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TOPOLOGICAL PROBLEMS ARISING WHEN SOLVING BOUNDARY VALUE PROBLEMS FOR ELLIPTIC PARTIAL DIFFERENTIAL EQUATIONS BY THE METHOD OF FINITE DIFFERENCES

By

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CONTENTS

1. Introduction and terminology

Let $\mathcal{R}$ be a bounded domain (open connected set) in the xy-plane with boundary $\partial\mathcal{R}$. Let u be the solution of a boundary value problem over $\mathcal{R}$, for example the Dirichlet problem

$$u_{xx} + u_{yy} = 0, \; (x,y), \in \mathcal{R} , \tag{1.1}$$

$$u = f, \; (x,y) \in \partial\mathcal{R} . \tag{1.2}$$

When the method of finite differences is used to compute an approximation to u, U say, the steps which are carried out can be summarized in the following algorithm (Forsythe and Wasow [12, p. 175], Greenspan [13, p. 14]; see also Appendix A, section A.1).

Algorithm 1

Step 1

Choose a set of netpoints, h.

Step 2

Set up a system of algebraic equations connecting the values of U at the points of h.

Step 3

Solve the system of algebraic equations set up in Step 2, thereby obtaining U.

There are several possible choices for the net h in Algorithm 1, some of which are discussed in Appendix A, section A.3. However, in the body of the present paper we will only consider rectangular nets, which are defined as follows. Let G be a rectangular grid, that is a set of orthogonal gridlines and corresponding gridpoints. Let $\mathfrak{n} = \mathfrak{n}(G, \mathcal{R})$ be the set consisting of the gridpoints and the points of intersection of gridlines will $\partial\mathcal{R}$. Then, $\mathfrak{n}$ is the <u>rectangular net</u> corresponding to G and $\mathcal{R}$.

Techniques for implementing Algorithm 1 as a program on a digital computer are discussed by Forsythe and Wasow [12, p. 357], and some additional comments will be found in Appendix A. It is desirable for the program implementing Algorithm 1 to be as general as possible and to require the minimum of human assistance. Ideally, the only input data needed should be information (possibly in the form of subroutines) about the domain $\mathcal{R}$, the differential equation, and the boundary conditions, together with information about the accuracy desired and the form in which the answers are required. However, the "ideal program" has yet to be written, primarily because it is hard to write a program which is capable of handling the topological complications which can arise when $\mathcal{R}$ is a general domain. In all programs of which the author is aware, assumptions are made about $\mathfrak{n}(G, \mathcal{R})$.

It is the purpose of the present paper to analyse three assumptions about $\mathfrak{n}(G, \mathcal{R})$ which are frequently made. We also provide a variety of information concerning the implementation of Algorithm 1 which is not available in the literature.

The most common assumption about $\mathfrak{n}(G, \mathcal{R})$ that is made is that $\mathfrak{n}(G, \mathcal{R})$ is <u>gridlike</u>, that is that all the points of $\mathfrak{n}$ are gridpoints. If $\mathfrak{n}$ is gridlike, the programming is substantially simplified because $\mathfrak{n}$ can then be represented as a two-dimensional array in the computer. However, in section 2 we show that $\mathfrak{n}(G, \mathcal{R})$ is gridlike only for very restricted classes of domains $\mathcal{R}$ and grids G.

Another assumption about $\mathfrak{n}$ which is sometimes made is that the "interior netpoints" are "gridconnected" (these concepts are defined below); in section 3 we obtain conditions upon $\mathcal{R}$ and G which ensure that this is the case.

Finally, in section 4, we analyse the assumption (which is implicit in many implementations of Algorithm 1) that the number of "irregular" netpoints is much smaller than the number of "regular" netpoints.

In the remainder of the present section, we introduce terminology which will be required later.

For a given rectangular grid G we introduce coordinates $x'y'$ such that the $x'y'$ - coordinate axes are parallel to the gridlines; we call $x'y'$ the grid coordinates. With respect to the grid coordinates, the gridlines of G are of the form

$$\left.\begin{array}{l} x' = x'_i \,, \quad 1 \le i \le I, \\ y' = y'_j \,, \quad 1 \le j \le J, \end{array}\right\} \tag{1.3}$$

while the set of gridpoints is given by

$$\mathcal{G} = \{(x'_i, y'_j);\ 1 \le i \le I,\ 1 \le j \le J\}. \tag{1.4}$$

Here, $x'_i \le x'_{i+1}$ and $y'_j \le y'_{j+1}$ for all i and j. We set,

$$\|G\| = \max\{\max_i |x'_{i+1} - x'_i|,\ \max |y'_{j+1} - y'_j|\}. \tag{1.5}$$

If $\mathcal{R}$ lies in the rectangle,

$$\{(x', y');\ x'_1 \le x' \le x'_I,\ y'_1 \le y' \le y'_J\}, \tag{1.6}$$

we say that G covers $\mathcal{R}$. If the gridlines are equally spaced so that

$$x'_i = x'_o + ih, \quad y'_j = y'_o + jk, \tag{1.7}$$

where h and k are positive constants, we call G a regular rectangular grid, and if $h = k$ so that

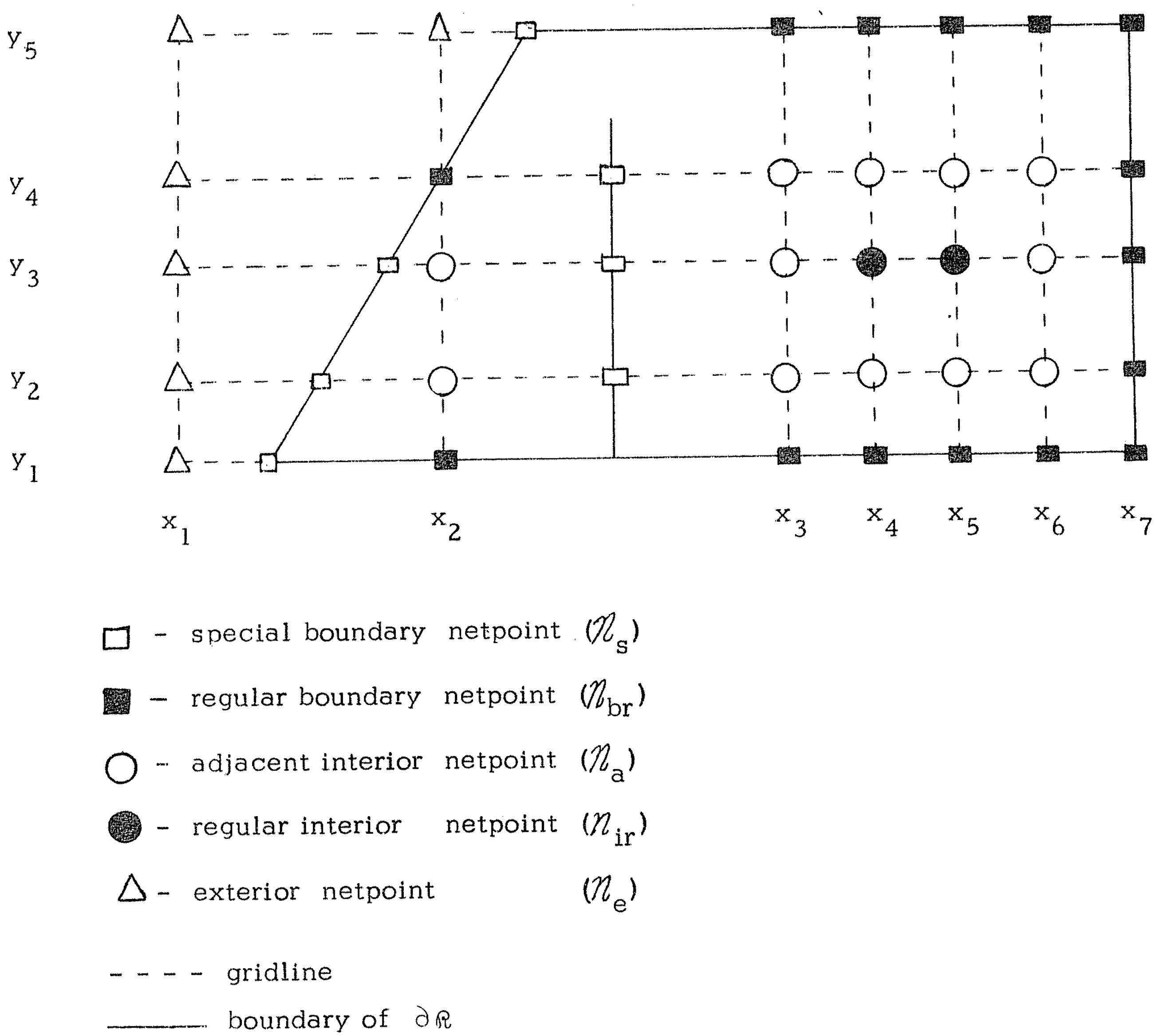
$$x'_i = x'_o + ih, \quad y'_j = y'_o + jh, \tag{1.8}$$

we call G a square grid with gridlength h.

If ℓ is a gridline, P is a <u>point of intersection</u> of ℓ with $\partial\mathcal{R}$ if $P \in \ell \cap \partial R$ and if in every neighbourhood of P there is a point $Q \in \ell$ which does not lie on $\partial\mathcal{R}$. Let $\mathcal{B} = \mathcal{B}(G, \mathcal{R})$ be the set of points of intersection of gridlines with $\partial\mathcal{R}$. We introduce the following sets (see Figure 1.1):

$$\left.\begin{array}{lll} \mathcal{N} & = \mathcal{B} \cup \mathcal{G} & \text{(netpoints)}, \\ \mathcal{N}_b & = \mathcal{B} & \text{(boundary netpoints)}, \\ \mathcal{N}_s & = \mathcal{B} - \mathcal{G} & \text{(special boundary netpoints)}, \\ \mathcal{N}_{br} & = \mathcal{B} \cap \mathcal{G} & \text{(regular boundary netpoints)}, \\ \mathcal{N}_e & = \mathcal{G} - \bar{\mathcal{R}} & \text{(exterior netpoints)}, \\ \mathcal{N}_i & = \mathcal{G} \cap \mathcal{R} & \text{(interior netpoints)}, \\ \mathcal{N}_{ir} & & \text{(regular interior netpoints)}, \\ \mathcal{N}_a & = \mathcal{N}_i - \mathcal{N}_{ir} & \text{(adjacent interior netpoints)}. \end{array}\right\} \qquad (1.9)$$

Here, an interior netpoint is <u>regular</u> if all its four neighbours are interior netpoints; otherwise it is <u>adjacent</u>. Note that $\mathcal{N}$ is gridlike iff $\mathcal{N}_s$ is empty.



<u>Figure 1.1</u>

Classification of netpoints.

An arc is a continuous map of the interval $[0, 1]$ into the xy - plane. (Ahlfors [1, p. 68]). A gridarc is an arc which consists of the union of segments of gridlines. We say that $\mathcal{R}_i$ is gridconnected if every two points of $\mathcal{R}_i$ can be joined by a gridarc which does not intersect $\partial\mathcal{R}$. For example, in Figure 1.1 $\mathcal{R}_i$ is not gridconnected.

If S is a finite set, $|S|$ denotes the number of elements of S. If $\mathcal{K}$ is a Lebesgue-measurable set in the xy-plane, then $\mu[\mathcal{K}]$ is the Lebesgue measure of $\mathcal{K}$. If $\overline{AB}$ is a line segment then $|AB|$ is the length of the segment. Finally, if $\mathcal{C}$ is a rectifiable arc (Ahlfors [1, p. 105]), then $\ell[\mathcal{C}]$ is the length of $\mathcal{C}$.

2. When is $\mathcal{R}$ gridlike?

As was pointed out in section 1, when Algorithm 1 is implemented it is often assumed that $\mathcal{R}$ is gridlike since then $\mathcal{R}$ can be represented in the computer as a two-dimensional array, which substantially simplifies the programming. In this section we show that it is only for very restricted classes of domains $\mathcal{R}$ and grids G that $\mathcal{R}(G,\mathcal{R})$ is gridlike.

First we show that if the orientation of the gridlines is fixed then $\mathcal{R}(G, \mathcal{R})$ may be gridlike only for trivial G, even if $\mathcal{R}$ is convex:

Theorem 2.1

Let $\mathcal{R}_1$ be the convex domain shown in Figure 2.1, where AB, CD, and DA, are straight lines, while BC is the curve

$$y = \sin[\pi(2-x)/2], \quad 1 \le x \le 2. \tag{2.1}$$

Let G be a grid with gridlines parallel to the xy-axes such that $\mathcal{h}(G, \mathcal{R}_1)$ is gridlike.

Then the gridlines of G which intersect $\bar{\mathcal{R}}_1$ are a subset of the lines

$$\left.\begin{array}{l} x = 0;\ x = 1;\ x = 2, \\ y = -1,\ y = 0;\ y = 1. \end{array}\right\} \qquad (2.2)$$

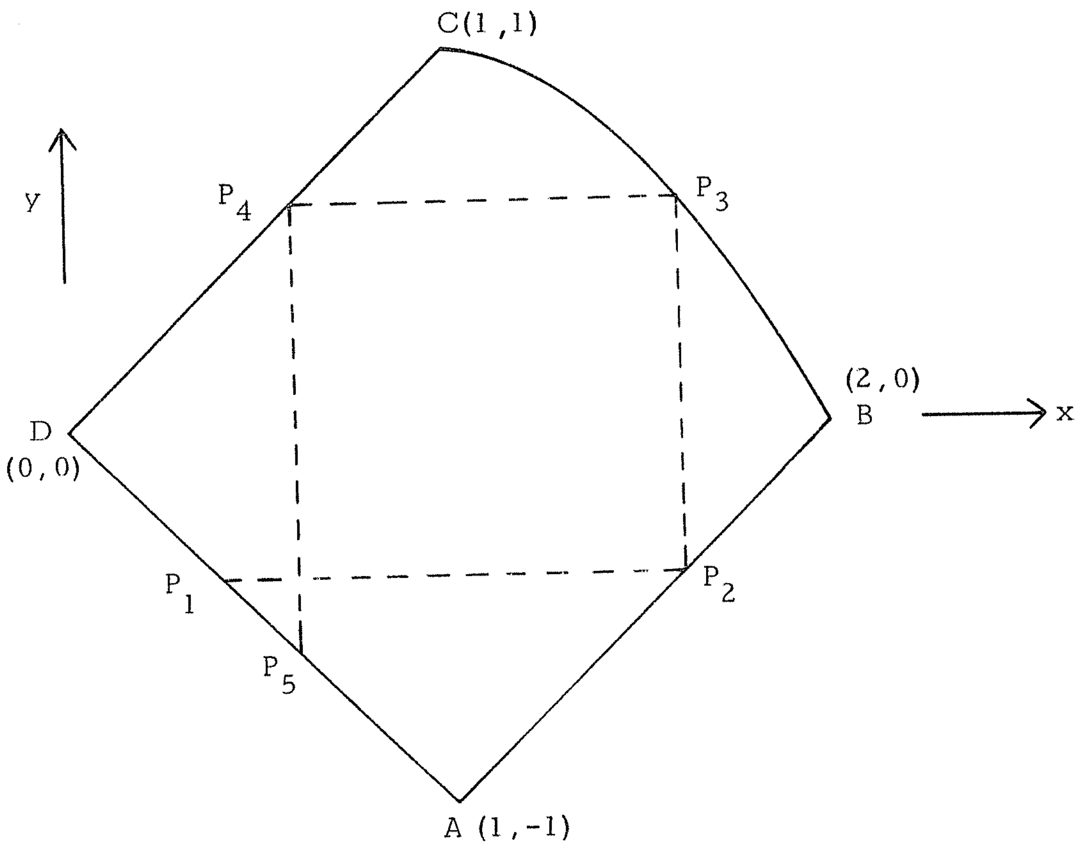


Figure 2.1

The domain $\mathcal{R}_1$.

