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THE NUMERICAL SOLUTION OF VOLTERRA FUNCTIONAL DIFFERENTIAL EQUATIONS BY EULER'S METHOD

by

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APPENDICES

THE NUMERICAL SOLUTION OF VOLTERRA FUNCTIONAL DIFFERENTIAL EQUATIONS BY EULER'S METHOD.

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1. Introduction.

Let $\alpha \leq a < b$ be real numbers, and let $C([t_1,t_2] \to E^n)$ denote the space of continuous functions on $[t_1,t_2]$ into E^n (n-dimensional Euclidean space). We shall be concerned with the Cauchy problem for Volterra functional differential equations:

$$\left.\begin{aligned} y'(t) &= F(y,t), \quad t \in [a,b], \\ y(t) &= g(t), \quad t \in [\alpha,a]. \end{aligned}\right\} \qquad (1.1)$$

Here, $F:C([\alpha,b] \to E^n) \times [a,b] \to E^n$ is a Volterra functional, that is, $F(y,t)$ depends on t and on $y(s)$ for $s \in [\alpha,t]$, but is independent of $y(s)$ for $s > t$; and the function $g \in C([\alpha,a] \to E^n)$ is a specified initial function. We will require F and g to satisfy certain continuity conditions which will be given later.

The problem (1.1) includes as special cases the initial value problems for ordinary differential equations, retarded ordinary differential equations, and Volterra integro-differential equations; some concrete examples of (1.1) are given below.

Among the many works dealing with Volterra functional differential equations we mention those of Volterra [V059], Driver [DR62], Bellman and Cooke [BC63], Zverkin et al [ZK62], and Oǧuztöreli [OU66]; further references to the literature will be found in Appendix A.

Numerical methods for solving special cases of (1.1) have been considered by several authors. Bellman et al [BB65], El'sgol'ts [EL57], Feldstein [FE64], and Zverkina [ZV65], have developed methods for solving retarded ordinary differential equations. Methods for solving Volterra integro-differential equations have been given by Pouzet [PO60], Feldstein and Sopka [FES68], and Linz [LI69]. A brief summary of all these methods will be found in Appendix A.

The most basic method for solving initial value problems for ordinary differential equations is Euler's method (Henrici[HE62, p. 9]). It is natural to seek a generalization of Euler's method which can be applied to the problem (1.1). Such a generalization has, for example, been suggested by Feldstein [FE64]. In the present paper we introduce another generalization of Euler's method. The major difference between the present approach and previous approaches is that the approximate solutions which we generate lie in $C([\alpha, b] \to E^n)$.

In the remainder of this section we describe the continuity conditions which F and g must satisfy. In section 2 we define the generalization of Euler's method, and in sections 3 and 4 it is proved that this method is convergent. In section 5 we consider the possibility that it is only possible to compute F and g approximately. In section 6 we generalize the "improved Euler method" (Henrici [HE62, p. 67]) and show that the resulting method is quadratically convergent. To conclude, we give a numerical example in section 7.

We shall use the following notation. For $v \in E^n$ the norm of v is denoted by $\|v\|$ and is defined by

$$\|v\| = \max_{1 \le i \le n} |v_i|.$$

For arbitrary real numbers t' and t, $t' \geq t$, the notation $C^p([t,t'] \to E^n)$ denotes the set of all functions on $[t,t']$ into E^n with p continuous derivatives. We shall write $C([t,t'] \to E^n)$ instead of $C^0([t,t'] \to E^n)$.

Let I be an interval in $[t,t']$. For $x \in C([t,t'] \to E^n)$ we define $\|x\|^I$ by

$$\|x\|^I = \max_{s \in I} \|x(s)\| .$$

The Banach space $C([\alpha,b] \to E^n)$ with the $\|\cdot\|^{[\alpha,b]}$ norm will be denoted by X.

Let $G: X \times [a,b] \to E^n$. If there exists a constant $L \geq 0$ such that

$$\|G(x,t) - G(y,t)\| \leq L\,\|x-y\|^{[\alpha,t]};\quad x,\ y \in X;\ t \in [a,b];$$

we write $G \in \mathrm{Lip}(X,L)$, (G is <u>uniformly Lipschitz</u> on X with Lipschitz constant L).

We denote by $\mathcal{C}(F,g,a)$ the Cauchy problem (1.1) with the following conditions

$$\left.\begin{array}{ll} \text{(a)} & F \in \mathrm{Lip}(X,L), \text{ for some } L \geq 0\ , \\ \text{(b)} & t \to F(x,t) \in C([a,b] \to E^n) \text{ for fixed } x \in X, \\ \text{(c)} & g \in C([\alpha,a] \to E^n)\ . \end{array}\right\} \qquad (1.2)$$

We give two examples to illustrate the significance of conditions (1.2).

<u>Example 1.1</u> Consider the scalar retarded ordinary differential equation

$$\left.\begin{array}{ll} y'(t) = f(y(t),\ y(u(t)),\ t), & t \in [a,b], \\ y(t) = g(t)\ , & t \in [\alpha,a]\ , \end{array}\right\} \qquad (1.3)$$

where $\alpha \leq u(t) \leq t$; g, f, and u are continuous; and f satisfies

$$|f(r_1,s_1,t) - f(r_2,s_2,t)| \leq L_1(|r_1 - r_2| + |s_1 - s_2|),$$

for $t \in [a,b]$; $r_1,r_2,s_1,s_2 \in E^1$. If we take X to be the space $C([\alpha,b] \to E^1)$ with the uniform norm and define $F(y,t) = f(y(t),\ y(u(t)),t)$, (1.3) satisfies conditions (1.2).

Proof: We have for $x, y \in X$, $t \in [a,b]$,

$$\|F(x,t) - F(y,t)\| = |f(x(t), x(u(t)), t) - f(y(t), y(u(t)), t)|,$$
$$\leq L_1 (|x(t) - y(t)| + |x(u(t)) - y(u(t))|),$$
$$\leq 2L_1 \|x-y\|^{[\alpha,t]},$$

which establishes (1.2a). For fixed $y \in X$, the mapping $t \to F(y,t)$ is continuous by the continuity of composition of continuous functions. This establishes (1.2b). Condition (1.2c) follows immediately.

Example 1.2 Consider the Volterra integro-differential equation

$$y'(t) = f(y(t), \int_a^t k(y(s), s, t)ds, t); \quad t \in [a,b], \tag{1.4}$$

where $y(a)$ is given, and f and k are continuous and satisfy

$$|f(r_1, s_1, t) - f(r_2, s_2, t)| \leq L_1(|r_1 - r_2| + |s_1 - s_2|),$$
$$|k(r_1, s, t) - k(r_2, s, t)| \leq L_2 |r_1 - r_2|,$$

for $r_1, r_2, s_1, s_2, s, t \in E^1$. If we take X as in Example 1.1, $F(y,t)$ equal to the right hand side of (1.4), $\alpha = a$, and define $g: \{a\} \to E^1$ by $g(a) = y(a)$, then (1.4) satisfies conditions (1.2).

Proof: We have for $x, y \in X$, $t \in [a,b]$,

$$\|F(x,t) - F(y,t)\| = |f(x(t), \int_a^t k(x(s), s, t)ds, t)$$
$$- f(y(t), \int_a^t k(y(s), s, t)ds, t)|,$$

$$\le L_1(|x(t) - y(t)| + \int_a^t |k(x(s),s,t) - k(y(s),s,t)|\, ds),$$

$$\le [L_1 + (b-a)\, L_2]\, \|x-y\|^{[a,t]},$$

which establishes (1.2a). Condition (1.2b) follows by the continuity conditions and the continuity of the integral in (1.4), and (1.2c) follows immediately.

2. Definition of Euler's method.

For a given $\mathcal{G}(F,g,a)$ and integer $N > 0$, the approximate solution $\tilde{y}$ corresponding to the step $h = (b-a)/N$ is constructed as follows. Set $t_i = a + i\,h$, $i = 0, \ldots, N$. Define $x_i \in X$, $i = 0, \ldots, N$, by

$$x_0(t) = \begin{cases} g(t), & \alpha \le t \le a, \\ g(a), & a < t \le b, \end{cases} \tag{2.1a}$$

and for $i = 0, \ldots, N-1$,

$$x_{i+1}(t) = \begin{cases} x_i(t), & \alpha \le t \le t_i, \\ x_i(t_i) + (t-t_i)\, F(x_i, t_i), & t_i < t \le b. \end{cases} \tag{2.1b}$$

Set $\tilde{y} = x_N$.

Remarks:

1. The method is independent of the manner in which x_i is extended beyond t_i; all that is required is that $x_i \in X$.

2. For the case where $\mathcal{G}(F,g,a)$ is an ordinary differential equation, the vectors $\tilde{y}(t_i)$ $i = 0, \ldots, N$, are identical with the approximate solution generated by the customary Euler method (Henrici [HE62, p. 9]).

