I to avoid the difficulties. Nevertheless, because of many questions asked by our interested friends and colleagues we include most of the details. We will assume the reader is familiar with Pimbley's work.

I wish to thank my many friends who discussed this matter with me. In particular, I am indebted to George Pimbley and Paul Rabinowitz.

2. POSITIVE SOLUTIONS

Our basic results of this section follow from some elementary lemmas.

Lemma 1: Suppose $K(s,t)$ satisfies condition M. Suppose $y(t)$ and $z(t)$ are two solutions of equation (1.) with

$$(2.1)\qquad \begin{cases} 0 \le y(t) \le z(t), & 0 < t < 1 \\ 0 \not\equiv y(t) & \end{cases} .$$

Then

$$(2.2)\qquad 0 < y(t) \equiv z(t), \quad 0 < t < 1 .$$

Proof: From equation (1.1) and conditions (2.1) we see that

$$0 < y(t), \quad 0 < t < 1 .$$

Thus, we need only prove that $y(t) \equiv z(t)$, $0 < t < 1$. Let

$$g_1(t) = H(t, y(t)), \quad g_2(t) = H(t, z(t)).$$

Then, condition (2.1) and (1.2c) imply that

$$g_2(t) \le g_1(t) .$$

Thus,

$$\mu(g_1) < \mu(g_2)$$

unless $g_1(t) \equiv g_2(t)$. However, the fact that $y(t)$ and $z(t)$ both satisfy equation (1.) and both are positive for $0 < t < 1$ means that

$$\lambda = \mu(g_1) = \mu(g_2) .$$

Thus

$$H(t, y(t)) \equiv H(t, z(t))$$

and we use (1.2c) to see that

$$y(t) \equiv z(t) .$$

Lemma 2. Let (1.5a) hold. Let $y_0(t)$ be a positive function which satisfies

(2.3) $$0 < c_0 \leq y_0(t) , \quad 0 \leq t \leq 1$$

for some constant c_0 . Suppose

(2.4) $$\lambda \int_0^1 K(s, t)\, f(t, y_0(t))\, dt \leq y_0(s) .$$

Consider the sequence of functions $y_n(t)$ generated by

(2.5) $$\begin{cases} y_0(t) = y_0(t) \\ y_{n+1}(s) = \lambda \int_0^1 K(s, t)\, f(t, y_n(t)) dt . \end{cases}$$

The functions $y_n(t)$ decrease with n and converge uniformly to a continuous function $\overline{y}(t)$ which satisfies equation (1.) and also satisfies

(2.6) $$0 < \overline{y}(t) \leq y_0(t), \quad 0 < t < 1 .$$

Proof: Condition (2.4) implies

$$0 < y_1(t) \leq y_0(t), \quad 0 < t < 1.$$

Assume that $y_n(t) \le y_{n-1}(t)$, $0 \le t \le 1$. Then

$$y_{n+1}(s) - y_n(s) = \lambda \int_0^1 K(s,t)\, f_x(t, \xi(t))\, [y_n(t) - y_{n-1}(t)]\, dt \le 0 .$$

Thus the functions $y_n(t)$ decrease with n.

The function $h_0(t)$, the non negative eigenfunction of equation (1.3), is the eigenfunction of a linear eigenvalue problem. Hence, it may be "scaled". We assume $h_0(t)$ has been chosen non negative and so small that

(2.7) $$\begin{cases} \lambda [H(t, h_0(t)) - \frac{\lambda_0}{\lambda} H(t,0)]\, h_0(t) > 0, & 0 < t < 1 \\ 0 < h_0(t) < c_0 \le y_0(t) . \end{cases}$$

Assume that

$$h_0(t) \le y_n(t) .$$

Then

$$y_{n+1}(s) - h_0(s) = \lambda \int_0^1 K(s,t)\, [f(t, y_n(t)) - f(t, h_0(t))]\, dt +$$

$$\lambda \int_0^1 K(s,t)\, [H(t, h_0(t)) - \frac{\lambda_0}{\lambda} H(t,0)]\, h_0(t)\, dt \ge 0 .$$

Thus, the functions $y_n(t)$ converge monotonically to a function $\bar{y}(t)$ which satisfies condition (2.6). Because the family $\{f(t, y_n(t))\}$ is bounded, the family $\{y_n(t)\}$ is compact and one sees that the convergence is uniform. Thus $\bar{y}(t) \in C[0,1]$ and satisfies equation (1.).