Proof: If the theorem is not true, there is a grid G and a point $P = (\xi, \eta)$ such that $\mathcal{N}(G, \mathfrak{R}_1)$ is gridlike, P is a point of intersection of a gridline with $\partial\mathfrak{R}_1$, and $P \neq A, B, C, D$. Since every intersection of a gridline with $\partial\mathfrak{R}_1$, is a gridpoint, it is then easy to see that there is a gridpoint $P_1 = (t_1, -t_1) \in DA$ where $0 < t_1 < 1$, and t_1 is equal to $|\xi|$, or $|2 - \xi|$. Consulting Figure 2.1 we see that the points P_2, P_3, P_4, and P_5, must also be gridpoints.

But, $P_5 = (t_5, -t_5)$ where $t_5 = \sin(\pi t_1/2)$. Since $s < \sin(\pi s/2) < 1$ for $0 < s < 1$, it follows that $t_1 < t_5 < 1$ so that the process may be repeated. Consequently $\mathcal{G}$ must contain the infinite sequence of distinct points P_1, P_5, P_9, ..., P_{4k+1}, ..., which is impossible. The proof of the theorem is thus complete.

Of course, by using gridlines which make an angle of 45° with the xy - axes, we can clearly construct an infinite number of grids G for which $\mathcal{N}(G, \mathfrak{R}_1)$ is grid-like. However, Theorem 2.1 does show that even for very simple domains $\mathfrak{R}$ the gridlines cannot be oriented arbitrarily if $\mathcal{N}$ is required to be gridlike. The author finds it difficult to conceive of a program which, given the specifications of $\mathfrak{R}$, (perhaps by means of a set of subroutines), could determine the correct orientation. Therefore, one consequence of Theorem 2.1 is that, with any implementation of Algorithm 1 for which it is assumed that $\mathcal{N}$ is gridlike, the orientation of the gridlines must be part of the input data provided by the user.

Next we show that for more complicated domains $\mathfrak{R}$ there may be no non-trivial grids for which $\mathcal{N}$ is grid-like:

Theorem 2.2

Let $\mathcal{R}_2$ be the convex domain shown in Figure 2.2. Here, each segment of the boundary is an arc of the circle with radius 5 and center the "opposite" corner. For example, AB is an arc of the circle with radius 5 and center D.

Then, there is a positive constant σ such that, irrespective of the orientation of the gridlines, if $\|G\| \leq \sigma$ and G covers $\mathcal{R}_2$, then $\mathcal{N}(G, \mathcal{R}_2)$ is not gridlike.

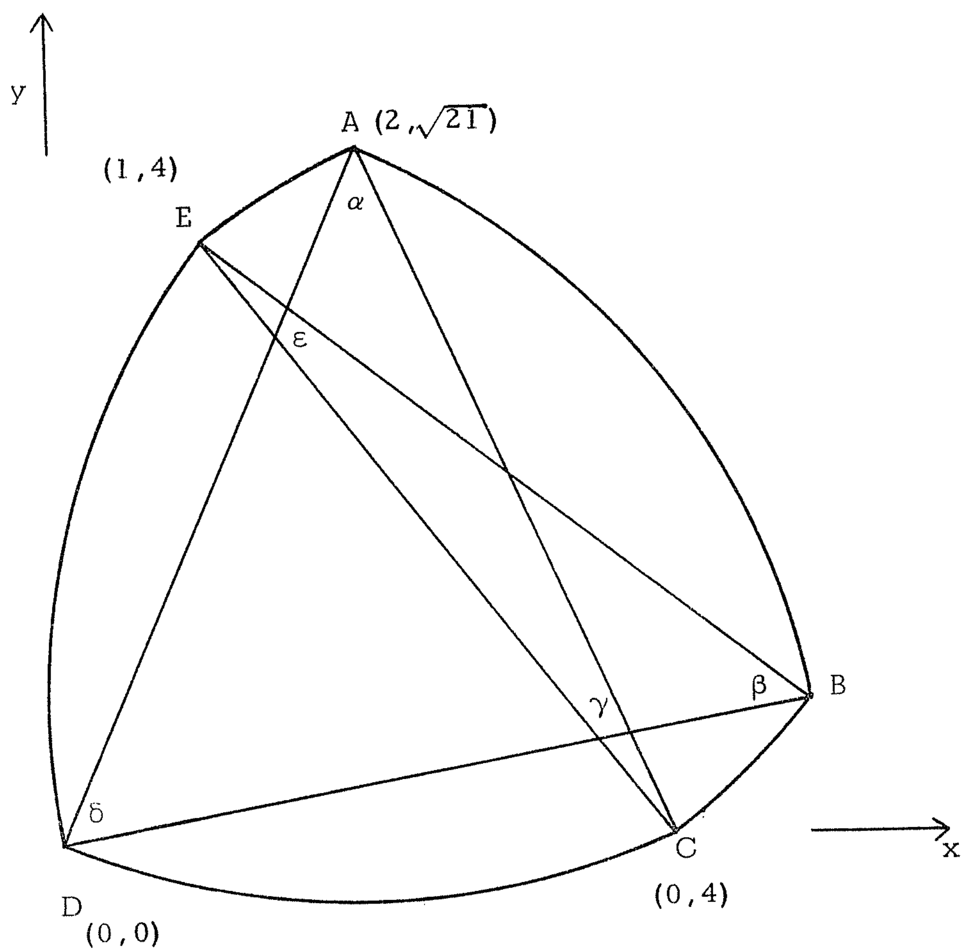


Figure 2.2

The domain $\mathcal{R}_2$.

Proof: We observe that $\mathfrak{R}_2$ is a Reuleaux polygon (Yaglom and Boltyanskii [35, p. 243], Eggleston [8, p. 128]).

It is easily seen that the coordinates of the vertices of $\partial\mathfrak{R}_2$ are:

$$A = (2, \sqrt{21}),$$

$$B = (\tfrac{1}{2} + 4\sqrt{83/68}\,, \quad 2 - \sqrt{83/68}\,),$$

$$C = (4,0), \quad D = (0,0), \quad E = (1,4).$$

From elementary trigonometry it then follows that

$$\left.\begin{aligned}
\alpha &= \text{arc cos } (.68) \approx 47.1563^\circ, \\
\beta &= \text{arc cos } (.66) \approx 48.7001^\circ, \\
\gamma &= \text{arc cos } (.24 + .16\sqrt{21}\,) \approx 13.2917^\circ, \\
\delta &= \text{arc cos } (.04 + .32\sqrt{83/68} + .08\sqrt{21} - .04\sqrt{\frac{21 \times 83}{68}}) \approx 56.1080^\circ, \\
\epsilon &= \text{arc cos } (.26 + .64\sqrt{83/68}\,) \approx 14.7437^\circ.
\end{aligned}\right\} \quad (2.3)$$

Inspection of Figure 2.2 shows that

$$\alpha + \beta + \gamma + \delta + \epsilon = 180^\circ. \quad (2.4)$$

Let τ be a positive angle such that $0 < \tau < \alpha/2$. Set $\varphi = \alpha/2 - \tau$, and let S and T be as in Figure 2.3. Let $d_{AD} = d_{AD}(\tau)$ be the largest number such that

$$\left.\begin{aligned}
&\text{(a)} \quad 2\, d_{AD} \leq |SC| \cos \varphi, \\
&\text{(b)} \quad 2\, d_{AD} \leq |AB| \sin (BAS), \\
&\text{(c)} \quad 2\, d_{AD} \leq (y_A - y_E) \sin (BAS), \\
&\text{(d)} \quad 2\, d_{AD} \leq |DT| \cos \varphi.
\end{aligned}\right\} \quad (2.5)$$

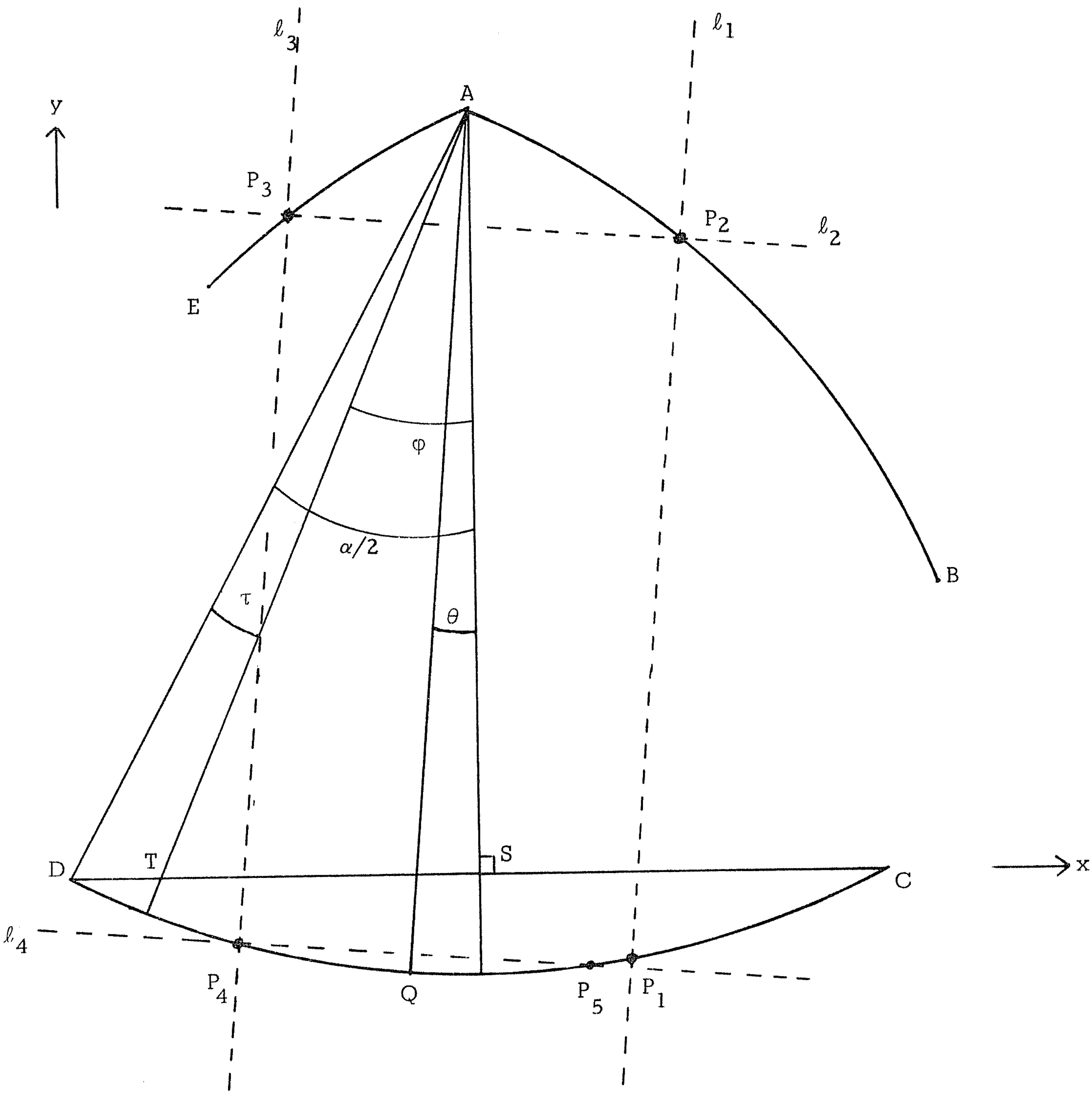


Figure 2.3

Intersections of gridlines with $\partial\mathbb{R}_2$.