3. The computability of Euler's method depends on the computability of F. For numerical computation F is replaced by a discrete approximation $\tilde{F}$ introducing an additional error. We call this the approximate Euler method and discuss it in section 5.

4. The implementation of Euler's method on a computer is discussed in Appendix B.

We will often be concerned not with a single approximate solution $\tilde{y}$ but with a sequence of approximations. Let $\{N_p\}$ be an increasing sequence of positive integers; for example, $N_p = 2^p$. Then we set

$$h_p = (b-a)/N_p, \tag{2.2}$$

and denote by y_p the approximate solution generated by Euler's method with step-size h_p.

Following Henrici [HE62, p. 16] we set

$$t_{i,p} = a + ih_p, \; 0 \leq i \leq N_p, \tag{2.3}$$

$$a_{[p]} = a, \tag{2.4}$$

and

$$t_{[p]} = a + k\, h_p, \text{ for } t \in (a,b], \tag{2.5}$$

where k is the largest integer such that $a + k\, h_p < t \leq a + (k+1)h_p$.

Then it follows from (2.1) through (2.5) that

$$\left.\begin{array}{l} y_p(t) = g(t),\; t \in [\alpha, a], \\ y_p(t) = y_p(t_{[p]}) + (t - t_{[p]})\, F(y_p, t_{[p]}),\; t \in [a,b]. \end{array}\right\} \tag{2.6}$$

3. Preliminary results.

Lemma 3.1 (Henrici [HE62, p. 18]) If the numbers q_i satisfy the inequality

$$|q_{i+1}| \leq A\,|q_i| + B, \quad i = 0,\ldots,N-1,$$

where $A, B \geq 0$, then

$$|q_i| \leq A^i\,|q_0| + \begin{cases} \dfrac{A^i - 1}{A - 1} B, & A \neq 1, \\[2ex] i\,B, & A = 1, \end{cases}$$

for $i = 0,\ldots,N$. Further, if $A = 1 + \delta$, $\delta \geq 0$, then

$$|q_i| \leq e^{i\delta}|q_0| + \begin{cases} \dfrac{e^{i\delta} - 1}{\delta} B, & \delta > 0, \\[2ex] i\,B, & \delta = 0, \end{cases}$$

for $i = 0,\ldots,N$.

Lemma 3.2 If $x \in C([\alpha,b] \to E^n)$, $N > 0$ is an integer, $h = (b-a)/N$, $t_i = a + i\,h$, $i = 0,\ldots,N$, and x satisfies the inequality

$$\|x(t)\| \leq (1 + hA)\,\|x\|^{[\alpha,t_i]} + B, \quad t \in (t_i, t_{i+1}],\ i = 0,\ldots,N-1,$$

where $A, B \geq 0$, then x satisfies the inequality

$$\|x\|^{[\alpha,b]} \leq e^{(b-a)A}\,\|x\|^{[\alpha,a]} + \begin{cases} \dfrac{e^{(b-a)A} - 1}{hA} B, & A \neq 0, \\[2ex] \dfrac{(b-a)}{h} B, & A = 0. \end{cases}$$

Proof: The bound for $\|x(t)\|$ is uniform for $t \in (t_i, t_{i+1}]$; therefore,

$$\|x\|^{[\alpha,t_{i+1}]} = \max\ \{\|x\|^{[\alpha,t_i]},\ \|x\|^{(t_i,t_{i+1}]}\},$$

$$\leq (1 + hA)\,\|x\|^{[\alpha,t_i]} + B,\ i = 0,\ldots,N-1.$$

By Lemma 3.1 we have

$$\|x\|^{[\alpha,t_i]} \le e^{ihA}\ \|x\|^{[\alpha,a]} + \begin{cases} \dfrac{e^{ihA}-1}{hA} B, & A \neq 0, \\ iB, & A = 0, \end{cases}$$

for $i = 0,\ldots,N$. The lemma follows by taking $i = N$.

We now turn to an investigation of the sequence of functions, $\{y_p\}$, generated by Euler's method with decreasing stepsize.

<u>Lemma 3.3</u> The sequence $\{y_p\}$ is uniformly bounded.

<u>Proof:</u> $0 \in X$ and $F \in Lip(X, L)$ for some $L > 0$, so that for $x \in X$ and $t \in [a,b]$ we have

$$\|F(x,t) - F(0,t)\| \le L\ \|x\|^{[\alpha,t]} .$$

Therefore,

$$\|F(x,t)\| \le L\ \|x\|^{[\alpha,t]} + C , \tag{3.1}$$

$$C = \max_{t\in[a,b]} \|F(0,t)\| .$$

By (2.6) and (3.1) we have, for $t \in (t_{i,p}, t_{i+1,p}]$,

$$\begin{aligned} \|y_p(t)\| &\le \|y_p(t_{i,p})\| + h_p\ \|F(y_p, t_{i,p})\| , \\ &\le \|y_p\|^{[\alpha,t_{i,p}]} + h_p\ L\ \|y_p\|^{[\alpha,t_{i,p}]} + h_p\ C , \\ &= (1 + h_p\ L)\ \|y_p\|^{[\alpha,t_{i,p}]} + h_p\ C . \end{aligned}$$

By Lemma 3.2 we have

$$\|y_p\|^{[\alpha,b]} \le e^{(b-a)L}\ \|g\|^{[\alpha,a]} + \frac{e^{(b-a)L}-1}{L}\ C ,$$

which gives a bound independent of p .

Lemma 3.4 $\{y_p\}$ is equicontinuous on $[\alpha, b]$.

Proof: Let $a \leq t \leq s \leq b$, so that $s_{[p]} - t_{[p]} = k\,h_p$ for some integer $k \geq 0$. Then

$$\|y_p(t) - y_p(s)\| \leq \|y_p(t) - y_p(t_{[p]} + h_p)\|$$
$$+ \|y_p(t_{[p]} + h_p) - y_p(t_{[p]} + 2h_p)\|$$
$$+ \dots + \|y_p(t_{[p]} + k\,h_p) - y_p(s)\| .$$

By (2.6) we have

$$\|y_p(t) - y_p(t_{[p]} + h_p)\| \leq (t_{[p]} + h_p - t)\,\|F(y_p, t_{[p]})\| ,$$

$$\|y_p(t_{[p]} + i\,h_p) - y_p(t_{[p]} + (i+1)h_p)\| \leq h_p\,\|F(y_p, t_{[p]} + i\,h_p)\| ,$$

for $i = 1, \dots, k-1$, and

$$\|y_p(t_{[p]} + k\,h_p) - y_p(s)\| \leq (s - s_{[p]})\ \|F(y_p, s_{[p]})\| .$$

By Lemma 3.3 we have a constant $M > 0$ such that

$$\|F(y_p, t)\| \leq M , \quad t \in [a, b] ,$$

so that, summing the above $k + 1$ inequalities, we obtain

$$\|y_p(t) - y_p(s)\| \leq (t_{[p]} + h_p - t + (k-1)h_p + s - s_{[p]})\ M ,$$
$$= |t - s|\ M .$$

We have, therefore, established that $\{y_p\}$ is Lipschitz equicontinuous on $[a, b]$. Since $y_p(t) = g(t)$ for $t \in [\alpha, a]$, the lemma follows.

Lemma 3.5 Let S be a bounded equicontinuous family of functions $x \in X$. For $h > 0$ define

$$\omega_S(h) = \sup \{ \| F(x,t) - F(x,s) \| : x \in S; t, s \in [a,b]; |t - s| \leq h \}. \quad (3.2)$$

Then $\omega_S(h) \to 0$ for fixed S.

Proof: Suppose the contrary. Then there exist sequences $\{(x_p, t_p)\}$ and $\{(x_p, s_p)\}$ in $S \times [a,b]$ such that

$$|t_p - s_p| \to 0 \text{ as } p \to \infty, \quad (3.3)$$

and, for some $\epsilon > 0$,

$$\| F(x_p, t_p) - F(x_p, s_p) \| > \epsilon, \quad p = 0, 1, \ldots . \quad (3.4)$$

Define $z_p : [\alpha, b] \to E^{n+1}$ by $z_p(t) = (x_p(t), t_p)$. Then $\{z_p\}$ is equicontinuous on the compact set $[\alpha, b]$ and by the Arzelá-Ascoli theorem it has a uniformly convergent subsequence $\{z_{p_k}\}$, $z_{p_k} \to (x,t) \in X \times E^1$. Therefore,

$$\| F(x_{p_k}, t_{p_k}) - F(x,t) \| \leq \| F(x_{p_k}, t_{p_k}) - F(x, t_{p_k}) \| + \| F(x, t_{p_k}) - F(x,t) \|,$$

$$\leq L \| x_{p_k} - x \|^{[\alpha, b]} + \| F(x, t_{p_k}) - F(x,t) \| \to 0 \text{ as } k \to \infty.$$

By the same argument we have $\| F(x_{p_k}, s_{p_k}) - F(x,t) \| \to 0$.