Corollary: Suppose there exist two constants y_0, α, $0 < \alpha < 1$ such that (1.5b) is satisfied. Suppose (1.5a) is satisfied. Then, there is a positive solution $\overline{y}(t)$ which satisfies

$$0 < \overline{y}(t) \le y_0 \quad .$$

Proof: Set $y_0(t) \equiv y_0$.

The normalization (2.7) is based on the ideas of Picard for nonlinear second order differential equations [5].

Lemma 3: Let $K(s,t)$ satisfy condition M. Suppose we can construct a function $y_0(t;c_0)$ which satisfies the hypotheses (2.3) and (2.4) of Lemma 2 for all choices of $c_0 > 0$. Suppose (1.5a) holds. Then there is only one nontrivial nonnegative solution $\overline{y}(t)$. This solution may be constructed via lemma 2 and is independent of the choice of $c_0 > 0$. Moreover, if $v(t)$ is any solution of equation (1.) then

$$v(t) \le \overline{y}(t).$$

Proof: Let $v(t)$ be any solution of equation (1.). Let

(2.8) $$c_0 = \max |v(t)| + 1.$$

Let the sequence $y_n(t)$ be constructed as in lemma 2 starting with $y_0(t) = y_0(t;c_0)$. Then, clearly $v(t) \le y_0(t) \le y_n(t)$. Then

$$v(s) - y_{n+1}(s) = \int_0^1 K(s,t)\,[f(t,v(t)) - f(t,y_n(t))]\,dt$$

$$= \int_0^1 K(s,t)\, f_x(t,\xi(t))\,[v(t) - y_n(t)]\,dt \le 0 \ .$$

Thus the functions $y_n(t)$ converge to a function $\overline{y}(t)$ which is also a solution of equation (1.) and

$$v(t) \le \overline{y}(t).$$

If $v(t) \ge 0$ and $v(t) \not\equiv 0$, then Lemma 1 gives us

$$v(t) \equiv \overline{y}(t) .$$

Thus there is only one positive solution, which dominates all other solutions.

These three lemmas immediately give our basic Theorem I.

<u>Theorem 2.1:</u> Suppose $K(s, t)$ satisfies condition M. Suppose there is a positive constant y_0 and an $\varepsilon_0 > 0$ such that

$$(2.9) \qquad \begin{cases} \lambda_0 < \lambda \\ \lambda H(t, y_0) \le (\lambda_0 - \varepsilon_0)\, H(t, 0) . \end{cases}$$

Let $f(t, x)$ be of the form (1.2) and satisfy (1.2a), (1.2b), (1.2c). Then there exists a positive solution of equation (1.), say $\overline{y}(t)$, with

$$(2.10) \qquad 0 < \overline{y}(t) \qquad 0 < t < 1 .$$

Moreover, if $v(t)$ is any other solution of equation (1.) then

$$(2.11) \qquad v(t) \le \overline{y}(t) .$$

Finally, if $z(t)$ is a nonnegative, nontrivial solution of equation (1.), then

$$(2.12) \qquad z(t) \equiv \overline{y}(t) .$$

<u>Proof:</u> Let

$$p(s) = \lambda y_0 \int_0^1 K(s,t)\, H(t,0)\, dt > 0 . \tag{2.13}$$

Let $u(x)$ be the unique solution of the linear integral equation

$$u(s) = \lambda \int_0^1 K(s,t)\, H(t,y_0)\, u(t)\, dt + p(s) . \tag{2.14}$$

It is easy to see that equation (2.14) has a unique solution since the minimal eigenvalue $\mu(\lambda H(t, y_0))$ satisfies

$$\mu(\lambda H(t,y_0)) \geq \mu((\lambda_0 - \varepsilon_0) H(t,0)) > 1 . \tag{2.15}$$