Now, let G be a grid one of whose gridlines is parallel to AQ (see Figure 2.3) where

$$0 < \theta = QAS \leq \varphi = \alpha/2 - \tau . \tag{2.6}$$

Furthermore, let $\mathcal{N}(G, \mathcal{R}_2)$ be gridlike and

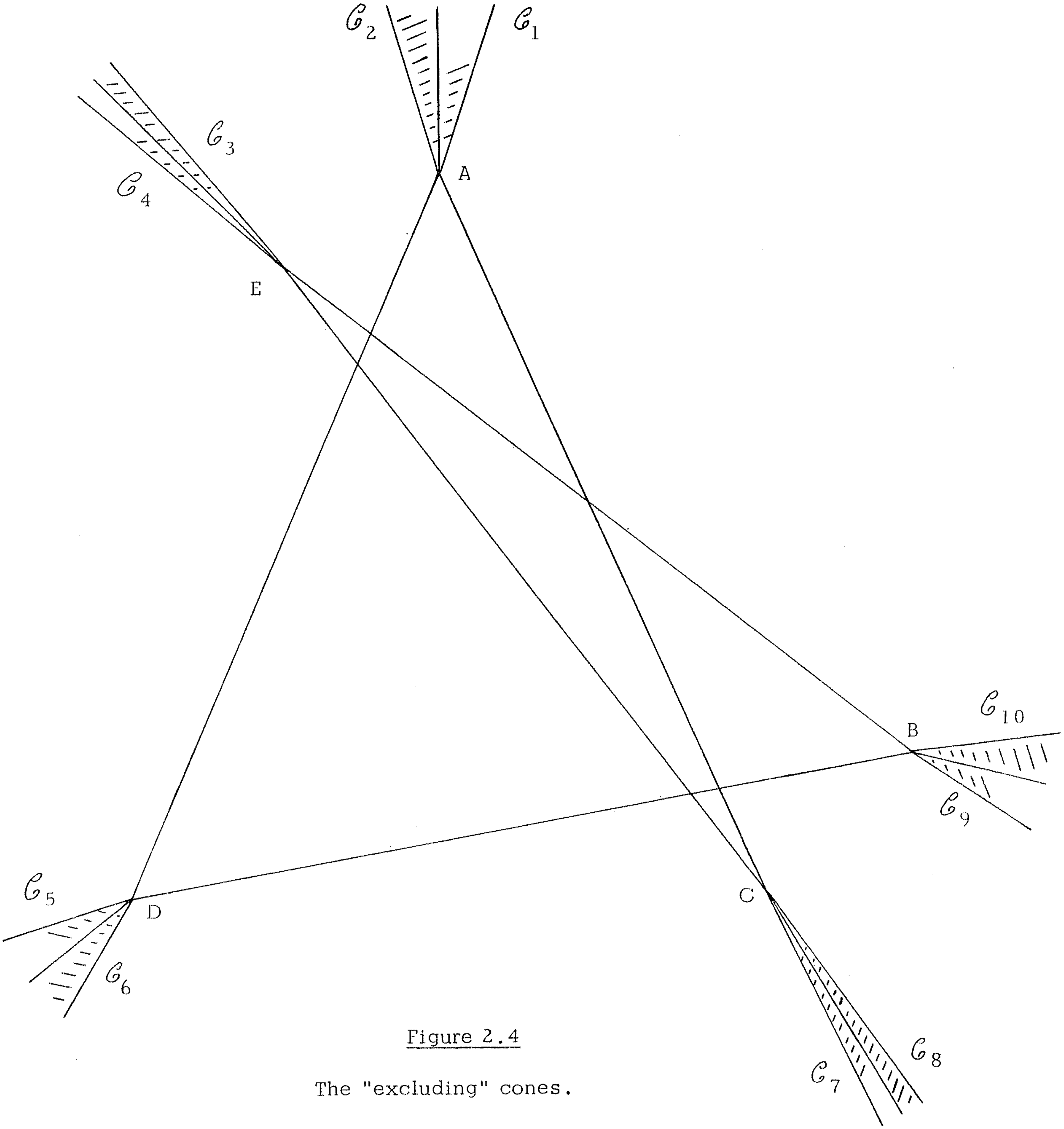
$$\| G \| \leq d_{AD}(\tau). \tag{2.7}$$

Consulting Figure 2.3, it follows from (2.5a) and (2.7) that the gridlines of G which are parallel to AQ intersect DC at points which are not more than $|SC|/2$ apart. Hence, there is a gridline ℓ_1 which is parallel to AQ, not further than d_{AD} from AQ, and intersects the arc QC at a point P_1. Similarly, using (2.5b) we see that ℓ_1 intersects the arc AB at a point P_2. Clearly,

$$y_A - y_{P_2} \leq d_{AD}/\sin(BAS).$$

Remembering that DP_2 is the normal to the arc AB at P_2, and noting (2.5c), we see that the gridline ℓ_2 intersects the arc AE at a point P_3. Using (2.5d) we see that the gridline ℓ_3 intersects the arc DQ at a point P_4. Finally, remembering that AQ is the normal to the arc DC at Q, we see that the gridline ℓ_4 intersects the arc QC at a point P_5. It is clear from the geometry that P_5 is an interior point of the arc QP_1, so that the process may be repeated. Hence there is an infinite number of gridpoints, P_1, P_5, P_9, ..., on the arc QC, which is impossible.

We have therefore proved that if $\mathcal{N}(G, \mathcal{R}_2)$ is gridlike and $\|G\| \leq d_{AD}(\tau)$, then none of the gridlines of G can be parallel to a line in the open cone $\mathcal{C}_1 = \mathcal{C}_1(\tau)$ (see Figure 2.4). Here, the bisector of DAC is one of the arms of $\mathcal{C}_1$, and the vertex angle of $\mathcal{C}_1$ is equal to $\alpha/2 - \tau$.



<u>Figure 2.4</u>

The "excluding" cones.

Repeating the above arguments, we see that if

$$0 < \tau < [\min\{\alpha, \beta, \gamma, \delta, \epsilon\}]/2,$$

then there is a positive constant $d(\tau)$ such that if

$$\|G\| \leq d(\tau), \tag{2.8}$$

and $\mathcal{H}(G, \mathcal{R}_2)$ is gridlike, then none of the gridlines of G can be parallel to a line in one of the ten cones of Figure 2.4.

Using (2.3) we find that to four decimal places the five sides and five bisectors of Figure 2.4 make the following angles with AD:

$$\left.\begin{array}{ll} AD:\ 0^\circ, & \\ bis(DAC): 23.5781^\circ, & AC: 47.1563^\circ, \\ bis(ACE): 71.5064^\circ, & CE: 95.8564^\circ, \\ bis(CEB): 102.5023^\circ, & EB: 109.1482^\circ, \\ bis(EBD): 137.2022^\circ, & BD: 165.2562^\circ, \\ bis(BDA): 172.6281^\circ. & \end{array}\right\} \tag{2.9}$$

Since all the angles of (2.9) are distinct, modulo 90°, it follows from (2.4) (after a little thought) that if τ is sufficiently small, $\tau \leq \tau_0$ say, then, given two orthogonal lines ℓ and n, either ℓ or n is parallel to a line in one of the ten cones of Figure 2.4. Hence, from (2.8), if

$$\|G\| \leq \sigma = d(\tau_0), \tag{2.10}$$

then $\mathcal{H}(G, \mathcal{R}_2)$ is not grid-like. The proof of the theorem is therefore complete.

When applying Algorithm 1 we must be able to use grids G for which $\|G\|$ is arbitrarily small. Therefore, it is a consequence of Theorem 2.2 that in any general implementation of Algorithm 1, provision must be made for special boundary points.

The domain $\Re_2$ of Theorem 2.2 has no axis of symmetry, and the following theorem shows that this lack of symmetry was essential.

Theorem 2.3

Let $\Re_3$ be a convex domain which has an axis of symmetry.

Then, for any $\epsilon > 0$ there is a grid G with $\|G\| \leq \epsilon$ for which $\mathcal{N}(G, \Re_3)$ is gridlike.

Proof: We may assume that the axis of symmetry is the y-axis so that $\Re_3$ is of the form shown in Figure 2.5. Here AA', B'C', D'D, and CB, are straight lines (which may be points). On the arcs AB and CD, $y = f(x)$ and $y = g(x)$, respectively, where $f(x)$ is a strictly decreasing function of x and $g(x)$ is a strictly increasing function of x.

Theorem 2.5

Let $\mathcal{R}_4$ be a strictly convex domain.

Then, there is a grid G_1 such that $\mathcal{G}_1 \cap \partial\mathcal{R}_4$ is not empty, $\mathcal{G}_1 \cap \mathcal{R}_4$ is empty, and $\mathcal{X}(G, \mathcal{R}_4)$ is gridlike.

Also, there is a grid G_2 such that $\mathcal{G}_2 \cap \mathcal{R}_4$ consists of exactly one point, and $\mathcal{X}(G_2, \mathcal{R}_4)$ is gridlike.

Proof: To prove the first part of Theorem 2.5, we note that, (Yaglom and Boltyanskii [35, p. 32]) it is possible to inscribe a square in $\bar{\mathcal{R}}_4$. That is, there is a square S such that $S \subset \bar{\mathcal{R}}_4$ and the four vertices of S lie on $\partial\mathcal{R}_4$. Extending the sides of S we obtain the desired grid G_1.

To prove the second part of the theorem, we note that to every angle α there corresponds a unique rectangle $TUVW(\alpha)$ which circumscribes $\mathcal{R}_4$ (see Figure 2.6). Now, the points $T(\alpha)$, $U(\alpha)$, $V(\alpha)$, and $W(\alpha)$, vary continuously with α (Eggleston [8, p. 32], Yaglom and Boltyanskii [35, p. 143]) so that the angles $WZT(\alpha)$ and $TZU(\alpha)$ are continuous functions of α. But,

$$WZT(\pi/2) - TZU(\pi/2) = -[WZT(o) - TZU(o)],$$

so that for some α, α_2 say, $WZT(\alpha_2) = TZU(\alpha_2)$. Since $WZT(\alpha) + TZU(\alpha) = \pi$,

$$WZT(\alpha_2) = TZU(\alpha_2) = \pi/2.$$

Extending the sides of $TUVW(\alpha_2)$, and the line segments $UW(\alpha_2)$ and $TV(\alpha_2)$, we obtain a grid G_2 satisfying the second part of the theorem.

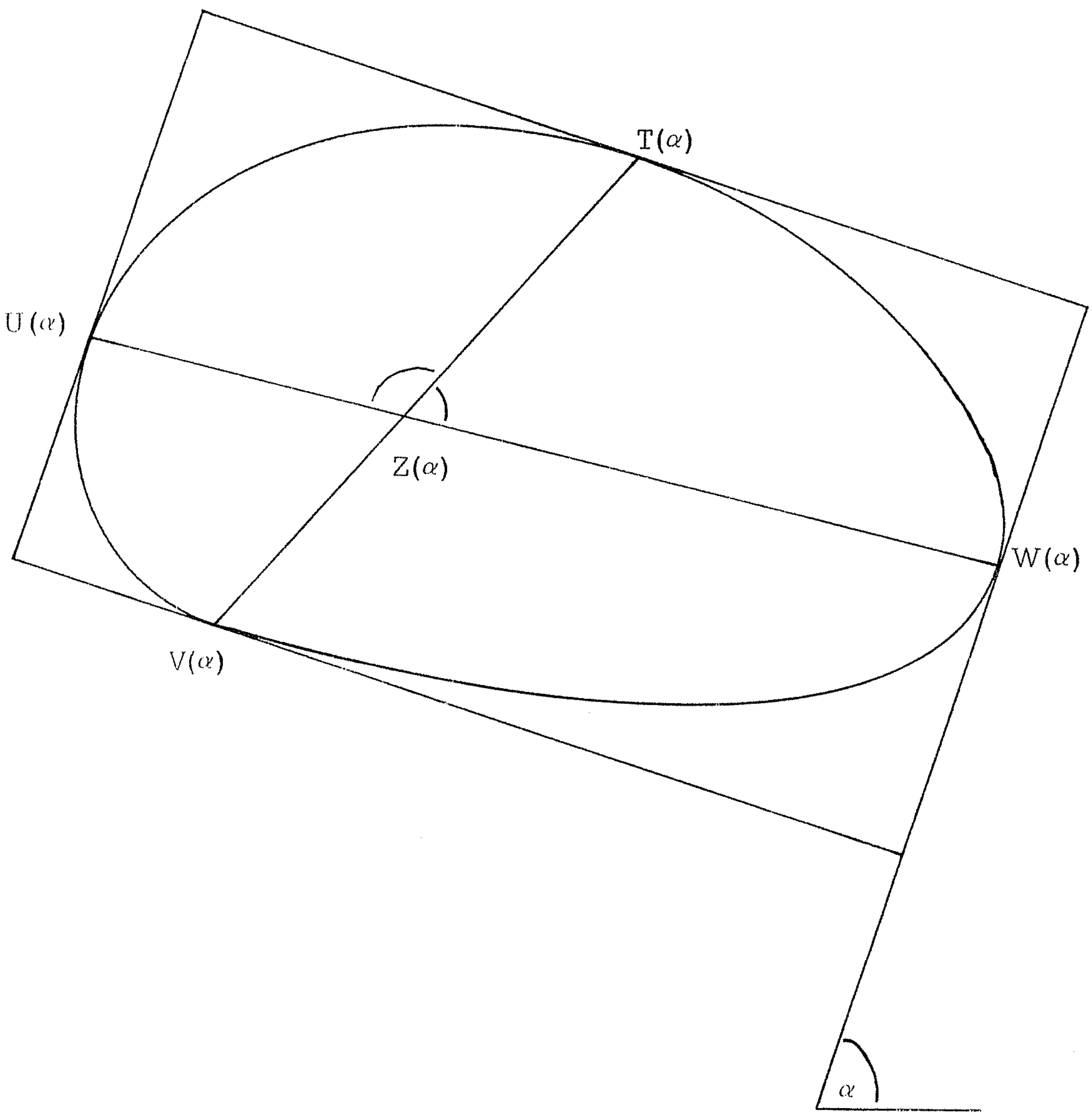


Figure 2.6

Rectangle circumscribed about $\mathcal{C}_4$.

Bearing Theorems 2.2 and 2.5 in mind we make the following conjecture:

Conjecture 2.6

There exists a strictly convex domain $\mathcal{R}$ with smooth boundary $\partial\mathcal{R}$ such that if G is any grid for which $\mathcal{G} \cap \mathcal{R}$ is not empty while $\mathcal{N}(G, \mathcal{R})$ is gridlike, then $\mathcal{G} \cap \mathcal{R}$ consists of at most one point.

3. When is $\mathcal{N}_i$ gridconnected?

When Algorithm 1 is implemented it is necessary to know which netpoints belong to $\mathcal{N}_i$. This information can be provided as input data by the user, but it is of course preferable for it to be generated by the program itself.

If $\mathcal{N}_i$ is gridconnected then there is a simple algorithm for determining the points of $\mathcal{N}_i$ (Forsythe and Wasow [12, p. 358]). Starting with a point $P_1 \in \mathcal{N}_i$, let P be a gridpoint which is adjacent to P_1. If the line segment PP_1 does not intersect $\partial\mathcal{R}$ then $P \in \mathcal{N}_i$. By repeating this procedure, we determine all the gridpoints which belong to $\mathcal{N}_i$ and can be connected to P_1 by a gridarc lying in $\mathcal{R}$. Since $\mathcal{N}_i$ is gridconnected, all the points of $\mathcal{N}_i$ have been found. The above algorithm is used, for example, in FREEBOUN (Cryer [7]).

In the present section we derive conditions which ensure that $\mathcal{N}_i$ is gridconnected.

Theorem 3.1

Let $\mathcal{R}$ satisfy

Condition A: There is a positive constant d such that if B_1 and B_2 are any two points on $\partial\mathcal{R}$ such that $|B_1 B_2| \leq d$, then there is an arc

$\alpha(B_1, B_2)$ (not necessarily belonging to $\partial\mathcal{R}$) which connects B_1 and B_2, has no points in common with $\mathcal{R}$, and lies in the closed disk with diameter $B_1 B_2$ (see Figure 3.1).

Let G be a grid which covers $\mathcal{R}$ and satisfies

$$\|G\| \leq d/\sqrt{2} \; . \tag{3.1}$$

Then, $\mathcal{N}_i(G,\mathcal{R})$ is gridconnected.