Therefore,

$$\| F(x_{p_k}, t_{p_k}) - F(x_{p_k}, s_{p_k}) \| \leq \| F(x_{p_k}, t_{p_k}) - F(x,t) \|$$

$$+ \| F(x_{p_k}, s_{p_k}) - F(x,t) \| \to 0 \text{ as } k \to \infty,$$

contradicting (3.4).

Lemma 3.6 The sequence $\{y_p\}$ converges.

Proof: Let p and q be arbitrary integers satisfying $0 \le p \le q$. Let $d(t) = y_p(t) - y_q(t)$, $t \in [\alpha, b]$. By (2.6) we have for any $t \in [a,b]$

$$y_q(t) = y_q(t_{[q]}) + (t - t_{[q]})\, F(y_q, t_{[q]}) , \tag{3.5}$$

$$y_p(t) = y_p(t_{[p]}) + (t - t_{[p]})\, F(y_p, t_{[p]}) , \tag{3.6}$$

Subtracting (3.5) from (3.6) we have,

$$\begin{aligned} y_p(t) - y_q(t) = {} & y_p(t_{[q]}) - y_q(t_{[q]}) \\ & + (t - t_{[q]})[F(y_p, t_{[p]}) - F(y_q, t_{[q]})] \\ & + y_p(t_{[p]}) + (t_{[q]} - t_{[p]})\, F(y_p, t_{[p]}) - y_p(t_{[q]}). \end{aligned} \tag{3.7}$$

If $t_{[q]} \ge t_{[p]}$ then $t_{[q]} \in [t_{[p]}, t]$ so that by (2.6),

$$y_p(t_{[q]}) - y_p(t_{[p]}) - (t_{[q]} - t_{[p]})\, F(y_p, t_{[p]}) = 0 . \tag{3.8}$$

Otherwise,

$$a \le t_{[p]} - h_p \le t_{[q]} < t_{[p]} < t,$$

so that, by (2.6) ,

$$\begin{aligned} & \| y_p(t_{[q]}) - y_p(t_{[p]}) - (t_{[q]} - t_{[p]})\, F(y_p, t_{[p]}) \| \\ & \quad = \| (t_{[q]} - t_{[p]})\, (F(y_p, t_{[p]} - h_p) - F(y_p, t_{[p]}) \| , \\ & \quad \le h_q\, \omega_S(h_p) , \end{aligned} \tag{3.9}$$

where $S = \{y_p\}$ and ω_S is defined by (3.2) .

Using (3.7), (3.8), and (3.9), and noting (3.2), we find that

$$\|d(t)\| \le \|d(t_{[q]})\| + (t - t_{[q]})\|F(y_p, t_{[p]}) - F(y_q, t_{[q]})\| + h_q\, \omega_s(h_p),$$

$$\le \|d(t_{[q]})\| + (t - t_{[q]})\| F(y_p, t_{[p]}) - F(y_p, t_{[q]})\|$$

$$+ (t - t_q)\| F(y_p, t_{[q]}) - F(y_q, t_{[q]})\|$$

$$+ h_q\, \omega_S(h_p) ,$$

so that,

$$\|d(t)\| \le (1 + h_q L)\|d\|^{[\alpha, t_{[q]}]} + 2h_q \omega_S(h_p) .$$

Hence,

$$\|d(t)\| \le (1 + h_q L)\|d\|^{[\alpha, t_{i,q}]} + 2h_q \omega_S(h_p),\ t \in (t_{i,q}, t_{i+1,q}] .$$

By Lemma 3.2 we have

$$\|d\|^{[\alpha, b]} \le e^{(b-a)L}\, \|d\|^{[\alpha, a]} + 2\frac{e^{(b-a)L} - 1}{L}\, \omega_S(h_p). \tag{3.10}$$

The sequence $\{y_p\}$ is bounded by Lemma 3.3 and is equicontinuous by Lemma 3.4. Therefore, by Lemma 3.5 given an $\epsilon > 0$ there exists a $\delta > 0$ such that $\omega_S(h_p) < \epsilon$ for all $h_p < \delta$. Since $\|d\|^{[\alpha, a]} = 0$, it follows from (3.10) that

$$\|y_p - y_q\|^{[\alpha, b]} < 2\frac{e^{(b-a)L} - 1}{L}\, \epsilon ,$$

for all sufficiently large p, $p \le q$. The bound is independent of q, and $\{y_p\}$ is therefore a Cauchy sequence in X. Convergence follows by the completeness of X.

4. Convergence of Euler's method and an existence-uniqueness theorem.

Theorem 4.1 $\mathscr{C}(F,g,a)$ has a unique solution in X; this solution is the limit of the sequence $\{y_p\}$ generated by Euler's method.

Proof: For $p = 0,1,2,\ldots$ define $f_p : [a,b] \to E^n$ by

$$f_p(t) = F(y_p, t_{[p]}) , \qquad t \in [a,b] . \tag{4.1}$$

By Lemma 3.6 $\{y_p\}$ has a limit $y \in X$. For $S = \{y_p\}$ we have

$$\|f_p(t) - F(y,t)\| \le \|F(y_p, t_{[p]}) - F(y_p,t)\| + \|F(y_p,t) - F(y,t)\|,$$
$$\le \omega_S(h_p) + L\|y_p - y\|^{[\alpha,t]} \to 0 \text{ as } p \to \infty .$$

Therefore $f_p(t) \to F(y,t)$ uniformly on $[a,b]$ as $p \to \infty$. From (2.6) and (4.1) it follows by induction that

$$y_p(t_{k,p}) = g(a) + h_p \sum_{i=0}^{k-1} f_p(t_{i+1,p}), \; k = 0,\ldots,N_p . \tag{4.2}$$

For arbitrary $t \in [a,b]$, $t_{[p]} = t_{k,p}$ for some k, $0 \le k \le N_p - 1$.

For this k , we have by (2.6) and (4.2)

$$y_p(t) = g(a) + h_p \sum_{i=0}^{k-1} f_p(t_{i+1,p}) + (t - t_{[p]}) f_p(t) .$$

Since f_p is constant on each interval $(t_{[p]}, t_{[p]} + h_p]$ we can write the above as

$$y_p(t) - g(a) = \int_a^t f_p(s)ds , \quad t \in [a,b] .$$

Taking the limit as $p \to \infty$ we have

$$y(t) - g(a) = \int_a^t F(y,s)\, ds . \tag{4.3}$$

Since $s \to F(y,s)$ is continuous, the right hand side of (4.3) is differentiable. Therefore the left hand side is also differentiable and we get

$$y'(t) = F(y,t) , \qquad t \in [a,b] .$$

The function y is therefore a solution of $\mathcal{C}(F,g,a)$. It remains to show that this solution is unique.

Suppose that x and y are two solutions of $\mathcal{C}(F,g,a)$. Define

$$\delta = x - y .$$

Then δ is bounded, $\|\delta(t)\| \leq K$, $t \in [\alpha,b]$.

We claim that

$$\|\delta(t)\| \leq KL^k (t-a)^k/k!, \quad t \in [a,b], \tag{4.4}$$

for $k = 0,1,2,\ldots$. This is now established by induction. We have

$$x(t) - g(a) = \int_a^t F(x,s)ds, \quad t \in [a,b] ,$$

$$y(t) - g(a) = \int_a^t F(y,s)\, ds , \quad t \in [a,b] ,$$

from which we get

$$\|\delta(t)\| \leq \int_a^t \|F(x,s) - F(y,s)\|\, ds , \quad t \in [a,b] . \tag{4.5}$$

The bound (4.4) holds for $k = 0$. Suppose it holds for $k = 0,1,\ldots, m$. By the induction hypothesis we have

$$\|F(x,s) - F(y,s)\| \leq L \, \|\delta\|^{[\alpha,s]} \leq KL^{m+1} (s-a)^m/m! \; , \; s \in [a,b].$$

Therefore, by (4.5),

$$\|\delta(t)\| \leq \int_a^t KL^{m+1} \frac{(s-a)^m}{m!} \, ds \leq KL^{m+1} \frac{(t-a)^{m+1}}{(m+1)!} \; , \; t \in [a,b] \; ,$$

which establishes (4.4) for all k .

Since $L^k (b-a)^k/k! \to 0$ as $k \to \infty$ and $\delta(t) = 0$ for $t \in [\alpha,a]$, it follows that $\|\delta\|^{[\alpha,b]} = 0$ so that the solution is unique.