Indeed, equation (2.15) also guarantees that we may compute $u(s)$ via the Neumann series. The positivity of $p(s)$ and $K(s,t)$ shows that

$$0 < u(s) , \qquad 0 < s < 1. \tag{2.16}$$

Let

$$y_0(t) = u(t) + y_0 .$$

Then

$$y_0(s) = \lambda \int_0^1 K(s,t) H(t, y_0 + u(t)) (u(t) + y_0)\, dt +$$
$$\lambda \int_0^1 K(s,t) [H(t,y_0) - H(t, y_0 + u(t))] (u(t) + y_0)\, dt +$$
$$\lambda \int_0^1 K(s,t) [H(t,0) - H(t,y_0)]\, y_0\, dt + y_0 .$$

That is,

$$\lambda \int_0^1 K(s,t)\ f(t, y_0(t))\ dt \le y_0(s) \ .$$

Thus, for $c_0 = y_0$ we have constructed the function $y_0(t, c_0)$ of Lemma 3. However, if $z_0 > y_0$ then of course, (2.9) holds with y_0 replaced by z_0. Thus we may construct $y_0(t, c_0)$ for all c_0. Hence we may apply Lemma 3 and the theorem follows at once.

Remark: The positive solution $\overline{y}(t) = \overline{y}(t, \lambda)$ is monotone in λ. To see this we need merely observe that $\delta > 0$ implies

$$\begin{aligned} \overline{y}(s, \lambda) &= \lambda \int_0^1 K(s,t)\ f(t, \overline{y}(t, \lambda))\ dt \\ &\le (\lambda + \delta) \int_0^1 K(s,t)\ f(t, \overline{y}(t,\lambda))\ dt. \end{aligned}$$

Hence if $c_0 = \max\ \overline{y}(t, \lambda)$, and $y_0(t, c_0)$ is used as a starting value when $\lambda + \delta$ replaces λ, the induction of Lemma 2 will easily prove that

$$\overline{y}(t, \lambda) \le \overline{y}(t, \lambda + \delta) \ .$$

Sometimes it is desirable to modify the functions $f(t, x)$, $H(t, x)$ and yet preserve the class of solutions of equation (1.). Let $a > 0$ be any constant. Let m be a positive constant such that

$$0 < m \le \frac{1}{4} \ . \tag{2.17}$$

Define

$$(2.17a) \qquad G(t, x) \equiv \begin{cases} H(t, x), & |x| < a \\ H(t, a) \{(1-m) + m e^{-\beta(t)(|x| - a)}\}, & |x| \geq a , \end{cases}$$

where

$$(2.17b) \qquad \beta(t) = \frac{-H_x(t, a)}{mH(t, a)} > 0 .$$

Lemma 4: Let

$$(2.18) \qquad F(t, x) = x\, G(t, x)$$

where $G(t, x)$ is given above by equations (2.17a) and (2.17b).

Then

$$(2.19a) \qquad G(t, x) = G(t, -x), \quad 0 \leq t \leq 1, \quad |x| < \infty$$

$$(2.19b) \qquad F_x(t, x) = G(t, x) + x\, G_x(t, x) > 0$$

$$(2.19c) \qquad x\, G_x(t, x) < 0 , \; x \neq 0$$

$$(2.19d) \qquad \lim_{|x| \to \infty} G(t, x) = (1-m) H(t, a) > 0$$

Proof: Clearly, it is only necessary to verify (2.19b) for $x > a$.

In that case

(2.20) $$F_x(t,x) = H(t,a)(1-m) + \{m\,H(t,a) + a\,H_x(t,a) + (x-a)H_x(t,a)\}e^{-\beta(t)(x-a)}.$$

A straight forward computation using (2.17) completes the proof.

Theorem 2.2: Let $K(s,t)$ be a positive kernel satisfying condition M. Suppose $f(t,x)$ is of the form (1.2) and satisfies (1.2a), (1.2b) and (1.2c) for $|x| < x_0$ where x_0 is some given positive constant. Suppose there are two constants $y_0 < x_0$, α with $0 < \alpha < 1$, such that (1.5a) and (1.5b) hold.