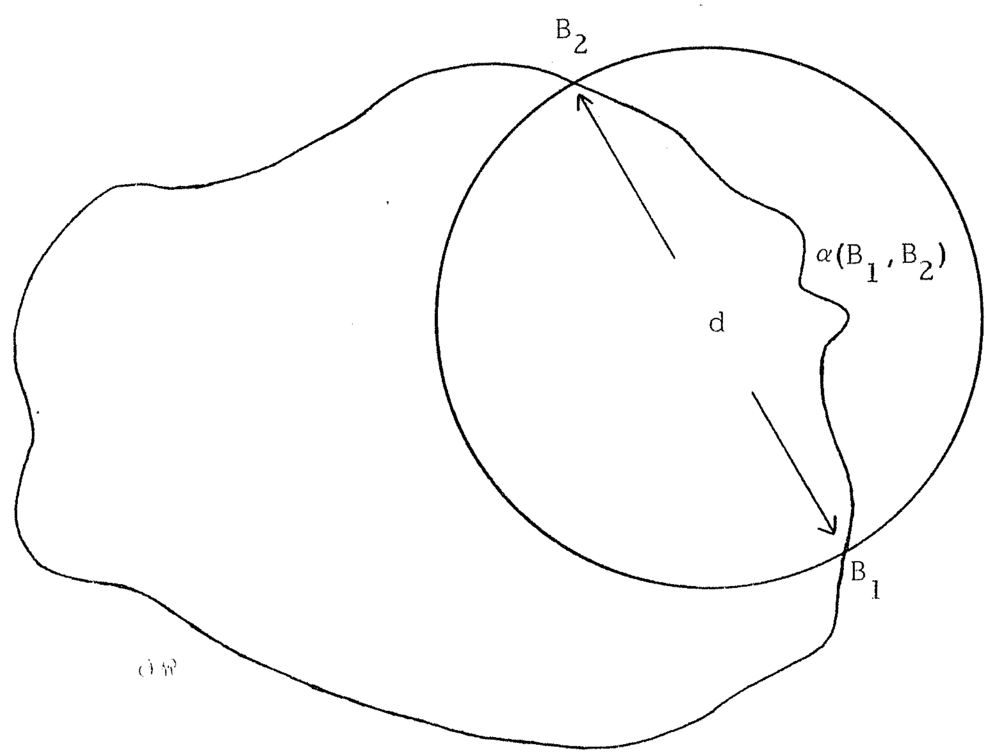


Figure 3.1

The arc $\alpha(B_1, B_2)$.

Proof: We remark that examples of domains $\mathcal{R}$ satisfying Condition A are given in Theorems 3.2 and 3.3.

Let $S, T \in \mathcal{N}_i(G, \mathcal{R})$. We must prove that S and T can be connected by a grid-arc, L_{ST} say, lying in $\mathcal{R}$.

Since $\mathcal{R}$ is open and connected, there is a polygon $P[S,T]$ which connects S and T and lies in $\mathcal{R}$ (Ahlfors [1, p. 56]); we may assume that the sides of $P[S,T]$ are parallel to the gridlines (Ahlfors [1, p. 57]).

If U and V lie on $P[S,T]$, let $P[U,V]$ be the subpolygon connecting U and V, $\mathcal{V}(P[U,V])$ the number of vertices on $P[U,V]$ (excluding U and V), and $\ell(P[U,V])$ the length of $P[U,V]$.

We construct L_{ST} by modifying $P[S,T]$. More precisely, we construct, by induction, a finite sequence of points, $S_1, S_2, \ldots, S_n$, such that

$$\left.\begin{array}{l} \text{(a)}\ S_1 = S, \quad S_n = T, \\ \text{(b)}\ S_i \in P[S,T], \text{ and } S_i \text{ lies on at least one gridline,} \\ \text{(c) There is a gridarc } L_i, \text{ lying in } \mathcal{R}, \text{ which connects } S_i \text{ and } S_{i+1}. \end{array}\right\} \tag{3.2}$$

Finally,

either

$$\left.\begin{array}{l} \text{(a)}\ \mathcal{V}(P[S_{i+1}, T]) \leq \mathcal{V}(P[S_i, T]) - 1, \\ \text{or} \\ \text{(b)}\ \ell(P[S_{i+1}, T]) \leq \ell(P[S_i, T]) - \|G\|. \end{array}\right\} \tag{3.3}$$

We set

$$\left.\begin{aligned} L[S, S_i] &= \bigcup_{k=1}^{i-1} L_i, \quad 1 \le i \le n-1, \\ L[S, T] &= \bigcup_{k=1}^{n-1} L_i . \end{aligned}\right\} \tag{3.4}$$

Let us assume that for some $i \geq 1$, points S_1 through S_i satisfying (3.2b), (3.2c), and (3.3) have been obtained. If $S_i = T$ we are finished. If $S_i \neq T$, we construct S_{i+1} as follows.

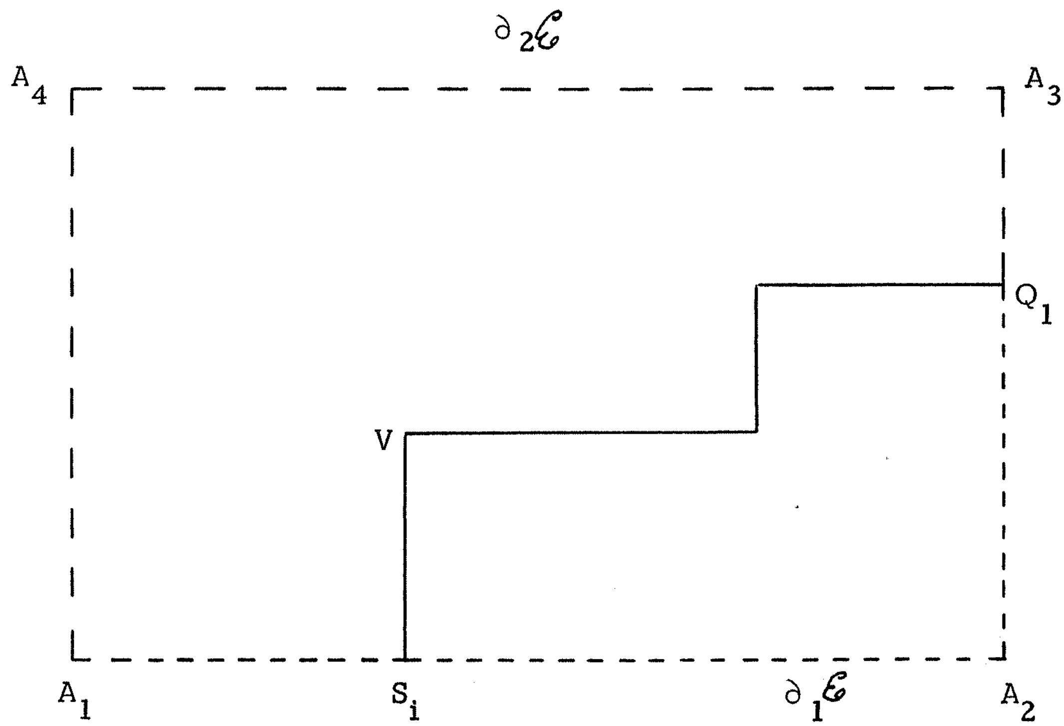
Let V be the first vertex beyond S_i on the polygon $P[S_i, T]$. If the line segment S_iV coincides with a gridline we set

$$S_{i+1} = V; \quad L_i = P[S_i, V], \tag{3.5}$$

so that (3.2b), (3.2c), and (3.3a), hold.

If S_iV does not coincide with a gridline, then S_i is not a gridpoint (since the sides of $P[S, T]$ are parallel to the gridlines). Let A_1 be the last gridpoint before S_i on the gridarc $L[S, S_i]$. Then S_iV is perpendicular to $A_1 S_i$, so that S_iV has points in common with exactly one of the four open gridrectangles adjoining A_1; call this gridrectangle $\mathcal{E}$. Without loss of generality, we may assume that $\mathcal{E}$ is the gridrectangle which has A_1 as its lower lefthand corner, and that S_i lies to the right of A_1 (see Figure 3.2). From the method of construction,

$$A_1S_i \subset L[S, S_i] \subset \mathfrak{R} . \tag{3.6}$$



<u>Figure 3.2</u>

The gridrectangle $\mathcal{E}$.

Let Q_1 be the first point of intersection after S_i of $P[S_i, T]$ with $\partial\mathcal{E}$ (see Figure 3.2). Then either $Q_1 \in S_iV$, in which case

$$\ell(P[Q_1, T]) \leq \ell(P[S_i, T]) - \|G\|, \tag{3.7}$$

or $Q_1 \notin S_iV$, in which case

$$\mathcal{V}(P[Q_1, T]) \leq \mathcal{V}(P[S_i, T]) - 1. \tag{3.8}$$

The points A_1 and Q_1 divide $\partial\mathcal{E}$ into the gridarcs $\partial_1\mathcal{E} = A_1Q_1$ and $\partial_2\mathcal{E} = Q_1A_1$ (see Figure 3.2). If $\partial_j\mathcal{E} \subset \mathcal{R}$ for $j = k$, we set

$$S_{i+1} = Q_1, \quad L_i = S_iA_1 \cup \partial_k\mathcal{E}. \tag{3.9}$$

Noting (3.6), (3.7), and (3.8), we see that (3.2b), (3.2c), and (3.3) hold.

There remains the possibility that $\partial\mathcal{R}$ intersects both $\partial_1\mathcal{E}$ and $\partial_2\mathcal{E}$. Then, since $\partial\mathcal{R}$ is compact, there are points $B_j \in \partial_j\mathcal{E} \cap \partial\mathcal{R}$, $j = 1, 2$, such that if $\partial_3\mathcal{E}$ is the gridarc B_2AB_1 then $\partial\mathcal{R} \cap \partial_3\mathcal{E} = \{B_1, B_2\}$ (in Figure 3.3, $\partial_3\mathcal{E}$ is the gridarc $B_2A_4A_1B_1$).

Let $\mathcal{E}_1$ be the open polytope with boundary $\partial_3\mathcal{E} \cup B_1B_2$; in Figure 3.3, $\mathcal{E}_1$ is the polytope $A_1B_1B_2A_4A_1$. Let D be the open disk with diameter B_1B_2. The points B_1 and B_2 divide $\partial\mathcal{D}$ into the arcs $\partial_1\mathcal{D} = B_1B_2$ and $\partial_2\mathcal{D} = B_2B_1$ (see Figure 3.3). Finally, let $\mathfrak{F}$ be the open set

$\mathcal{E}_1 \cup \mathcal{D}$ (in Figure 3.3 $\partial\mathfrak{F}$ is the solid line).

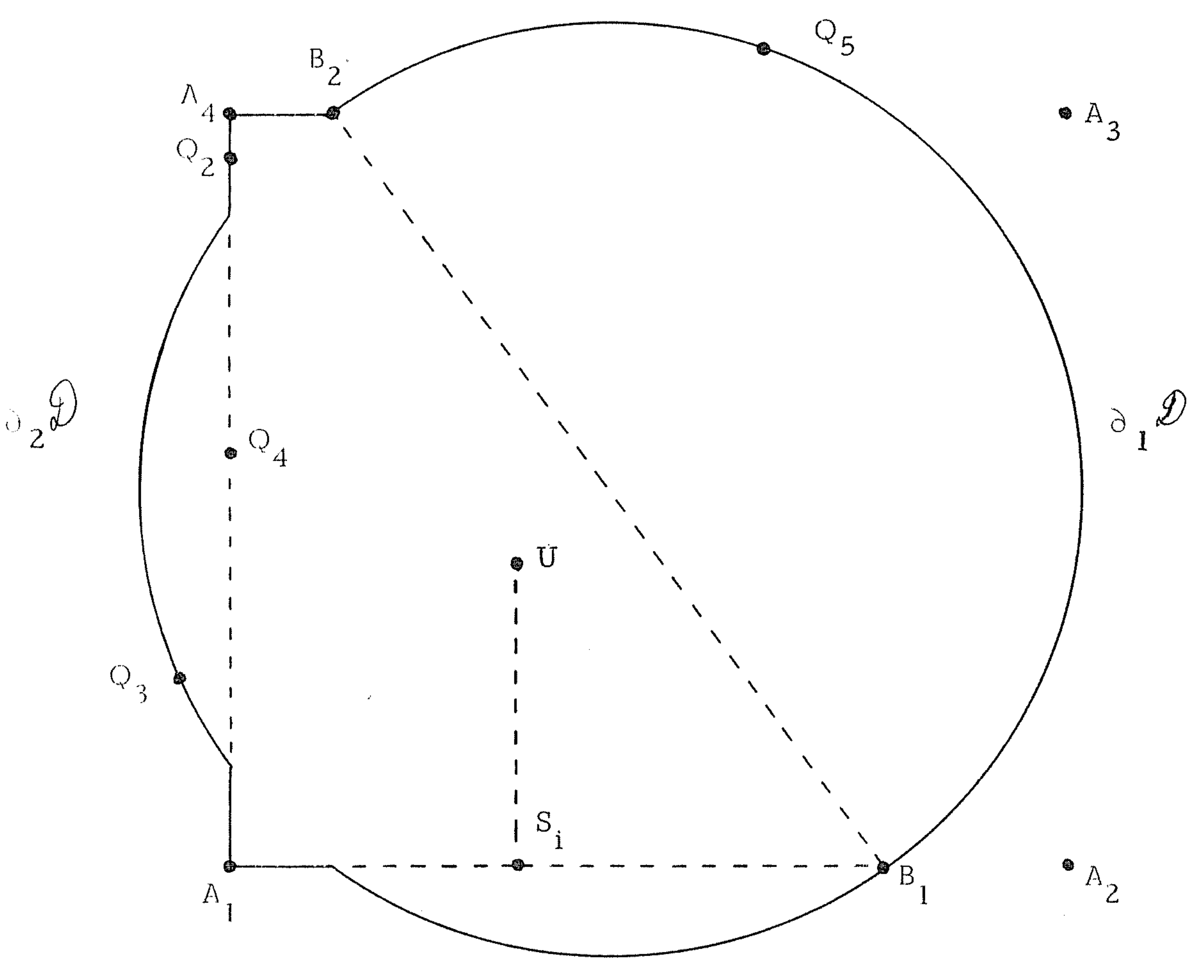


Figure 3.3

The domain $\mathcal{K}$.