5. The approximate Euler method.

A significant difference between an ordinary differential equation $y' = f(x,y)$ and the functional differential equation (1.1) is that usually f can be evaluated to arbitrary accuracy whereas the functional F , in general, must be replaced by a discrete approximation for numerical work. We now consider the case where in Euler's method g and F are replaced by approximations $\tilde{g}$ and $\tilde{F}$. Let $\tilde{g}$: $[\alpha,a] \to E^n$, $\tilde{F}: X \times [a,b] \to E^n$ be approximations to g and F corresponding to the stepsize $h = (b-a)/N$, $N > 0$ an integer. Instead of using (2.1) we compute an approximate Euler function $\tilde{y}$ by the relations

$$x_0(t) = \begin{cases} \tilde{g}(t) \; , & \alpha \leq t \leq a \; , \\ \tilde{g}(a) \; , & a < t \leq b \; , \end{cases} \tag{5.1a}$$

$$x_{i+1}(t) = \begin{cases} x_i(t)\ , & \alpha \le t \le t_i\ , \\ x_i(t) + (t - t_i)\,\tilde{F}(x_i, t_i),\ & t_i < t \le b\ , \end{cases} \tag{5.1b}$$

for $i = 0, \ldots, N-1$. We set $\tilde{y} = x_N$.

We denote by y_p the approximate solution computed by (5.1) with $N = N_p$, $h = h_p$, $\tilde{g} = g_p$, and $\tilde{F} = F_p$.

Theorem 5.1

Let $\{y_p\}$ be the sequence of approximate solutions generated by the approximate Euler method. Let y be the solution of $\mathcal{G}(F,g,a)$.

If y is twice continuously differentiable on $[a,b]$; $F_p \in \mathrm{Lip}(X,M)$ for some $M > 0$; $g_p(t) = g(t) + O(h_p)$, $t \in [\alpha, a]$; and $F_p(y,t) = F(y,t) + O(h_p)$, $t \in [a,b]$; then

$$y_p(t) = y(t) + O(h_p),\ t \in [\alpha, b]\ .$$

Proof: Since y is twice continuously differentiable, the mapping $t \to F(y,t)$ is continuously differentiable on $[a,b]$. Hence,

$$A_p = \max_{\substack{|s-t| \le h_p \\ s,t \in [a,b]}} \| F(y,s) - F(y,t) \| = O(h_p)\ .$$

For $r \in [0,1]$, $i = 0, \ldots, N_p - 1$, y and y_p satisfy the relations

$$y(t_{i,p} + rh_p) = y(t_{i,p}) + rh_p F(y, s_{i,r}),\ t_{i,p} < s_{i,r} < t_{i+1,p}\ ,$$

$$y_p(t_{i,p} + rh_p) = y_p(t_{i,p}) + rh_p F_p(y_p, t_{i,p})\ ,$$

giving

$$\|(y-y_p)(t_{i,p}+rh_p)\| \le \|y-y_p\|^{[\alpha,t_{i,p}]} + h_p\|F(y,s_{i,r}) - F(y,t_{i,p})\|$$

$$+ h_p\|F(y,t_{i,p}) - F_p(y,t_{i,p})\| + h_p\|F_p(y,t_{i,p}) - F_p(y_p,t_{i,p})\| ,$$

$$\le \|y-y_p\|^{[\alpha,t_{i,p}]} + h_pA_p + h_p\|F(y,t_{i,p})-F_p(y,t_{i,p})\|$$

$$+ h_p M\|y-y_p\|^{[\alpha,t_{i,p}]} ,$$

$$\le (1+h_pM)\|y-y_p\|^{[\alpha,t_{i,p}]} + h_p(A_p + C_p),$$

where

$$C_p = \sup_{t \in [a,b]} \|F(y,t) - F_p(y,t)\| . \tag{5.2}$$

By Lemma 3.2 we get

$$\|y-y_p\|^{[\alpha,b]} \le e^{(b-a)M} \|g-g_p\|^{[\alpha,a]} + \frac{e^{(b-a)M} - 1}{M}[A_p + C_p] .$$

The theorem follows .

6. The improved Euler method.

We shall consider the case of approximate computation directly. The improved Euler method for $\mathcal{O}(F,g,a)$ and stepsize $h = (b-a)/N$, $N > 0$ an integer, generates an approximation $\tilde{y}$ using the relations

$$x_0(t) = \begin{cases} \tilde{g}(t) , & \alpha \le t \le a , \\ g(a) , & a < t \le b , \end{cases} \tag{6.1a}$$

$$x_{i+1}(t) = \begin{cases} x_i(t)\ , & \alpha \leq t \leq t_i\ , \\ x_i(t_i) + rh\,\tilde{F}\,(x_i, t_i) + \frac{1}{2} r^2\, h\,[\tilde{F}(Q_{t_{i,h}}(x_i),\ t_i + h) \\ \quad - \tilde{F}(x_i, t_i)],\ r = (t-t_i)/h,\ t_i < t \leq b\ , \end{cases} \tag{6.1b}$$

for i = 0, . . . , N-1. The operators $Q_{t,h} : X \to X$ are defined by

$$[Q_{t,h}(x)](s) = \begin{cases} x(s), & \alpha \leq s \leq t\ , \\ x(t) + (t-s)\,\tilde{F}(x,t), & t < s \leq b\ . \end{cases} \tag{6.1c}$$

$\tilde{g}$ and $\tilde{F}$ are approximations to g and F respectively, as in the approximate Euler method. We set $\tilde{y} = x_N$.

We call the above method the improved Euler method because it corresponds to the improved Euler method (or Heun's method) for ordinary differential equations, (Henrici [HE62, p. 67]), namely,

$$y_{i+1} = y_i + \frac{h}{2}\,[f(y_i, t_i) + f(y_i + hf(y_i, t_i),\ t_i + h)].$$

The relationship between the two methods is made clearer by noting that

1. x_{i+1} is a quadratic polynomial on $[t_i, t_{i+1}]$,

2. $$x_{i+1}(t_i) = x_i(t_i)\ , \quad \lim_{t \downarrow t_i} x'_{i+1}(t) = \tilde{F}(x_i, t_i)\ ,$$

3. $x_{i+1}(t_{i+1}) = x_i(t_i) + \frac{h}{2}\,[F(x_i, t_i) + F(Q_{t_{i,h}}(x_i),\ t_i + h)]$,

   $[Q_{t_{i,h}}(x_i)]\ (t_i + h) = x_i(t_i) + h\ \tilde{F}(x_i, t_i)$.

We denote by y_p the approximate solution computed by (6.1) with $N = N_p$, $h = h_p$, $\tilde{g} = g_p$, and $\tilde{F} = F_p$. We have the following theorem.

Theorem 6.1 Let $\{y_p\}$ be the sequence of approximate solutions generated by the improved Euler method. Let y be the solution of $\mathcal{C}(F,g,a)$.

If $y \in C^3([a,b] \to E^n)$; $F_p \in \mathrm{Lip}(X,M)$, for some $M > 0$; $g_p(t) = g(t) + O(h_p^2)$, $t \in [\alpha,a]$; and $F_p(y,t) = F(y,t) + O(h_p^2)$, $t \in [a,b]$; then

$$y_p(t) = y(t) + O(h_p^2),\ t \in [\alpha,b].$$

Proof: For $r \in [0,1]$, $i = 0,\dots, N_p - 1$, y satisfies the relations

$$y(t_{i,p} + rh_p) = y(t_{i,p}) + rh_p y'(t_{i,p}) + \frac{r^2h_p^2}{2} y''(t_{i,p}) + O(h_p^3),$$

$$y''(t_{i,p}) = \frac{1}{h_p}\,[F(y,t_{i,p} + h_p) - F(y,t_{i,p})] + O(h_p),$$

giving

$$y(t_{i,p} + rh_p) = y(t_{i,p}) + rh_p F(y,t_{i,p}) + \frac{r^2 h_p}{2}[F(y,t_{i,p} + h_p) - F(y,t_{i,p})] + O(h_p^3). \qquad (6.2)$$

By (6.1), with $y^*_{i,p} = Q_{t_{i,p},h_p}(y_p)$, we have

$$y_p(t_{i,p} + rh_p) = y_p(t_{i,p}) + rh_p F_p(y_p,t_{i,p}) + \frac{r^2 h_p}{2}\,[F_p(y^*_{i,p},t_{i,p} + h_p) - F_p(y_p,t_{i,p})]. \qquad (6.3)$$

By subtracting (6.3) from 6.2) we get

$$\|(y-y_p)(t_{i,p}+rh_p)\| \le \|y-y_p\|^{[\alpha,t_{i,p}]} + \frac{h_p}{2}\,\|F(y,t_{i,p}) - F_p(y_p,t_{i,p})\|$$

$$+\frac{h_p}{2}\,\|F(y,t_{i,p}+h_p) - F_p(y^*_{i,p},t_{i,p}+h_p)\| + O(h_p^3)\,,$$

$$\le \|y-y_p\|^{[\alpha,t_{i,p}]} + \frac{h_p}{2}\,\|F(y,t_{i,p}) - F_p(y,t_{i,p})\|$$

$$+\frac{h_p}{2}\,\|F_p(y,t_{i,p}) - F_p(y_p,t_{i,p})\|$$

$$+\frac{h_p}{2}\,\|F(y,t_{i,p}+h_p) - F_p(y,t_{i,p}+h_p)\|$$

$$+\frac{h_p}{2}\,\|F_p(y,t_{i,p}+h_p) - F_p(y^*_{i,p},t_{i,p}+h_p)\| + O(h_p^3)\,,$$

$$\le \|y-y_p\|^{[\alpha,t_{i,p}]} + \frac{h_p}{2}\,C_p + \frac{h_p}{2}\,M\|y-y_p\|^{[\alpha,t_{i,p}]}$$

$$+\frac{h_p}{2}\,C_p + \frac{h_p}{2}\,M\|y-y^*_{i,p}\|^{[\alpha,t_{i,p}+h_p]} + O(h_p^3)\,, \tag{6.4}$$

where C_p is defined by (5.2).