Then there is a positive solution $\bar{y}(t)$ of equation (1.). Moreover

$$0 < \bar{y}(t) < y_0 < x_0 .$$

Finally, if $v(t)$ is any nontrivial solution of equation (1.) satisfying

(2.21) $$|v(t)| < x_0$$

then

(2.22) $$v(t) \le \bar{y}(t) .$$

Of course, if $v(t)$ is also nonnegative,

$$v(t) \equiv \bar{y}(t) .$$

Proof: Let $y_0 = a$ and construct the functions $F(t,x)$, $G(t,x)$ as above. Let $\bar{v}(t)$ be a solution of

$$\overline{v}(s) = \lambda \int_0^1 K(s,t)\, F(t, \overline{v}(t))\, dt \,. \tag{2.23}$$

Then, by the above results, there is a function $\overline{y}(t)$ which satisfies equation (2.23), such that

$$\begin{cases} 0 < \overline{y}(t) \leq y_0 \,, & 0 < t < 1 \\ \overline{v}(t) \leq \overline{y}(t) \leq y_0 \end{cases} \tag{2.24}$$

But then, $\overline{y}(t)$ and $\overline{v}(t)$ are both solutions of equation (1.) which satisfy (2.21).

Conversely, if $v(t)$ is a solution of equation (1.) such that (2.21) holds, let

$$b = \max\,(\max |v(t)|,\ y_0) < x_0 \,. \tag{2.25}$$

And, let $\widetilde{F}(t,x)$, $\widetilde{G}(t,x)$ be the functions constructed above and modified for $|x| > b$. The argument leading to theorem I shows that all solutions of

$$x(t) = \lambda \int_0^1 K(s,t)\, \widetilde{F}(t, x(t))\, dt \tag{2.26}$$

satisfy

$$x(t) \leq y_0 \,.$$

Thus, all solutions of (2.26) are solutions of (2.23) and satisfy (2.24). But, of course, $v(t)$ satisfies (2.26) by the choice of b. Thus, the theorem is proven.

3. THEOREM II

Let $K(s,t)$ be a symmetric oscillation kernel [2]. Then, by the remarks of section 1, $K(s,t)$ satisfies condition M. The difficulties in the proof of [7] arise because $H(t,x) \to 0$ as $x \to \infty$. However, we avoid this difficulty by using the modification described in section 2. That is, let there be a positive constant y_0 such that (1.5b), and (1.9) hold. Assume

$$(3.1) \qquad \lambda_n < \lambda$$

where λ_n is the n^{th} smallest eigenvalue of the linearized problem (1.3). Let $a = y_0$ and let $F(t,x)$, $G(t,x)$ be constructed as in (2.17), (2.17a), (2.17b) and (2.18). Clearly any solution of equation (1.) is a solution of equation (1.) when $f(t,x)$ is replaced by $F(t,x)$ and vice-versa. Hence we may assume

$$(3.2) \qquad \lim_{|y|\to\infty} H(t,y) = (1-m)\,H(t,y_0) > 0$$

uniformly for $t \in [0,1]$.

We follow Pimbley's argument. Let

$$(3.3) \qquad A \equiv \{q(t) \in C[0,1];\ (1-m)H(t,y_0) \le q(t) \le H(t,0)\}\,.$$

Let $j \le n$. For every α, $0 \le \alpha < \infty$ and every $q(t) \in A$ we construct a mapping $L_j(\alpha)$ taking A into A. Let $V_j(t)$, μ_j be the jth eigenfunction and eigenvalue of the linear integral equation

(3.4) $$V_j(s) = \lambda \mu_j \int_0^1 K(s,t)\, q(t)\, V_j(t)\, dt \; .$$

The theory of oscillation kernels allows us to conclude that μ_j and $V_j(s)$ are unique (up to a scale factor in $V_j(s)$), and $V_j(s)$ has exactly j interior nodal zeros and no other interior zeros. We normalize $V_j(s)$ so that

(3.5) $$\|V_j\|_\infty = \max_{0 \le t \le 1} |V_j(t)| = 1 \, .$$

Moreover, if C_j is the jth eigenvalue associated with the kernel $K(s,t)\, H(t, y_0)(1-m)$, then

(3.6) $$0 < \mu_j \le C_j \, .$$

Thus the functions $V_j(s)$ are uniformly bounded and equicontinuous.