The precise geometry of $\mathfrak{F}$ depends upon the location of the points B_1 and B_2. However, it is readily verified that for all $\mathfrak{F}$,

$$\left.\begin{array}{l}\text{(a) } \mathfrak{F} \text{ is a Jordan domain (that is, } \partial\mathfrak{F} \text{ is a simple closed curve),} \\ \text{(b) } B_1 \text{ and } B_2 \text{ separate } \partial\mathfrak{F} \text{ into two components, } \partial_1\mathfrak{F} \text{ and} \\ \quad \partial_2\mathfrak{F}\text{, where } \partial_1\mathfrak{F} = \partial_1\mathcal{D} \text{ and } \partial_2\mathfrak{F} \subset \partial_2\mathcal{D} \cup \partial_3\mathcal{E}.\end{array}\right\} \quad (3.10)$$

Next, we observe that $\mathcal{D}$ contains no gridpoints. This is obviously the case if B_1 and B_2 lie on the same side of $\mathcal{E}$. If B_1 and B_2 lie on opposite or adjacent sides of $\mathcal{E}$, it follows from the observation that if $W \in \mathcal{D}$ then the diameter $B_1 B_2$ subtends an angle greater than 90° at W.

Since $\mathfrak{F} \subset \mathcal{E} \cup \mathcal{D}$, we see that $\mathfrak{F}$ contains no gridpoints, and, in particular, $T \notin \mathfrak{F}$.

Now, $S_i V$ is perpendicular to $A_1 S_i$. Furthermore, by (3.6), $S_i \in \partial_3 \mathcal{E}$. It follows easily that there is a point $U \in P[S_i, Q_1] \cap \mathfrak{F}$. But $T \notin \mathfrak{F}$. Therefore, $P[S_i, T] \cap \partial\mathfrak{F}$ is not empty. Let Q be the first point of intersection after S_i of $P[S_i, T]$ with $\partial\mathfrak{F}$. Noting (3.10b), we see that either $Q \in \partial_1 \mathcal{D}$, or $Q \in \partial_2 \mathcal{D}$, or $Q \in \partial_3 \mathcal{E}$.

If $Q = Q_2 \in \partial_3 \mathcal{E}$ (see Figure 3.3) we set

$$S_{i+1} = Q_2, \quad L_i = S_i A_1 \cup A_1 Q_2. \tag{3.11}$$

Since $Q_2 \in \partial\mathcal{E}$, we have that $Q_2 \in P[Q_1, T]$. Hence, it follows from (3.7) and (3.8) that (3.2b), (3.2c), and (3.3) hold.

Next, suppose that $Q = Q_3 \in \partial_2 \mathcal{D}$ (see Figure 3.3). Then it is geometrically obvious, and can be proved rigorously with the aid of Lemma 3.6, that $P[U, Q_3]$ intersects $\partial_3\mathcal{E}$ at some point Q_4 (see Figure 3.3). We set

$$S_{i+1} = Q_4, \quad L_i = S_i A_1 \cup A_1 Q_4. \tag{3.12}$$

As for (3.11), conditions (3.2b), (3.2c), and (3.3) hold.

Finally, we show that $Q \notin \partial_1 \mathcal{D}$. For, suppose that $Q = Q_5 \in \partial_1 \mathcal{D}$ (see Figure 3.3). Then $A_1 S_i \cup P[S_i, Q_5]$ lies in $\bar{\mathfrak{F}}$ and connects the boundary points A_1 and Q_5. By Condition A of Theorem 3.1 and (3.1), there is an arc $\alpha(B_1, B_2)$ which connects B_1 and B_2 and is such that

$$\alpha(B_1, B_2) \subset \bar{\mathcal{D}} \subset \bar{\mathfrak{F}}.$$

It is geometrically obvious, and is proved in Lemma 3.6, that $A_1 S_i \cup P[S_i, Q_5]$ must intersect $\alpha(B_1, B_2)$, which is impossible. Therefore, as asserted above, $Q \notin \partial_1 \mathcal{D}$.

To sum up, if points S_1 through S_i satisfying (3.2b), (3.2c), and (3.3) have been obtained, and if $S_i \neq T$, we can construct S_{i+1} using either (3.5), (3.9), (3.11), or (3.12). Noting (3.3) we see that this process must terminate after a finite number of steps. The proof of the theorem is therefore complete.

The following theorems, which follow immediately from Theorem 3.1, show that $\mathcal{N}_i(G, \mathfrak{R})$ is gridconnected in many important cases.

Theorem 3.2

Let $\mathfrak{R}$ be a domain whose boundary $\partial\mathfrak{R}$ is the union of a finite number of disjoint simple closed curves and simple arcs. Let each curve and arc have a continuously turning tangent.

Then there is a positive constant ε such that $\mathcal{N}_i(G,\mathcal{R})$ is gridconnected if G covers $\mathcal{R}$ and

$$\|G\| \leq \epsilon . \tag{3.13}$$

Theorem 3.3

Let $\mathcal{R}$ be a domain whose boundary $\partial\mathcal{R}$ is the union of a finite number of simple polygonal arcs. Let each interior vertex angle θ of $\partial\mathcal{R}$ satisfy the inequality,

$$\theta \geq \pi/2. \tag{3.14}$$

Then there is a positive constant ϵ such that $\mathcal{N}_i(G,\mathcal{R})$ is gridconnected if G covers $\mathcal{R}$ and

$$\|G\| \leq \epsilon . \tag{3.15}$$

It is natural to ask whether Theorems 3.1, 3.2, and 3.3, can be strengthened. We now show that this cannot be done.

First, consider the grid G_5 and domain $\mathcal{R}_5$ of Figure 3.4. Here, $\partial\mathcal{R}_5$ consists of two semi-circles of radius $1-\eta$ joined by two line segments of length 4η. Clearly, $\mathcal{N}_i(G_5,\mathcal{R}_5)$ is not gridconnected. But $\mathcal{R}_5$ satisfies condition A of Theorem 3.1 with $d = 2 + 2\eta$ and

$$\|G_5\| = \sqrt{2} \leq (1+\eta)\, d/\sqrt{2} .$$

This example therefore shows that conditions (3.1) of Theorem 3.1 and (3.13) of Theorem 3.2 are necessary.

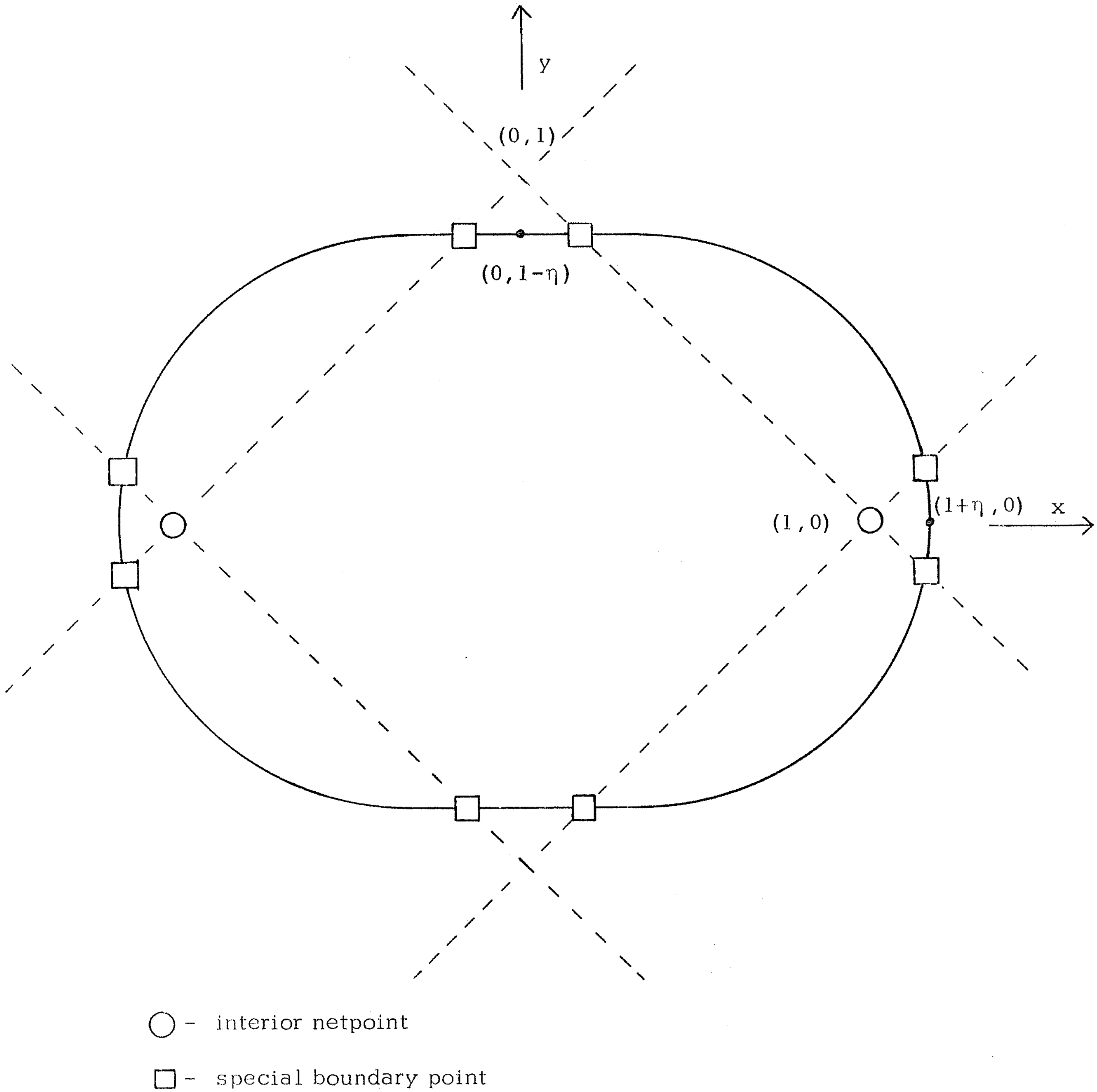


Figure 3.4

The grid G_5 and domain $\mathcal{R}_5$.

Next, consider the domain $\mathcal{R}_6$ of Figure 3.5. $\mathcal{R}_6$ is an isosceles triangle with vertex at the origin, and vertex angle φ. $\mathcal{R}_6$ is symmetric about the line $x = y$. The base of $\mathcal{R}_6$ is a segment of the line $x + y = 5$. We will consider nets $\mathcal{N}(G(a,p), \mathcal{R}_6)$ where $G(a,p)$ is the square grid with gridlines

$$\left.\begin{aligned} x &= x_i = -a + i\,h_p, \quad i \geq 0, \\ y &= y_j = -a + j\,h_p, \quad j \geq 0. \end{aligned}\right\} \tag{3.16}$$

Here, $a \geq 0$, $h_p = 2^{-p}$, and p is a non-negative integer.

The first interior netpoint on the line $y = x$ will be denoted by $Q_p = (z_p, z_p)$ (see Figure 3.5). Note that

$$z_p = h_p\,([a/h_p] + 1 - a/h_p), \tag{3.17}$$

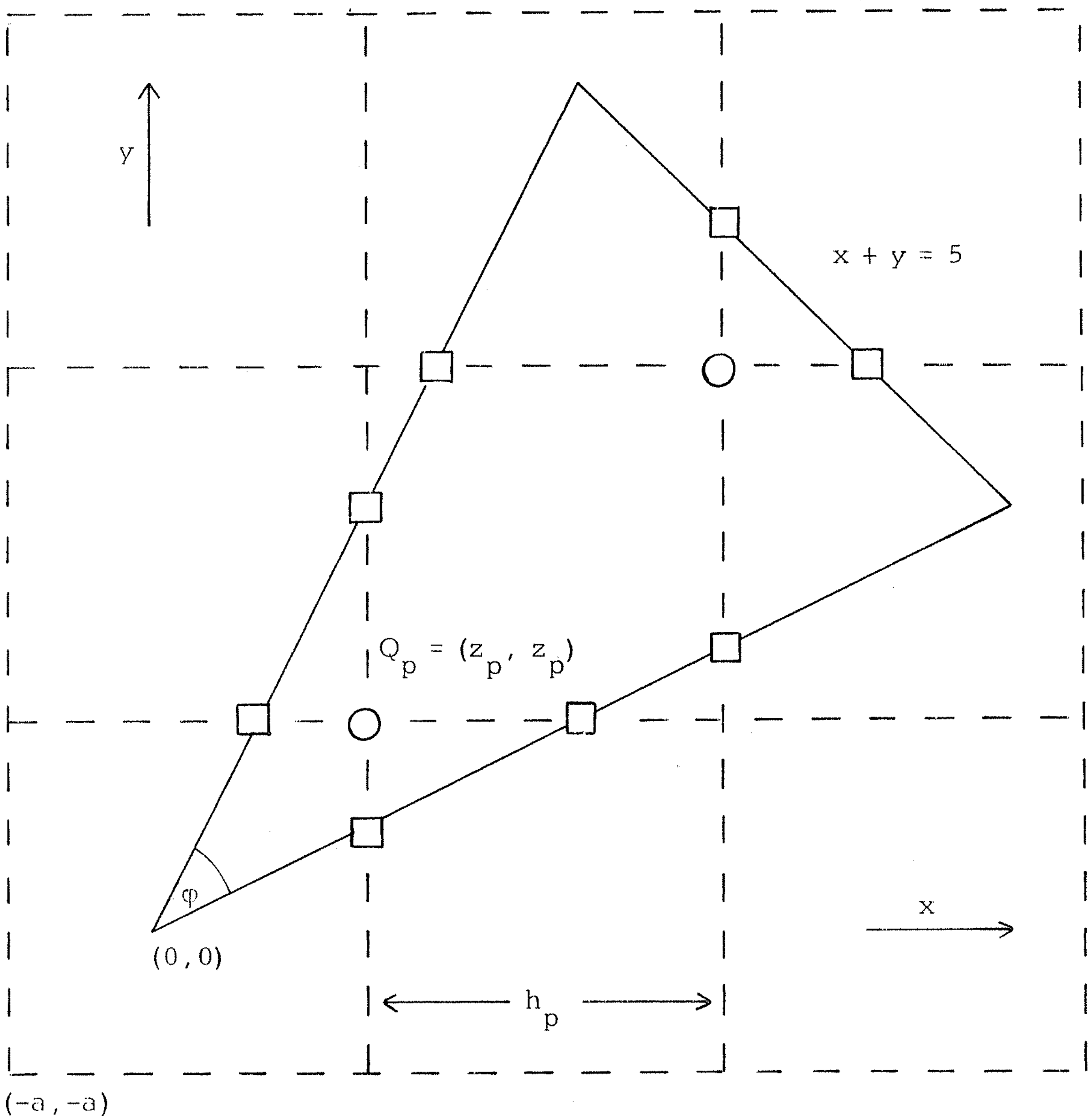
$$0 < z_p \leq h_p, \tag{3.18}$$

where $[a/h_p]$ denotes the integer part of a/h_p.