We now compute a bound for the term involving $y^*_{i,p}$. By (6.1c) we have

$$y^*_{i,p}(t) = \begin{cases} y_p(t), & \alpha \le t \le t_{i,p}\,, \\ y_p(t_{i,p}) + (t-t_{i,p})\,F_p(y_p,t_{i,p}), & t_{i,p} < t \le b\,. \end{cases}$$

Writing $y(t) = y(t_{i,p}) + rh_pF(y,t_{i,p}) + O(h_p^2)$ we get

$$\|(y-y^*_{i,p})(t_{i,p}+rh_p)\| = \|y(t_{i,p}) + rh_pF(y,t_{i,p}) + O(h_p^2)$$
$$- y_p(t_{i,p}) - rh_pF_p(y_p,t_{i,p})\| ,$$

$$\leq \|y-y_p\|^{[\alpha,t_{i,p}]} + h_p\|F(y,t_{i,p}) - F_p(y,t_{i,p})\|$$
$$+ h_p\|F_p(y,t_{i,p}) - F_p(y_p,t_{i,p})\| + O(h_p^2) ,$$
$$\leq \|y-y_p\|^{[\alpha,t_{i,p}]} + h_pC_p + h_pM\|y-y_p\|^{[\alpha,t_{i,p}]} + O(h_p^2) .$$

This bound is uniform in r , $0 < r \leq 1$. Since $\|y-y^*_{i,p}\|^{[\alpha,t_{i,p}]} =$

$\|y-y_p\|^{[\alpha,t_{i,p}]}$, we get

$$\|y-y^*_{i,p}\|^{[\alpha,t_{i,p}+h_p]} \leq (1+h_pM)\|y-y_p\|^{[\alpha,t_{i,p}]} + h_p C_p + O(h_p^2) . \quad (6.5)$$

By (6.4) and (6.5) we get

$$\|(y-y_p)(t_{i,p}+rh_p)\| \leq (1+\frac{h_p}{2}M)\|y-y_p\|^{[\alpha,t_{i,p}]} + h_p C_p$$
$$+ \frac{h_p}{2} M(1+h_pM)\|y-y_p\|^{[\alpha,t_{i,p}]} + \frac{h_p^2}{2} MC_p + O(h_p^3) ,$$
$$\leq (1+h_pM_0)\|y-y_p\|^{[\alpha,t_{i,p}]} + h_p(1+\frac{h_p}{2}M)C_p + O(h_p^3) ,$$

where $M_0 = M(1+\frac{h_0M}{2})$. By Lemma 3.2 we get

$$\|y-y_p\|^{[\alpha,b]} \leq e^{(b-a)M_0}\|g-g_p\|^{[\alpha,a]}$$
$$+ \frac{e^{(b-a)M_0} - 1}{M_0} (1+\frac{h_p}{2} M) C_p + O(h_p^2) .$$

Since $C_p = O(h_p^2)$, the theorem follows.

7. A numerical example.

In this section we present numerical results for the scalar retarded differential equation

$$\left.\begin{aligned} y'(t) &= [y(t/u(t))]^{u(t)} , \\ y(o) &= 1 , \end{aligned}\right\} \qquad (7.1)$$

where

$$u(t) = (1 + 2t)^2 .$$

It is easily verified that the solution of (7.1) is $y(t) = e^t$.

An interesting feature of (7.1) is that $t - t/u(t) \to 0$ as $t \to 0$, so that the methods of Bellman et al [BB65], El'sgol'ts [EL57], and Zverkina[ZV65], cannot be used.

In Table 7.1 we give the values of $y_p(1)$ for both Euler's method (2.1) and the improved Euler method (6.1), N_p being equal to 2^p in each case. The exact value of $y(1)$ is $e \approx 2.71828$.

For comparison, we also give the results of Feldstein. Since Feldstein presents three variants of Euler's method, we have used for our comparison the most accurate of the three. Feldstein published his results to four digits. After rounding our results for Euler's method to four figures we get exact agreement. The extrapolated values computed by Feldstein are obtained in the usual manner by computing the solution with stepsize h and 2h. This involves 3N derivative

$$\|(y-y^*_{i,p})(t_{i,p}+rh_p)\| = \|y(t_{i,p}) + rh_pF(y,t_{i,p}) + O(h_p^2)$$

$$- y_p(t_{i,p}) - rh_pF_p(y_p,t_{i,p})\| ,$$

$$\leq \|y-y_p\|^{[\alpha,t_{i,p}]} + h_p\|F(y,t_{i,p}) - F_p(y,t_{i,p})\|$$

$$+ h_p\|F_p(y,t_{i,p}) - F_p(y_p,t_{i,p})\| + O(h_p^2) ,$$

$$\leq \|y-y_p\|^{[\alpha,t_{i,p}]} + h_pC_p + h_pM\|y-y_p\|^{[\alpha,t_{i,p}]} + O(h_p^2) .$$

This bound is uniform in r , $0 < r \leq 1$. Since $\|y-y^*_{i,p}\|^{[\alpha,t_{i,p}]} =$

$\|y-y_p\|^{[\alpha,t_{i,p}]}$, we get

$$\|y-y^*_{i,p}\|^{[\alpha,t_{i,p}+h_p]} \leq (1+h_pM)\|y-y_p\|^{[\alpha,t_{i,p}]} + h_p\,C_p + O(h_p^2) . \quad (6.5)$$

By (6.4) and (6.5) we get

$$\|(y-y_p)(t_{i,p}+rh_p)\| \leq (1+\frac{h_p}{2}M)\|y-y_p\|^{[\alpha,t_{i,p}]} + h_p\,C_p$$

$$+ \frac{h_p}{2}M(1+h_pM)\|y-y_p\|^{[\alpha,t_{i,p}]} + \frac{h_p^2}{2}MC_p + O(h_p^3) ,$$

$$\leq (1+h_pM_0)\|y-y_p\|^{[\alpha,t_{i,p}]} + h_p(1+\frac{h_p}{2}M)C_p + O(h_p^3) ,$$

where $M_0 = M(1+\frac{h_0M}{2})$. By Lemma 3.2 we get

$$\|y-y_p\|^{[\alpha,b]} \leq e^{(b-a)M_0}\|g-g_p\|^{[\alpha,a]}$$

$$+ \frac{e^{(b-a)M_0}-1}{M_0}(1+\frac{h_p}{2}M)C_p + O(h_p^2) .$$

Since $C_p = O(h_p^2)$, the theorem follows.

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where

$$u(t) = (1 + 2t)^2 .$$

It is easily verified that the solution of (7.1) is $y(t) = e^t$.

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	FELDSTEIN		CONTINUOUS APPROXIMATION	
p	Euler	Extrapolated	Euler	Improved Euler
1	2.301		2.301	2.7496
2	2.475	2.649	2.475	2.7267
3	2.568	2.661	2.568	2.7192
4	2.639	2.711	2.639	2.71866
5	2.678	2.716	2.678	2.71841
6	2.698	2.717	2.698	2.71832
7	2.708	2.718	2.708	2.718292

Table 7.1

Values of $y_p(1)$ for (7.1).

Appendix A

Survey of Volterra functional differential equations and numerical methods for solving them.

1. Examples of Volterra functional differential equations

Retarded ordinary differential equations. These are differential equations of the form

$$y'(t) = f(y(t),\ y(u_1(t)),\ldots,y(u_m(t)),t),\ t \in [a,b]\ , \qquad (1.1)$$

where $f{:}E^{n(m+1)+1} \to E^n$, $\alpha \le u_i(t) \le t$, $i = 1,\ldots,m$, and $y(t)$ is specified on $[\alpha,a]$ for the Cauchy problem. Among the many expository works in this area are the papers of Myshkis [MY49], Hahn [HA54], Zverkin, Kemenskii, Norkin, and El'sgol'ts [ZK62], Myshkis and El'sgol'ts [MYE67], and the books of Pinney [PI59], Bellman and Cooke [BC63], and El'sgol'ts [EL66]. Theorems on existence and uniqueness for this problem have also been given by Franklin [FR54] and Sansone [SA55] .

The more general equation

$$y'(t) = f(y(t),\ y(u_1(y(t),t)),\ldots,\ y(u_m(y(t),t)),t),\ t \ge a \qquad (1.2)$$

where $u_i(y(t),t) \le t$, $i = 1,\ldots,m$, has been studied by Driver [DR60, DR63.1, DR63.2] in connection with a problem of electrodynamics. (See also [DR63.3], and Bullock [BU67].)