Having obtained $V_j(t)$, μ_j we set

(3.7) $$[L_j(\alpha)q](t) = H(t, \alpha V_j(t)) . \; .$$

Clearly,

(3.8) $$L_j(\alpha): A \rightarrow A \quad \forall \; \alpha \in [0, \infty) \, .$$

Lemma 5. For every j, $0 \le j \le n$, and every $\alpha \in [0,\infty)$, $L_j(\alpha)$ is a continuous mapping of A into itself.

Proof: See Lemma 1 of [7] .

Let $\sigma_j(\alpha)$, $\psi_j(t)$ be the eigenvalue and eigenfunction of the linear eigenvalue problem

$$\text{(3.9)} \qquad \begin{aligned} \psi_j(s) &= \lambda\ \sigma_j(\alpha) \int_0^1 K(s,t)\ H(t, \alpha V_j(t))\ \psi_j(t)\,dt. \\ \|\psi_j\|_\infty &= 1 \end{aligned}$$

For each $\delta \geq 0$ let $\rho_j(\delta)$ be the j^{th} smallest eigenvalue of the linear eigenvalue problem

$$\text{(3.10)} \qquad \varphi_j(s) = \lambda\ \rho_j(\delta) \int_0^1 K(s,t)\ H(t,\delta)\, \varphi_j(t)\,dt\ .$$

Clearly $\rho_j(0) = \lambda_j/\lambda$ and ρ_j increases as δ increases. Moreover,

$$\lim_{\delta \to \infty} \rho_j(\delta) = C_j \cdot \lambda > \lambda$$

because of condition (1.9). Thus there is a unique δ_j such that

$$\rho_j(\delta_j) = \frac{\lambda_j + \lambda}{2\lambda} < 1$$

<u>Lemma 6:</u> There is a unique $\alpha = \tilde{\alpha}(q) \in [\delta_j, \infty)$ such that

$$\text{(3.11)} \qquad \sigma_j(\tilde{\alpha}(q)) = 1\ .$$

<u>Proof:</u> For fixed $q \in A$, hence for fixed $V_j(t)$, the eigenvalue $\sigma_j(\alpha)$ is a strictly monotone increasing function of α (see Lemma 2 of [7]). Since

$$\lambda\, H(t, \delta_j V_j(t)) \geq \lambda\, H(t, \delta_j)$$

we see that

$$(3.12) \qquad \sigma_j(\delta_j) \le \frac{\lambda_j + \lambda}{2\lambda} < 1.$$

As $\alpha \to \infty$ the functions $H(t, \alpha V_j(t))$ converge to $H(t, \infty) = (1-m)H(t, y_0)$ monotonically, almost everywhere, and uniformly on compact subintervals not containing a zero of $V_j(t)$. Hence

$$\sigma_j(\alpha) \to C_j > 1 \quad \text{as} \quad \alpha \to \infty .$$

Thus there is a unique $\tilde{\alpha} = \tilde{\alpha}(q)$, $\delta_j \le \tilde{\alpha} < \infty$, and equation (3.11) holds.

<u>Lemma 7</u>. The quantities $\tilde{\alpha}(q)$ are bounded for $q \in A$.