Since $(2,2) \in \mathcal{R}_6$, $\mathcal{N}_i(G(a,p), \mathcal{R}_6)$ always contains at least two points. Then (see Figure 3.5), $\mathcal{N}_i(G(a,p), \mathcal{R}_6)$ is not gridconnected if

$$z_p/(z_p + h_p) \leq \tan\{(\pi/2 - \varphi)/2\}. \tag{3.19}$$

Using (3.19) we obtain two theorems, Theorems 3.4 and 3.5. Theorem 3.4 shows that condition (3.14) of Theorem 3.3 cannot be relaxed. Theorem 3.5 shows that if $\partial\mathcal{R}$ has corners then $\mathcal{N}_i(G, \mathcal{R})$ may not be gridconnected even if G is chosen so that all the vertices of $\partial\mathcal{R}$ are gridpoints.



○ - interior netpoint

□ - special boundary point

Figure 3.5

The domain $\mathcal{R}_6$.

Theorem 3.4

Let $\varphi < \pi/2$, so that for some integer k, $k \geq 2$,

$$\tan\{(\pi/2 - \varphi)/2\} > 1/(2^k - 1).$$

Set

$$a = 1 - \sum_{r=1}^{\infty} 2^{-rk},$$

$$p_m = mk, \quad m = 1, 2, \ldots .$$

Then, for all m, $\mathcal{N}_i(G(a, p_m), \Re_6)$ is not grid-connected.

Proof: It is readily verified using (3.17) that

$$z_{p_m}/h_{p_m} = 1/(2^k - 1),$$

which implies (3.19).

Theorem 3.5

Let

$$\varphi \leq \varphi_o = \pi/2 - 2\,[\text{arc tan } 1/2],$$

so that, approximately,

$$\varphi \leq 36.8698^o.$$

Then, for all a and p, $\mathcal{N}_i(G(a,p), \Re_6)$ is not grid-connected.

Proof: Using (3.18) we see that

$$\begin{aligned} z_p/(z_p + h_p) &\leq 1/2, \\ &= \tan\{(\pi/2 - \varphi_o)/2\}, \\ &\leq \tan\{(\pi/2 - \varphi)/2\}. \end{aligned}$$

Finally, we prove the following lemma which was used in the proof of Theorem 3.1.

Lemma 3.6

Let $\mathfrak{F}_1$ be a Jordan domain. Let P_1 and P_2 be two points on $\partial\mathfrak{F}_1$ which are connected by a path p in $\bar{\mathfrak{F}}_1$ (see Figure 3.6). The boundary $\partial\mathfrak{F}_1$ is divided into two open arcs, $\partial_1\mathfrak{F}_1$ and $\partial_2\mathfrak{F}_1$ say, by P_1 and P_2. Let $Q_1 \in \partial_1\mathfrak{F}_1$ and $Q_2 \in \partial_2\mathfrak{F}_1$ be joined by a path q in $\bar{\mathfrak{F}}_1$.

Then p and q intersect.

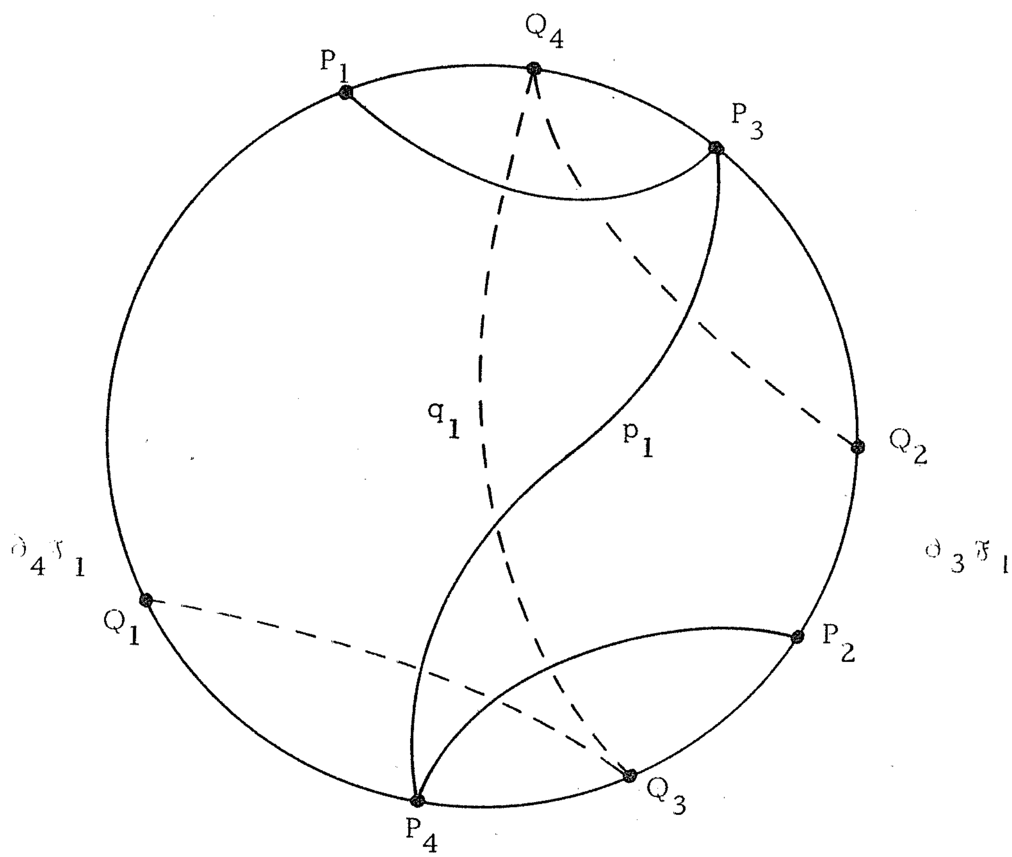


Figure 3.6

The domain $\mathfrak{F}_1$.

Proof: This lemma is geometrically obvious and is undoubtedly well known. However, we have been unable to locate a proof in the literature.

First we note that we may assume that P_1 and P_2 do not lie on q and that Q_1 and Q_2 do not lie on p, since otherwise the lemma is trivial.

Next, let Q_3 be the last point of intersection of q with $\overline{\partial_1 \mathfrak{F}_1}$ and Q_4 the first point of intersection of q with $\overline{\partial_2 \mathfrak{F}_1}$ (see Figure 3.6). Q_3 and Q_4 exist since q, $\overline{\partial_1 \mathfrak{F}_1}$ and $\overline{\partial_2 \mathfrak{F}_1}$ are compact. Moreover, neither Q_3 nor Q_4 is equal to P_1 or P_2, so that $Q_3 \in \partial_1 \mathfrak{F}_1$ and $Q_4 \in \partial_2 \mathfrak{F}_1$. Let q_1 be the segment of q which connects Q_3 and Q_4. Clearly, q_1 is a crosscut, that is, a simple arc which, except for its endpoints, lies in $\mathfrak{F}_1$ (Newman [23, p. 118]).

Let Q_3 and Q_4 divide $\partial \mathfrak{F}_1$ into the open arcs $\partial_3 \mathfrak{F}_1$ and $\partial_4 \mathfrak{F}_1$ where $P_1 \in \partial_3 \mathfrak{F}_1$ and $P_2 \in \partial_4 \mathfrak{F}_1$. Let P_3 be the last point of intersection of p with $\overline{\partial_3 \mathfrak{F}_1}$ and P_4 the first point of intersection of p with $\overline{\partial_4 \mathfrak{F}_1}$. If either P_3 or P_4 is equal to either Q_3 or Q_4, then p and q intersect; if not, $P_3 \in \partial_3 \mathfrak{F}_1$ and $P_4 \in \partial_4 \mathfrak{F}_1$ and P_3 and P_4 are connected by a crosscut $p_1 \subset p$. (see Figure 3.6).

The crosscut q_1 divides $\mathfrak{F}_1$ into two disjoint simply connected domains $\mathfrak{F}_3$ and $\mathfrak{F}_4$ with boundaries $q_1 \cup \partial_3 \mathfrak{F}_1$ and $q_1 \cup \partial_4 \mathfrak{F}_1$, respectively (Newman [23, p. 119 and p. 145]).

Now assume that p_1 and q_1 do not intersect. Let $p_1' = p_1 - \{P_3, P_4\}$. Then $p_1' \subset \mathfrak{F}_3 \cup \mathfrak{F}_4$. Since p_1' is a connected set, $p_1' \subset \mathfrak{F}_j$ for $j = 3$ or

$j = 4$, so that $p_1 \in \overline{\mathfrak{F}_j}$ for $j = 3$ or $j = 4$. But this is impossible since $P_3 \notin \overline{\mathfrak{F}_4}$ and $P_4 \notin \overline{\mathfrak{F}_3}$. Therefore p_1 and q_1, and hence p and q, intersect.

4. <u>The number of netpoints</u>

When Algorithm 1 is implemented in a straightforward manner, the amount of storage required may exceed the capacity of the high speed store. It is possible to reduce the amount of storage required by allocating different amounts of storage to different types of netpoints (see Appendix A, section A.2). When doing so, it is necessary to have estimates for the number of netpoints of each type, and such estimates are obtained in the present section.

Throughout the present section it will be assumed that

(a) $\mathcal{N} = \mathcal{N}(G, \mathfrak{R})$ where G is a square grid with gridlength h which covers $\mathfrak{R}$.

(b) $$\partial \mathfrak{R} = \bigcup_{k=1}^{\alpha} \partial_k \mathfrak{R}, \tag{4.1}$$

where α is an integer and each $\partial_k \mathfrak{R}$ is a rectifiable arc (Ahlfors [1, p. 104]). Furthermore, any two distinct arcs $\partial_k \mathfrak{R}$ and $\partial_\ell \mathfrak{R}$ have at most two points in common.

In addition, we will sometimes assume that

(c) The gridlines of G are of the form,

$$\begin{aligned} x &= x_i = x_o + ih, \quad 1 \le i \le I, \\ y &= y_j = y_o + jh, \quad 1 \le j \le J, \end{aligned} \tag{4.2}$$

and

$$\partial \mathcal{R} = \bigcup_{k=1}^{\beta} \partial_{xk} \mathcal{R} = \bigcup_{k=1}^{\gamma} \partial_{yk} \mathcal{R} , \tag{4.3}$$

where β and γ are integers. Each $\partial_{xk}\mathcal{R}$ is an arc on which y is either a constant or a strictly monotone function of x. Each $\partial_{yk}\mathcal{R}$ is an arc on which x is either a constant or a strictly monotone function of y .

If $\mathcal{K}$ is any set in the xy-plane, we denote by $\mathcal{K}(\epsilon)$ the set of points not further than ϵ from $\mathcal{K}$. We remind the reader that $|\mathcal{N}|$ denotes the number of points in $\mathcal{N}$, $\mu[\mathcal{R}]$ denotes the Lebesgue measure of $\mathcal{R}$, and $\ell[\partial\mathcal{R}]$ denotes the length of $\partial\mathcal{R}$.

The main result of this section is:

Theorem 4.1

$$|\mathcal{N}_i| = \mu[\mathcal{R}]h^{-2} + O(h^{-1}),$$

$$|\mathcal{N}_{br}|, \ |\mathcal{N}_a| = O(h^{-1}).$$

If condition (c) holds then

$$|\mathcal{N}_s| = O(h^{-1}).$$

Proof: Theorem 4.1 follows from Lemmas 4.2, 4.5, 4.6, and 4.7.

Lemma 4.2

$$|\mathcal{N}_{br}| \leq (\ell[\partial\mathcal{R}]/h) + \alpha.$$

Proof: Let n_k denote the number of gridpoints on the arc $\partial_k\mathcal{R}$. The n_k gridpoints form a polygonal line P_k with $(n_k - 1)$ segments. Since each segment has length at least h, we have, from the definition of the length of a rectifiable curve, that

$$(n_k - 1)h \leq \ell[P_k] \leq \ell[\partial_k \mathcal{R}].$$

Summing over k the lemma follows.

Lemma 4.3

If $\mathcal{C}$ is a rectifiable arc,

$$\mu[\mathcal{C}(\epsilon)] \leq \pi\epsilon^2 + 2\epsilon\, \ell[\mathcal{C}]. \tag{4.4}$$

Proof: This lemma is the case $n = 2$ of a formula due to Estermann [10]; see also Hornich [15]; Verblunsky [31]. For the convenience of the reader we give a modified version of the proof of Hornich.

First, (4.4) is proved by induction for the case when $\mathcal{C}$ is a polygonal line, $P_0 P_1 P_2 \ldots P_r$. Inequality (4.4) clearly holds when $r = 0$ or $r = 1$, for then $\mathcal{C}$ is either a point or a line segment. Assuming that (4.4) holds for $r \leq n$, let p be the polygonal line $P_0 P_1 \ldots P_{n+1}$. Set $p_1 = P_0 \ldots P_n$, $p_2 = P_n P_{n+1}$. Then,

$$\mu[p(\epsilon)] = \mu[p_1(\epsilon) \cup p_2(\epsilon)],$$

$$= \mu[p_1(\epsilon)] + \mu[p_2(\epsilon)] - \mu[p_1(\epsilon) \cap p_2(\epsilon)].$$

Since $P_n \in p_1 \cap p_2$,

$$\mu[p_1(\epsilon) \cap p_2(\epsilon)] \geq \pi\epsilon^2,$$

so that

$$\mu[p(\epsilon)] \leq [\pi\epsilon^2 + 2\ell[p_1]] + [\pi\epsilon^2 + 2\ell[p_2]] - \pi\epsilon^2,$$

$$= \pi\epsilon^2 + 2\ell[p].$$

Therefore, (4.4) holds for all polygonal lines.

Now let $\mathcal{G}$ be an arbitrary rectifiable arc. Choose $\eta > 0$. Then there is a sequence of points on $\mathcal{G}$, $P_o, P_1, \ldots, P_n$, say, such that P_o and P_n are the endpoints of $\mathcal{G}$, and such that every point of the arc P_iP_{i+1} is at a distance less than η from the chord P_iP_{i+1}. Let p denote the polygonal line $P_o \ldots P_n$. A point which is at a distance $\leq \epsilon$ from $\mathcal{G}$ is at a distance $\leq \epsilon + \eta$ from p. Hence

$$\mu[\mathcal{G}(\epsilon)] \leq \mu[p(\varepsilon + \eta)],$$

$$\leq \pi(\epsilon + \eta)^2 + 2(\epsilon + \eta)\ \ell[p].$$

But, $\ell[p] \leq \ell[\mathcal{G}]$ and η is arbitrary, so that (4.4) follows.

Lemma 4.4

$$\mu[\partial\mathcal{R}(\epsilon)] \leq \alpha\pi\epsilon^2 + 2\epsilon\ \ell[\partial\mathcal{R}].$$

Proof: Follows immediately from Lemma 4.3 and (4.1).