Volterra integro-differential equations. An example is given by the scalar equation

$$y'(t) = f(y(t),\ \int_a^t k(y(s),\ s,\ t)ds,t),\ \ t \in [a,b], \qquad (1.3)$$

where y(a) is specified for the Cauchy problem. A discussion of equations of this type and a bibliography of earlier works are to be found in [VO59]. A survey of integro-differential equations is given in [SAA67].

A classical example of Volterra integro-differential equations is given by the Volterra population equations [VO59, p. 207]

$$y'_1(t) = y_1(t)\,[a_1 - b_1\, y_2(t) - \int_{t-T_0}^{t} k_1(t-s)\, y_2(s)ds] ,$$

$$y'_2(t) = y_2(t)[-a_1 + b_2\, y_1(t) + \int_{t-T_0}^{t} k_2(t-s)y_1(s)ds] ,$$

which are found in the mathematical theory of two species living together. Applications of Volterra integro-differential equations to servomechanisms and a nuclear reactor problem are given by Beneš [BE61, BE63], Levin and Nohel [LEN60] and Nohel [NO64], respectively.

Volterra functional differential equations have found applications in fields such as ballistics, control theory, economics, oscillation theory, statistics, electrodynamics, elasticity, magnetic hysteresis, biomathematics, number theory, and neutron transport problems. Extensive bibliographies are found in [MY49, MY50, HA54, ZK62, BD54, BC63, EL66, CH60]. The book of Oğuztöreli [OU66] is devoted to the theory of control processes described by Volterra functional differential equations.

2. Existence and uniqueness theorems for Volterra functional differential equations.

Driver [DR62] considered the equation

$$y'(t) = F(y,t), \quad a < t < b, \tag{2.1}$$

where $F(y,t) \in E^n$ is defined for $t \in [a,b)$ and $y \in C([\alpha,t] \to D)$, and D is an open connected set in E^n. The case $\alpha = -\infty$ is included by replacing $[\alpha,t]$ by $(-\infty,t]$ and replacing $y \in C([\alpha,t] \to D)$ by $y \in C((-\infty,t] \to D_y)$, where $D_y \subset D$ is compact. He proved the following theorem, (the vector norm $\|\cdot\|$ denotes any norm in E^n).

Theorem 2.1 If the following conditions hold.

1. For fixed $x \in C([\alpha,t] \to D)$, $t \to F(x,t)$ is continuous on $[a,b)$.
2. For every $s \in [a,b)$ and every compact $G \subset D$ there exists a constant $L_{s,G}$ such that

$$\|F(x,t) - F(y,t)\| \le L_{s,G} \max_{\alpha \le t' \le t} \|x(t') - y(t')\|$$

whenever $t \in [a,s)$ and $x,y \in C([\alpha,t] \to G)$.

3. $g \in C([\alpha,a] \to D)$.

Then there exists a unique solution y on $[\alpha,\beta)$, $a < \beta \le b$, and if $\beta < b$ and β cannot be increased, then for any compact $G \subset D$ there is a sequence of numbers $a = t_0 < t_1 < t_2 < \dots \uparrow \beta$ such that

$$y(t_i) \in D - G \text{ for } i = 1,2,\dots,$$

i.e. either $y(t)$ comes arbitrarily close to the boundary of D or $y(t)$ is unbounded.

Driver remarks that it is not true, in general, that $y(t)$ approaches the boundary of D as $t \uparrow \beta$ as would be the case for ordinary differential equations. We shall discuss this point in some detail: Consider the ordinary differential equation

$$y'(t) = f(y(t), t), \qquad t \geq a, \tag{2.2}$$

where $y(t) \in E^n$, and $y(a)$ is given. The usual statement of the Picard-Lindelöf existence-uniqueness theorem for this equation is given below.

Theorem 2.2 Let $R = \{v \in E^n : \|v - y(a)\| \leq b\}$, $I = [a, a+h]$, $h > 0$. If f is continuous on $R \times I$ and satisfies the uniform Lipschitz condition

$$\|f(u,t) - f(v,t)\| \leq L \|u - v\|,$$

for $u, v \in R$, $t \in I$, then (2.2) has a unique solution on $[a, a+h_0]$, where $h_0 = \min\{h, b/M\}$, and M is a bound for $\|f(v,t)\|$ over $R \times I$.

Suppose that f satisfies the hypotheses of Theorem 2.2. For $y \in C(I \to R)$ and $t \in I$, define $F(y,t) = f(y(t), t)$. Then for fixed y, $t \to F(y,t)$ is continuous because it is a continuous function of a continuous function. Furthermore, for $x, y \in C(I \to R)$ we have

$$\|F(x,t) - F(y,t)\| = \|f(x(t),t) - f(y(t),t)\| \leq L \max_{a \leq t' \leq t} \|x(t') - y(t')\|,$$

where L is the Lipschitz constant in Theorem 2.2. The above holds for all $t \in I$. Therefore F satisfies the hypotheses of Theorem 2.1. However for ordinary differential equations we have a stronger theorem on the extension of solutions (which is independent of Theorem 2.2):

Theorem 2.3 ([HAR64, Corollary 3.2, p. 14]) Let D be an open connected set in E^{n+1}, and let f be continuous on the closure $\bar{D}$ of D. If (2.2) has a solution y (with $(y(t),t) \in \bar{D})$) on the interval J and J is a maximal interval, then one of the following three conditions must occur:

1. $J = [a, \infty)$.
2. $J = [a,b)$, $b < \infty$, and $\|y(t)\| \to \infty$ as $t \uparrow b$.
3. $J = [a,b]$ and $(y(b),b) \in \partial D$, the boundary of D.

That is, if the solution cannot be extended, it is either unbounded or it terminates on the boundary of D.

The above theorem is not true even for retarded ordinary differential equations, where a fourth case may occur: $J = [a,b)$ can be a maximal interval of existence of the solution y, and there exist sequences $t_1 < t_2 < \ldots \uparrow b$ and $t'_1 < t'_2 < \ldots \uparrow b$ such that $y(t_i)$ approaches the boundary of D as $i \to \infty$, but $y(t'_i)$ moves away from the boundary of D as $i \to \infty$. An actual example which illustrates this case was constructed by Myshkis [MY49, translation pp. 19-23]. In addition, Myshkis proved that for equation (1.2) this case can only occur if there exist indices j and k such that

(i) $\lim_{t \uparrow \beta} \sup f_j = +\infty$, $\lim_{t \uparrow \beta} \inf f_j = -\infty$,

(ii) $u_k(\beta) = \beta$.

3. <u>Numerical methods for retarded ordinary differential equations</u>.

Difference-differential equations with constant delay can be solved by standard numerical methods for ordinary differential equations. To illustrate this technique consider the scalar equation

$$y'(t) = f(y(t), y(t - \tau), t), \quad t \geq a, \tag{3.1}$$

where $\tau > 0$. We require that y satisfy the initial condition $y(t) = g(t)$, $t \varepsilon [a - \tau, a]$, where g is given. This problem can be reduced to a sequence of initial value problems in ordinary differential equations. For $t \varepsilon [a, a + \tau]$, $y(t - \tau) = g(t - \tau)$ is known. On this interval the solution of (3.1) can be computed by solving the differential equation

$$y_1'(t) = f_1(y_1(t), t), \quad t \varepsilon [a, a + \tau], \tag{3.2}$$

where $f_1(s,t) = f(s, g(t - \tau), t)$, and $y_1(a) = g(a)$. Having solved (3.2) one can compute y on $[a + \tau, a + 2\tau]$ by solving the differential equation

$$y_2'(t) = f_2(y_2(t), t), \quad t \varepsilon [a + \tau, a + 2\tau], \tag{3.3}$$

where $f_2(s,t) = f(s, y_1(t - \tau), t)$, and $y_2(a + \tau) = y_1(a + \tau)$. Repeating this reduction process we obtain the sequence of initial value problems

$$y_i'(t) = f_i(y_i(t), t), \quad t \varepsilon [a + (i - 1)\tau, a + i\tau], \tag{3.4}$$

where $f_i(s,t) = f(s, y_{i-1}(t - \tau), t)$, and $y_i(a + (i - 1)\tau) = y_{i-1}(a + (i-1)\tau)$, $i = 1, 2, \ldots$

If the constant delay τ in (3.1) is replaced by a variable delay $\tau(t)$, the difference-differential equation which results,

$$y'(t) = f(y(t), y(t - \tau(t)), t), \quad t \geq a, \tag{3.5}$$

where $\tau(t) > 0$, can be reduced to a sequence of ordinary differential equations similar to (3.4) if $\tau(t)$ satisfies suitable conditions.