<u>Proof:</u> Suppose there is a sequence $q^n(t) \in A$ such that

$$\tilde{\alpha}^n = \tilde{\alpha}(q^n) \to \infty \quad \text{as} \quad n \to \infty .$$

The sequence $\{q^n\}$ is weakly compact in $L^2[0,1]$. Hence there is a subsequence $q^{n'}(t)$ which converges weakly to a function $\bar{q}(t)$. While $\bar{q}(t)$ may not be continuous, we have

$$(3.13) \qquad H(t,\infty) \le \bar{q}(t) \le H(t,0), \quad \text{a.e.}$$

By extracting enough subsequences we may assume

$$\mu_j(q^n) \to \bar{\mu}$$

$$V_j^n \to \bar{V}(t) \qquad \text{uniformly}$$

$$\psi_j^n \to \bar{\psi}(t) \qquad \text{uniformly}$$

$$q^n(t) \to \bar{q}(t) \qquad \text{weakly in } L^2 .$$

Then

$$\bar{V}(s) = \lambda \bar{\mu} \int_0^1 K(s,t)\, \bar{q}(t)\, \bar{V}(t)\, dt . \tag{3.14}$$

Since $\|\bar{V}\| = 1$, we see that $\bar{V}(t)$ is an eigenfunction of an integral equation with an oscillation kernel relative to the positive measure $\bar{q}(t)\,dt$. But then $\bar{V}(t)$ has only a finite number of zeros. As in the argument above

$$H(t, \tilde{\alpha}^n V_j^n(t)) \to H(t, \alpha)$$

almost everywhere and uniformly on compact sets not including zeros of $\bar{V}(t)$. Furthermore,

$$\bar{\psi}(s) = \lambda \int_0^1 K(s,t)\, H(t,\alpha)\, \bar{\psi}(t)\, dt . \tag{3.15}$$

But then, (1.9) implies that

$$\bar{\psi}(t) \equiv 0 .$$

However, $\|\psi\|_\infty = 1$.

Lemma 8. The function $\tilde{\alpha}(q)$ is a continuous function of $q \in A$.

Proof: Let q^k, $k = 1, \ldots$ be a sequence in A which converge uniformly to a function $\overline{q}(t) \in A$. The associated eigenvalues and eigenfunctions μ^k, $V_j^k(t)$ also converge to an eigenvalue and eigenfunction $\overline{\mu}^k$, $\overline{V}_j(t)$ associated with $\overline{q}(t)$ (see lemma 1 of [7]).

The constants $\tilde{\alpha}^k = \tilde{\alpha}(q^k)$ satisfy

$$0 < \alpha_0 \le \tilde{\alpha}^k \le \alpha_1 < \infty .$$

Let $\hat{\alpha}$ be a limit point of this sequence. We must show that

$$\hat{\alpha} = \tilde{\alpha}(\overline{q}) . \tag{3.16}$$

After extracting a subsequence (k') we have

$$\tilde{\alpha}^{k'} \to \hat{\alpha}$$

$$V_j^{k'}(t) \to \overline{V}_j(t) \quad \text{uniformly.}$$

Thus the functions $H(t, \tilde{\alpha}^{k'} V_j^{k'}(t))$ converge uniformly to $H(t, \hat{\alpha}\, \overline{V}_j(t))$. By lemma 1 of [7] we see that the associated eigenfunctions $\psi_j^{(h)}(t)$ converge to the j'th eigenfunction $\overline{\psi}_j^{(t)}$ of the limit equation. That is

$$\overline{\psi}_j(s) = \lambda \int_0^1 k(s, t)\, H(t, \hat{\alpha}\, \overline{V}_j(t))\, \overline{\psi}_j(t)\, dt .$$

Using the unicity of $\tilde{\alpha}(q)$ we see that (3.16) holds and the lemma is proven.

Let M_j be the mapping of A into A defined by

(3.17) $$(M_j\ q)(t) = H(t, \tilde{\alpha}(q)\, V_j(t)) = (L_j(\tilde{\alpha}(q))\, q)(t) \ .$$

It is now an easy matter to verify that M_j is a completely continuous operator (see [7]) and hence there is a fixed point $q(t)$. Then

(3.18) $$q(t) = H(t, \tilde{\alpha}\, V_j(t))$$

where

(3.19) $$V_j(s) = \lambda \int_0^1 K(s,t)\; H(t, \tilde{\alpha}\; V_j(t))\; V_j(t)\, dt \ .$$

Hence

$$x_j(t) = \tilde{\alpha}\, V_j(t)$$

is a solution of equation (1.) having exactly j interior nodel zeros and no other zeros.

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