Lemma 4.5

$$|\mathcal{N}_a| \leq \pi\alpha(\sqrt{2} + 1)^2/2 + (2 + \sqrt{2})\ \ell[\partial\mathcal{R}]/h.$$

Proof: Let $P \in \mathcal{N}_a$. Let

$\mathcal{K}_P = \{(x', y');\ |x' - x'_P|,\ |y' - y'_P| < h/2\}$, where x'y' are the grid-coordinates corresponding to G (see 1.3).

Then, $\mathcal{K}_P \in \partial\mathcal{R}(h + h/\sqrt{2})$. Also, $\mathcal{K}_P = \mathcal{K}_{P'}$ iff $P = P'$. Hence,

$$h^2|\mathcal{N}_a| = \mu[\bigcup_{P\in\mathcal{N}_a} \mathcal{K}_P] \leq \mu[\partial\mathcal{R}(h + h/\sqrt{2})].$$

Applying Lemma 4.4, the lemma follows.

Lemma 4.6

$$\mu[\mathcal{R}]/h^2 - \{\pi\alpha(\sqrt{2}+1)^2/2 + (2+\sqrt{2})\ \ell[\partial\mathcal{R}]/h\},$$

$$\leq \ |\mathcal{N}_{ir}|,$$

$$\leq \ \mu\ [\mathcal{R}]/h^2 + \{\pi\alpha/2 + \sqrt{2}\ \ \ell[\partial\mathcal{R}]/h\}.$$

Proof: For $P \in \mathcal{N}_{ir}$ define $\mathcal{K}_P$ as in Lemma 4.5. Let

$$\mathcal{K} = \bigcup_{P \in \mathcal{N}_{ir}} \mathcal{K}_P .$$

Then,

$$\mathcal{R} - \partial\mathcal{R}(h + h/\sqrt{2}\,) \subset \bar{\mathcal{K}} \subset \mathcal{R} \cup \partial\mathcal{R}(h/\sqrt{2}\,),$$

so that

$$\mu[\mathcal{R}] - \mu[\partial\mathcal{R}(h + h/\sqrt{2})] \leq h^2\ |\mathcal{N}_{ir}| \leq \mu[\mathcal{R}] + \mu[\partial\mathcal{R}\ (h/\sqrt{2}\,)].$$

Applying Lemma 4.4 the lemma follows.

At first sight one might suppose that $\mathcal{N}_s$ is a finite set if $\partial\mathcal{R}$ is rectifiable. However, this is certainly not the case. Consider, for example, the case when the arc $y = x^{10}\sin(1/x)$, $0 \leq x \leq 1$, is part of $\partial\mathcal{R}$. This arc is not only rectifiable but also of bounded curvature; yet it intersects the line $y = 0$ infinitely often. Therefore, to ensure that $\mathcal{N}_s$ is finite we must impose additional conditions upon $\partial\mathcal{R}$, as is done in the following lemma.

Lemma 4.7

If condition (c) holds, then $\mathcal{N}_s$ is a finite set and

$$|\mathcal{N}_s| \leq \beta + \gamma + 4 + 2\ \ell[\partial\mathcal{R}]/h.$$

Proof: Noting (4.3), let

$$\partial \mathcal{R} = [\bigcup_{k=1}^{\beta'} \partial'_{xk} \mathcal{R}] \cup [\bigcup_{k=1}^{\beta''} \partial''_{xk} \mathcal{R}],$$

where y is a strictly monotone function of x on each $\partial'_{xk} \mathcal{R}$ and y is constant on each $\partial''_{xk} \mathcal{R}$. Then, as in Lemma 4.2,

$$(n'_{xk} - 1)h \leq \ell[\partial'_{xk} \mathcal{R}],$$

where n'_{xk} is the number of intersections of gridlines parallel to the x-axis with $\partial'_{xk}\mathcal{R}$. Hence,

$$\sum_{k=1}^{\beta'} n'_{xk} \leq \beta + \ell[\partial \mathcal{R}]/h.$$

Remembering that y is constant on each $\partial''_{xk} \mathcal{R}$, we see that the number of intersections of gridlines parallel to the x-axis with $\partial\mathcal{R}$ is not greater than

$$\beta + \ell[\partial \mathcal{R}]/h + 2.$$

The lemma follows immediately.

To conclude this section we mention some related results in the literature.

The first result provides an alternative method for estimating $|\mathcal{N}_a|$:

Lemma 4.8

The number of gridsquares having points in common with $\partial\mathcal{R}$ is not greater than $4(\alpha + \ell[\partial \mathcal{R}]/h)$.

Proof: See Michael [21] and Potts [27]. The lemma is an easy consequence of the observation that an arc of length less than h has points in common with at most four gridsquares.

Corollary 4.9

$$|\mathcal{N}_a| \leq 16(\alpha + \ell[\partial\mathcal{R}]/h).$$

Proof: Apply Lemma 4.8 remembering that a square has four corners.

The next result shows that even if condition (c) does not hold, G can be chosen so that $\mathcal{N}_s$ is finite.

Lemma 4.10

Given $h > 0$ there are numbers x_o and y_o such that none of the lines $x = x_o + ih$, $y = y_o + jh$, $i, j = 0, \pm 1, \pm 2, \ldots$, meets $\partial\mathcal{R}$ in an infinite number of points.

Proof: This lemma is due to Estermann [11]. The proof of Estermann is for the case when $\partial\mathcal{R}$ is a Jordan curve, but examination of the proof shows that it holds for $\partial\mathcal{R}$ of the form (4.1).

Next, let

$$m(\mathcal{R}) = \inf |\mathcal{N}_i|,$$

$$M(\mathcal{R}) = \sup |\mathcal{N}_i|,$$

where the infinum and supremum are taken over all possible square grids G with gridlength h. Then, among other results, Niven and Zuckerman [24] prove that

$$m(\mathcal{R}) \leq \mu[\mathcal{R}]/h^2 \leq M(\mathcal{R}). \tag{4.5}$$

In a certain sense, this result complements Lemma 4.6.

Finally, it should be pointed out that there is a connection between the results of the present section and the theory of the geometry of numbers. For example, the theorem of Siegel (Cassels [4, p. 175]) is related to the inequality (4.5). However, in the theory of the geometry of numbers, interest is usually focussed upon properties which hold for all "lattices" whose "determinant" is equal to a fixed quantity, whereas in the present paper we are concerned with "rectangular lattices."

APPENDIX A

Remarks on the implementation of Algorithm 1

A. 1. Expanded description of Algorithm 1

Step 1

Construct a net $\mathcal{N}$. "Classify" each netpoint, P, and determine its set of "neighbours", M(P).

$\mathcal{N}$ is often constructed with the use of a grid G, and this approach was described in section 1. As in section 1 we then write $\mathcal{N} = \mathcal{N}(G, \mathcal{R})$.

The definition of M(P) depends upon the boundary value problem being solved, the method of constructing the algebraic equations (A. 1.1), and the classification of P. When $\mathcal{N}$ is constructed using a grid, and the Dirichlet problem {(1.1), (1.2)} is being solved, M(P) usually consists of five points, namely P and the points adjacent to P in the North, South, East, and West directions, respectively. For differential equations such as

$$u_{xx} + u_{xy} + u_{yy} = 0,$$

or

$$u_{xxxx} + 2u_{xxyy} + u_{yyyy} = 0,$$

M(P) is of course more complicated.

Step 2

Set up a system of algebraic equations connecting the values of the approximate solution, U, at the netpoints. A typical equation is of the form

$$\sum_{Q \in M(P)} A(P,Q)\, U(Q) = b(P). \tag{A.1.1}$$

Here, A(P,Q) and b(P) are constants whose values depend upon the differential equation (1.1), the boundary conditions (1.2), and the classification and location of P and its neighbours.

Step 3

Solve the system of algebraic equations set up in Step 2, thereby obtaining U.

A. 2. Storage requirements

In this section we discuss the amount of the storage required to implement Algorithm 1, and possible ways of reducing it.

In step 2 of Algorithm 1 the following information is needed about each netpoint P:

(a) A list of the neighbours of P.

(b) A list of the coordinates of the neighbours of P.

(c) Information about the type of equation to be set up at P.

For example, if

$$\mathcal{N} = \mathcal{N}(G, R) \qquad \text{(A.2.1)}$$

then it is necessary to know whether P lies in $\mathcal{N}_e$, $\mathcal{N}_{br}$, $\mathcal{N}_s$, $\mathcal{N}_a$, or $\mathcal{N}_{ir}$. (see Figure 1.1).

If $P \in \mathcal{N}_b$ we must also know the type of boundary condition (Dirichlet, Neumann, etc.) which holds at P.

We shall refer to this information as the net data.

In step 3 of Algorithm 1 it is necessary to know the equation corresponding to each netpoint P. That is, the following information is needed about each netpoint P:

$$\left.\begin{array}{l}\text{(a) A list of the netpoints with non-zero coefficients in the} \\ \quad\ \text{equation corresponding to P.} \\ \text{(b) A list of the non-zero coefficients.}\end{array}\right\} \qquad \text{(A.2.2)}$$

We shall refer to this information as the <u>equation data</u>.

If $|\mathcal{N}|$ is large, then it may be difficult or impossible to store all the net data and equation data in the high speed memory of the computer. For example, if $\mathcal{N} = \mathcal{N}(G, R)$ where G is a square grid with gridlength h, then, from Theorem 4.1,

$$|\mathcal{N}| = \mu[\mathcal{R}]h^{-2} + O(h^{-1}). \qquad \text{(A.2.3)}$$

Since there are in general at least five non-zero coefficients in (A.1.1), we see that if $h = 1/100$ and $\mu[\mathcal{R}] = 1$ it is impossible to store all the equation data in the high speed store of a computer with a 32,000 word memory.

The difficulty of storing the net data and equation data lessens as the size of the high speed memory increases. On early computers such as EDSAC II (which had a memory of 1000 words) it was a tremendous problem (Cryer [6]). On present day computers with 132,000 word memories, it is far less of a problem. However, even on a computer with a 132,000 word memory, difficulties arise when h is very small. In addition, difficulties arise on all computers when $\mathcal{R}$ is a three-dimensional region, and many of our remarks below apply, mutatis mutandis, to the three-dimensional case.

Of course, it is almost always possible to store the net data and equation data in auxiliary storage. However, the access time and read/write time for auxiliary storage is considerably greater than for core storage; this is illustrated in Table A.2.1 for the UNIVAC 1108. For comparison, floating point multiplication and division on the UNIVAC 1108 require 2.6 μs and 8.25 μs, respectively.

Type of storage	Average access time	Read/write time per word	Capacity (words)
Core	none	.75 μs	262,000 (maximum possible)
Drum (FH 432)	4.25 ms	4 μs	262,000 (per drum)
Drum (FH 1782)	17 ms	4 μs	2,000,000 (per drum)
Drum (FASTRAND II)	92 ms	40 μs	22,000,000 (per drum)
Tape (UNISERVO VIC)	variable	180 μs	2,000,000 (per tape, approximately)

Table A.2.1

Storage characteristics of the UNIVAC 1108.

If in step 3 of Algorithm 1 the system of algebraic equations is solved by iteration (as is usually the case), each equation is referred to many times. It is therefore highly desirable to avoid the use of auxiliary storage.

The amount of storage required by the net data and equation data can be reduced in a number of ways (usually at the cost of more programming and slower execution):

1. Certain data does not require a full word, and can be packed. For example, the net data (A.2.1c) requires only a few bits. In many early programs (Forsythe and Wasow [12, p. 359]), this data was stored in the last few bits of the mantissa of U (P).

2. If the system of algebraic equations is symmetric, then the possibility exists of reducing the amount of storage required for the coefficients by about half.

3. If $\mathcal{N}$ has a regular structure, then substantial savings in storage can be achieved. We illustrate this for the case when $\mathcal{N} = \mathcal{N}(G, \mathcal{R})$ where G is a square grid of gridlength h. (Other possible regular nets are regular triangular nets and regular hexagonal nets; see section A.3). We will assume that the boundary value problem to be solved is of the form

$$\frac{\partial}{\partial x}\left(a \frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(b \frac{\partial u}{\partial y}\right) = 0, \quad (x,y) \in \mathcal{R}, \tag{A.2.4}$$

$$\alpha \frac{\partial u}{\partial n} + \beta u + \gamma = 0, \quad (x,y) \in \partial\mathcal{R}, \tag{A.2.5}$$

and that a five-point formula is used to approximate (A.2.4).

We note that $\mathcal{N}(G, \mathcal{R})$ can be represented in the computer as two arrays, a one-dimensional array for the points in $\mathcal{N}_s$ and a two-dimensional array for the remaining netpoints. (In some early programs $\mathcal{N}_s$ was embedded in $\mathcal{N}_e$, a special boundary point being associated with the nearest external netpoint. This approach sometimes leads to difficulties since, if $\partial\mathcal{R}$ has corners, two special boundary points may be associated with the same external netpoint.)

Because of the correspondence between the method of representing $\mathcal{N}$ in the computer and the topological structure of $\mathcal{N}$, there is no need to store the data (A.2.1a), (A.2.1b), and (A.2.2a), for points in $\mathcal{N}_{ir}$. Furthermore, for points in $\mathcal{N}_{ir}$ the coefficients (A.2.2b) can easily be generated when required, and indeed it may be faster to generate these coefficients than to read them in from auxiliary storage. Hence, for $P \in \mathcal{N}_{ir}$ only the data (A.2.1c) need be stored, and as we have already mentioned, this data can if necessary be stored in the last few bits of the mantissa of $U(P)$.

Therefore, the amount of storage required for the net data and equation data is proportional to $|\mathcal{N}_a \cup \mathcal{N}_b|$. Since, by Theorem 4.1, $|\mathcal{N}_a \cup \mathcal{N}_b| = O(h^{-1})$ while $|\mathcal{N}_{ir}| = O(h^{-2})$, a very substantial saving in storage has been achieved.

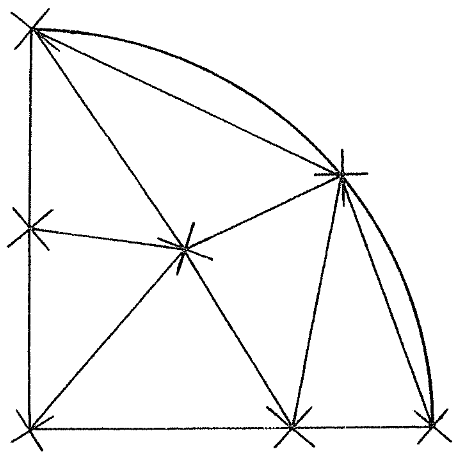
A. 3. Commonly used nets

In this section we describe the various nets which have been used, starting with nets with no structure and ending with nets with very regular structure.