The method of Bellman, Buell, and Kalaba. Bellman, Buell, and Kalaba [BB65] considered the case where τ is differentiable and monotone decreasing. They obtain a sequence similar to (3.4) where the i-th differential equation is solved on the interval $[t_i, t_{i+1}]$ with $t_0 = a$, $t_{i+1} - \tau(t_{i+1}) = t_i$, $i = 0,1,\ldots$ To be able to evaluate the right hand side of (3.5), a reduction method of this type would generally require that the values of the approximate solution (at the grid points) be saved so that the second argument of f can be evaluated by interpolating on these values. To avoid doing this, Bellman, Buell and Kalaba proposed the following. Define $L_0(t) = t$, $L_1(t) = t - \tau(t)$, and $L_{i+1}(t) = L_1(L_i(t))$ for $i = 1,2,\ldots$ The solution of (3.5) can be computed on $[t_0, t_1]$ by solving

$$y'(t) = f(y(t), g(t - \tau(t)), t), \quad t \in [t_0, t_1].$$

Suppose that (3.5) has been solved for $t \in [t_0, t_i]$. Define the real functions y_j on $[t_j, t_{j+1}]$ by

$$y_{i-j}(t) = y(L_j(t)), \ t_i \leq t \leq t_{i+1}, \ j = 0, \ldots, i. \tag{3.6}$$

Since τ is assumed to be differentiable, L_j is also differentiable. From (3.6) we have

$$y'_{i-j}(t) = y'(L_j(t))L'_j(t), \quad j = 0,\ldots,i\ .$$

From the above and (3.5) we obtain the system of $i + 1$ differential equations

$$y'_{i-j}(t) = L'_j(t)f(y_{i-j}(t), y_{i-j-1}(t), L_j(t)),\ j = 0,\ldots,i, \tag{3.7}$$

where $t \varepsilon [t_i, t_{i+1}]$. Now, $L_0(t_i) = t_i$, $L_1(t_i) = t_i - \tau(t_i) = t_{i-1}$, and, in general $L_j(t_i) = t_{i-j}$. Hence, $y_{i-j}(t_i) = y(t_{i-j})$. The solution of (3.5) on $[t_i, t_{i+1}]$ is given by y_i. The initial values $y(t_0),\ldots,y(t_i)$ are known since the solution y is known on $[t_0, t_i]$. This method gives the solution on $[t_0, t_{i+1}]$ at the expense of solving $1 + 2 + \ldots (i+1) = (i+1)(i+2)/2$ ordinary differential equations.

El'sgol'ts' method. Prior to the work of Bellman, Buell, and Kalaba, El'sgol'ts [EL55, Translation pp. 164-165, EL57, Translation pp. 283-284] proposed a different scheme which avoids both the use of interpolation and the solution of an increasingly large number of differential equations. His method is more direct: If one can find a sequence $a = t_0, t_1, t_2, \ldots$ such that either $t_i - \tau(t_i) < a$ or $t_i - \tau(t_i) = t_{j(i)} \geq a$ for some $j(i) < i$, (3.5) can be solved by Euler's method for ordinary differential equations:

$$y_0 = y(a)$$

$$y_{i+1} = y_i + \begin{cases} h_i f(y_i,\ g(t_i - \tau(t_i), t_i), & \text{if } t_i - \tau(t_i) \le a\ , \\ h_i f(y_i,\ y_{j(i)}, t_i)\ , & \text{if } t_i - \tau(t_i) > a\ , \end{cases}$$

where $h_i = t_{i+1} - t_i$, $i = 1,2,\ldots$

It is difficult to generalize the method of El'sgol'ts to obtain higher order formulas because of the variable stepsize. Further, higher order methods may not give good results when used to start the solution because, even if f and τ have arbitrary differentiability, y may not have the required number of derivatives until t is sufficiently large [EL57, translation pp. 282-283, EL55, translation pp. 159-161]. This is illustrated by the simple example

$$y'(t) = y(t-1), \qquad t \ge 0\ ,$$
$$y(t) = 1\ , \qquad t \le 0\ .$$

For this problem (3.4) reduces to

$$y'_i(t) = y_{i-1}(t-1),\ \ t \quad [i-1,\ i],\ \ i = 1,2,\ldots,$$

where $y_0(t) = 1$. The first three terms of the sequence are

$$y_1(t) = 1 + t\ ,$$
$$y_2(t) = (3 + t^2)/2\ ,$$
$$y_3(t) = [(t-1)^3 + 9t + 2]/6.$$

We find $y_1''(1) = 0$ and $y_2''(1) = 1$, so that $y''(1)$ does not exist. But $y_2''(2) = y_3''(2) = 1$, so that $y''(2)$ exists. In general, one finds that the solution of (3.1) has k continuous derivatives for all $t \ge a + (k-1)\tau$, provided that f has $k - 1$ continuous derivatives.

Zverkina's integration formulas. Zverkina [ZV62, ZV65] has developed numerical methods which take into account the jumps of the derivatives. Suppose that a real function g on $[a,b]$ has $m+1$ continuous derivatives in the intervals $[a,t_1], [t_1,t_2], \ldots, [t_k,b]$, where $a < t_1 < \ldots < t_k < b$. Let

$$c_{i,j} = \lim_{\varepsilon \uparrow 0} \left[g^{(j)}(t_i + \varepsilon) - g^{(j)}(t_i - \varepsilon) \right],$$

and define $I(t) = \{i : t_i < t\}$. Then g has the expansion

$$g(t) = \sum_{j=0}^{m} \frac{g^{(j)}(a)}{j!} (t-a)^j + \sum_{i \varepsilon I(t)} \sum_{j=1}^{m} \frac{c_{i,j}}{j!} (t - t_i)^j + R_m(t), \tag{3.8}$$

where $R_m(t)$ is the remainder, analogous to the remainder in Taylor's expansion. Finite difference formulas can be constructed based on (3.8) rather than on Taylor's expansion. This is done by Zverkina in [ZV65] where she generalizes the classical Adams' formulas to handle the scalar equation of the neutral type given by

$$y'(t) = f(y(t), y(t - \tau(t)), y'(t - \tau(t)), t), \quad t \varepsilon [a,b], \ y(t) = g(t), \ t \leq a .$$

The methods discussed so far are not applicable to the equation

$$y'(t) = y(t/2), \quad t \geq 0 . \tag{3.9}$$

These methods require that $t_1 - \tau(t_1) = t_0 = 0$; since $\tau(t) = t/2$ for equation (3.9), this implies that $t_1 = 0$ and the solution cannot be started. The methods discussed so far are in general not applicable to problems where $\tau(t) = 0$ for any t in the interval where the solution is desired.

Feldstein's method. Feldstein [FE64] considered the scalar equation

$$y'(t) = f(y(t), y(u(t)), t), \quad t \geq a , \tag{3.10}$$

where $a \leq u(t) \leq t$. This excludes the difference-differential equation (3.1) but does include problems such as (3.9) and $y'(t) = y(t^2)$, $t \in [0,1]$. Feldstein developed Euler's method for (3.10) and, under suitable conditions, developed an asymptotic expansion for the error, thereby allowing the use of Richardson's extrapolation to refine the approximate solution.

Feldstein defines Euler's method for (3.10) as follows. Let $h = (b - a)/p$ for some integer $p > 0$, and let $t_i = a + ih$, $i = 0, \ldots, p$. Define $q(i) = \mathrm{int}((u(t_i) - a)/h)$, $r(i) = (u(t_i) - a)/h - q(i)$, for $i = 0, \ldots, p$, where int denotes the greatest integer function. The sequence $\{y_i\}$, where y_i approximates $y(t_i)$, is defined by

$$y_0 = y(a), \quad x_0 = y(a),$$

$$x_i = y_{q(i)} + hr(i)f_{q(i)} ,$$

$$f_i = f(y_i, x_i, t_i) ,$$

$$y_{i+1} = y_i + hf_i ,$$

for $i = 0, \ldots, p-1$. The algorithm is obviously explicit for $q(i) < i$. If $q(i) = i$, then $u(t_i) - a = ih$ which gives $r(i) = 0$ and $x_i = y_i$, so that the algorithm is explicit for all i. This scheme can therefore handle cases such as $y'(t) = y(t/2)$, $t \geq 0$. Feldstein also explores various modifications of the above scheme.

4. Numerical methods for Volterra integro-differential equations.

The Runge-Kutta-Pouzet formulas. Pouzet [PO60] considers the scalar equati

$$y^{(p)}(t) = f(y(t),y'(t),\ldots,y^{(p-1)}(t),t) + \int_a^t g(y(t),y'(t),\ldots,y^{(p)}(t),t,s)$$

for $t \geq a$,

where $y(a), y'(a),\ldots,y^{(p-1)}(a)$ are given. This type of initial value prob is not included in (1.1) because $y^{(p)}$ appears on both sides of the equatio Pouzet developed Runge-Kutta formulas for (4.1). For the case $p = 1$, to a from the point a to the point $a + h$ he gives the formula

$$z_i = f(y_i, a + \theta_i h) + h \sum_{j=0}^{i-1} B_{i,j}\, g(y_j, z_j, a + \theta_i h, a + \theta_j h) ,$$

$$y_i = y(a) + h \sum_{j=0}^{i-1} A_{i,j} z_j ; \quad i = 1,2,\ldots q .$$

Here, $z_0 = f(y_0, a)$, $y_0 = y(a)$. The value y_q is the computed ap to $y(a + h)$.

The coefficients θ_i, $A_{i,j}$, $B_{i,j}$ satisfy, among others, the relati

$$\theta_0 < \theta_1 \leq \theta_2 \leq \ldots \leq \theta_q , \quad \theta_0 = 0 , \quad \theta_q = 1 ,$$

$$\sum_{j=0}^{i-1} A_{i,j} = \theta_i , \qquad \sum_{j=0}^{i-1} B_{i,j} = \theta_i .$$

The differential equation solving routine can now compute a numerical solution based on the information provided by the above FORTRAN functions. Details of the routine are given in Figures B.1 and B.2. The arguments A and B are, respectively, the left and right endpoints of the interval over which the approximate solution is to be computed. N is the number of subdivisions of this interval. P and Q are one-dimensional arrays of at least $N + 1$ elements, and INT denotes the greatest integer function. At the end of the computation the function Y can be used to evaluate the approximate solution <u>at any point</u> in the interval [A,B] .

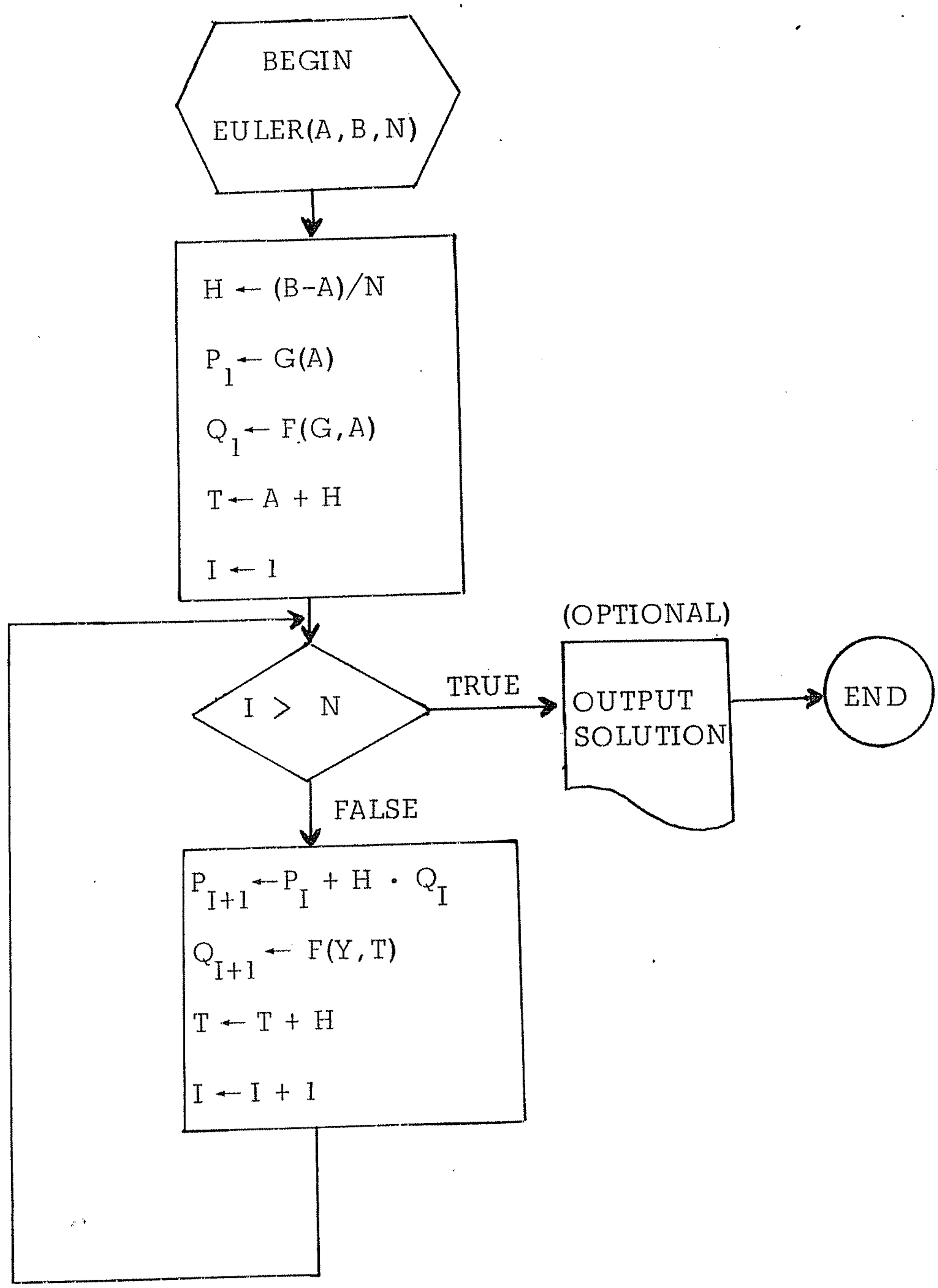

Figure B.1. Euler's Method: Main Routine.

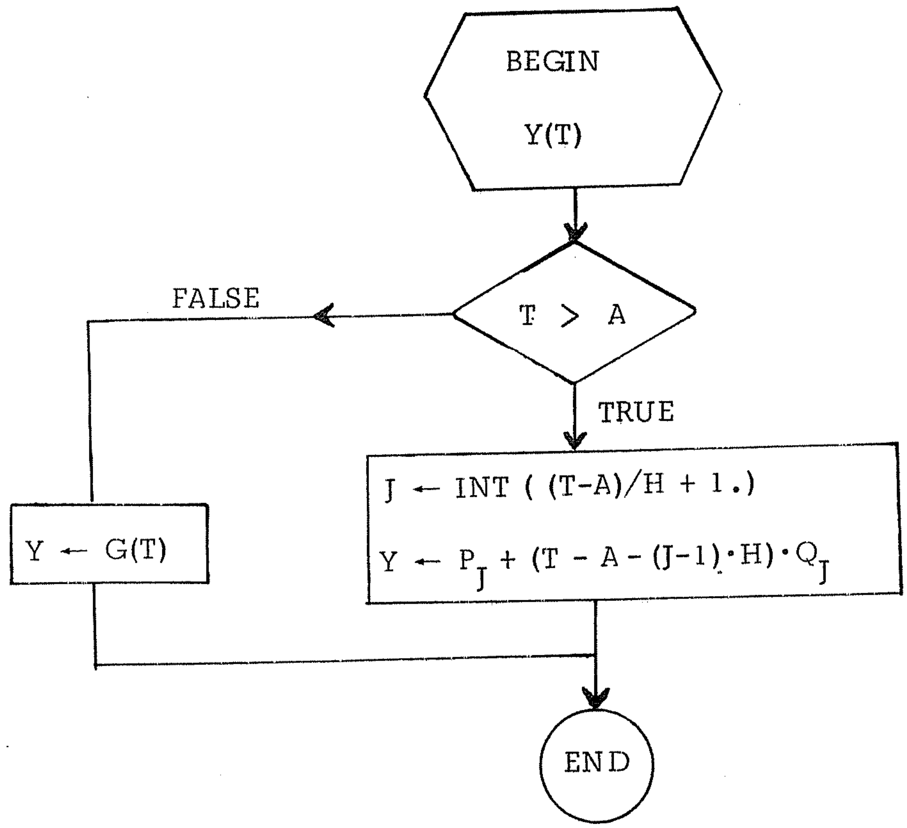

Figure B.2. Euler's Method: Function Subprogram.

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13. ABSTRACT

Let $\alpha \leq a < b$ be real numbers and let $C([t_1, t_2] \to E^n)$ denote the space of continuous functions on $[t_1, t_2]$ into E^n (n-dimensional Euclidean space). We consider the Cauchy problem for Volterra functional differential equations:

$$y'(t) = F(y,t), \ t \in [a,b]; \ y(t) = g(t), \ t \in [\alpha,a] .$$

Here, $F : C([\alpha,b] \to E^n) \times [a,b] \to E^n$ is a Volterra functional, that is, $F(y,t)$ depends on t and on $y(s)$ for $s \in [\alpha,t]$, but is independent of $y(s)$ for $s > t$; and $g \in C([\alpha,a] \to E^n)$ is a specified initial function. This problem includes as special cases the initial value problems for ordinary differential equations, retarded ordinary differential equations, and Volterra integro-differential equations.

We develop two methods for numerically solving the Cauchy problem for Volterra functional equations; these methods are generalizations of the methods of Euler and Heun for ordinary differential equations. It is proved that these methods converge and a numerical example is given.

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Retarded differential equations						
Numerical methods						
Euler's method						
Heun's method						
Improved Euler method.						

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