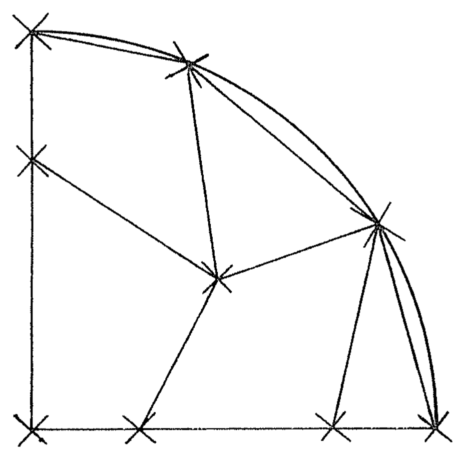
Irregular nets

Irregular nets (see Figure A.3.1) were first suggested by MacNeal [20]. They have the advantage of great flexibility, so that oddly shaped regions can be easily handled, and the density of netpoints can be increased in areas where the solution is expected to change rapidly.

The major disadvantage of these nets is that every netpoint must be treated as a special point, so that large amounts of storage are needed, and it is only recently that computers have had large enough stores for this choice of net to be feasible.



(a) Triangulated.



(b) Polygonal.

Figure A.3.1

Irregular nets.

Even so, the number of netpoints is severely restricted. For example, the programs of Wilson [32] and Taylor [30], written for IBM 7090 computers with 32,000 word memories, can handle a maximum of 340 and 500 netpoints, respectively.

Nevertheless, irregular nets have become increasingly popular, especially among civil engineers using the "finite element" method (Zienkiewicz [37]). Perhaps this is because the problems of civil engineers often involve irregular boundaries. Irregular nets have also been widely used at the Lawrence Radiation Laboratory (Noh [25], Winslow [33, 34]).

The discretization error for irregular triangulated nets has been studied by Kellogg [16].

Graded Nets

The basic idea of a graded net (see Figure A.3.2) is to combine the advantages of a rectangular net with the advantages of a high density of netpoints in regions, such as the neighbourhood of a corner on $\partial \mathcal{R}$, when the solution is changing rapidly. This type of net was used very effectively by Southwell and his coworkers (Southwell [29, p. 98]). However, it is difficult to implement on a computer, and has seldom been used. Recently, Young and Whiteman [36] have used graded nets, but it is the author's understanding that their program makes no use of the structure of the net to reduce storage requirements.

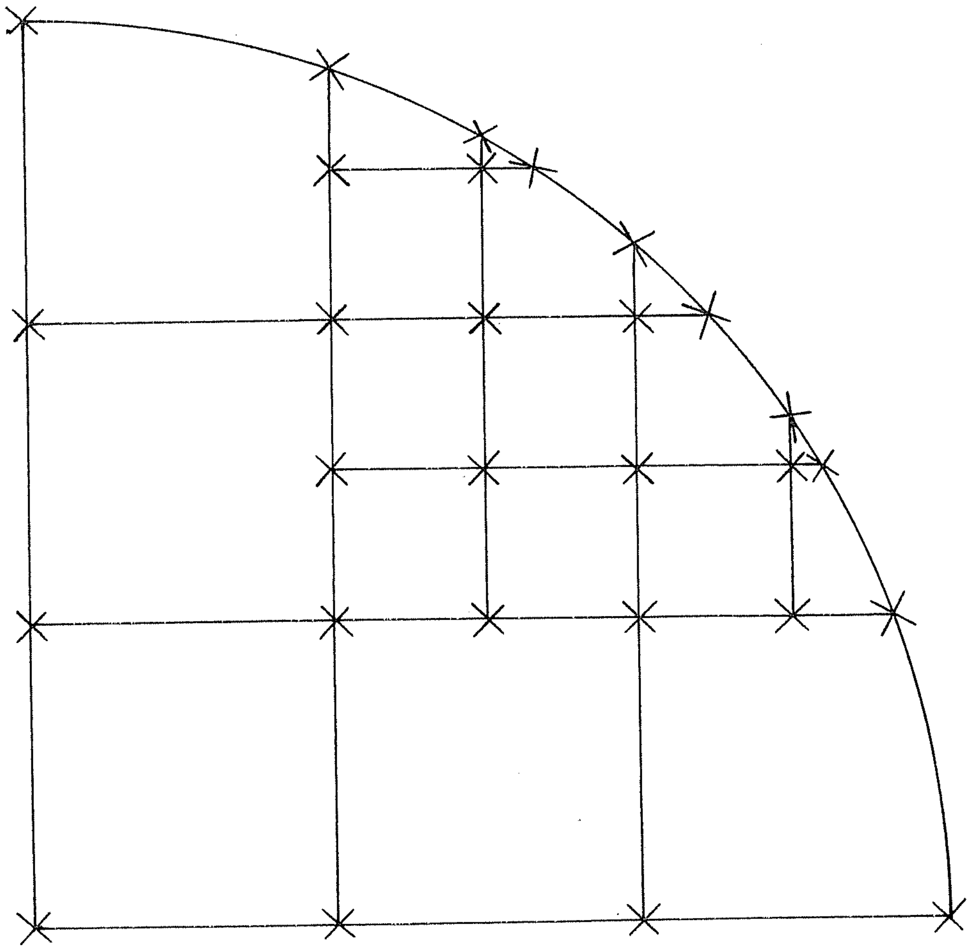


Figure A.3.2

A graded net.

Rectangular nets

Rectangular nets, regular rectangular nets, and gridlike rectangular nets, were defined in section 1 and will not be discussed here.

Regular nets

We have seen in section A.2 that substantial simplifications result if the net has a regular structure. The regular nets considered so far have been constructed using regular rectangular grids. Two other possibilities - regular triangular grids and regular hexagonal grids - are shown in Figures A.3.3 and A.3.4. The following lemma is of interest, since it shows that no other regular grids are possible.

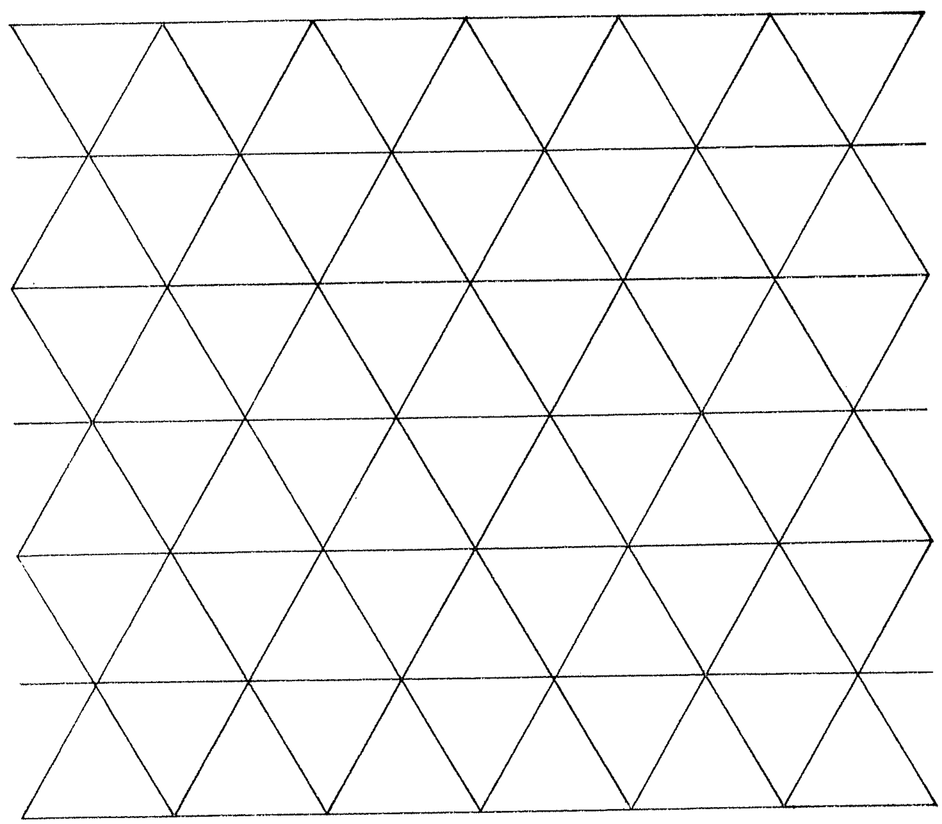


Figure A.3.3

A regular triangular grid.

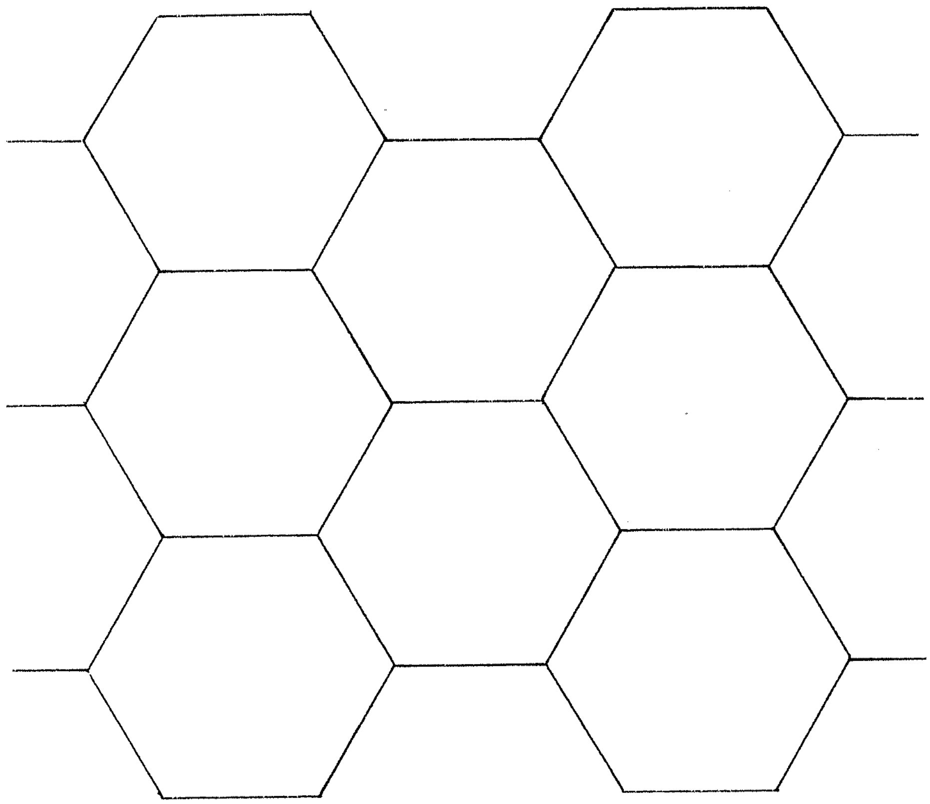


Figure A.3.4

A regular hexagonal grid.

Lemma A.3.1

Let G be a grid which covers the plane, and which is built up out of copies of a regular n-sided polytope P.

Then G is either a square ($n = 4$), triangular ($n = 3$), or hexagonal ($n = 6$) grid.

Proof: This lemma is well-known. See Coxeter [5, p. 58] and Southwell [29, p. 10].

Let m polytopes meet at every gridpoint, and let each interior angle of P be equal to α.

Then,

$$m\alpha = 2\pi,$$

and

$$n\alpha + 2\pi = n\pi.$$

Eliminating α, we find that

$$2n + 2m = nm \quad . \tag{A.3.1}$$

Assume that $n \geq m$. Then $nm \leq 4n$ so that $m \leq 4$. Setting $m = 1, 2, 3,$ and 4, and remembering that n is an integer, we obtain the solutions

$$m = 3,\ n = 6;\quad m = 4,\ n = 4.$$

Interchanging m and n we obtain the solution

$$m = 6,\ n = 3.$$

These are all the possible integer solutions of (A.3.1). The proof of the lemma is therefore complete.

Given a regular triangular grid or a regular hexagonal grid, the corresponding net $\mathcal{N}$ can be constructed in a manner analogous to that used for rectangular grids; we call the resulting nets regular triangular nets and regular hexagonal nets, respectively.

Regular triangular nets were used by handworkers such as Southwell [29, p. 49]; however, Southwell [29, p. 54] did not advocate the use of regular hexagonal nets. At present, regular hexagonal nets are used in programs for nuclear reactor calculations such as KARE (Archibald and Teaford [2]), and PDQ-7 (Cadwell [3]); see also Kellogg [17].

A. 4. General purpose programs

In this section we discuss a few of the programs which have been written to implement Algorithm 1.

First, it should be noted that the three steps of the algorithm require completely different types of programming.

Step 3 is easily implemented and is highly suitable for a digital computer, since it involves an immense amount of repetitive computation.

Step 2 is also easy to implement. It may be thought of as a problem in information processing, the input being the net data (A.2.1) and the output the equation data (A.2.2).

Step 1 is difficult to implement, and most general purpose programs for solving boundary value problems require substantial human assistance in this step.

One possible approach to implementing Algorithm 1 is typified by the program of Engeli [9]. Engeli avoids step 1 by requiring the user to provide much of the equation data and net data. However, his program has many features which simplify the preparation of the data.

The program FREEBOUN (Cryer [7]) contains, as a subprogram, an implementation of Algorithm 1. The input data to the subprogram is provided through subroutines; the reason for this is that FREEBOUN itself generates the boundary value problems to be solved. The differential equation, the boundary, and the boundary conditions, must be specified. A net $\mathcal{N}(G, \mathcal{R})$ is used, and both the gridlines and the boundary netpoints must also be specified. It is assumed that $\mathcal{N}$ is gridlike and that $\mathcal{N}_i$ is gridconnected.

The programs KARE (Archibald and Teaford [2]) and PDQ-7 (Cadwell [3]) are general purpose programs for nuclear reactor problems. Both programs are the latest of a series (there is a more recent version of KARE which is at present still classified). Both regular rectangular nets and regular hexagonal grids are used. It is assumed that $\mathcal{N}$ is gridlike, and $\partial\mathcal{R}$ is required to be a polygonal line. There is provision for specifying $\partial\mathcal{R}$ in a very compact manner. An interesting feature of nuclear reactor problems is that

$$\mathcal{R} = \bigcup_{k=1}^{n} \mathcal{R}_k$$

where different equations hold in different regions $\mathcal{R}_k$. To specify the domains $\mathcal{R}_k$ in the input data both KARE and PDQ-7 use the technique of "overlays".

Finally, it should be remarked that the problems involved in implementing step 1 of Algorithm 1, are similar to those encountered in certain data-processing problems. (Loomis [18], Morse [22], Nordbeck and Rystedt [38]). It may well prove possible to apply techniques developed for these data-processing problems to step 1.

APPENDIX B

Two examples of non-Jordan domains $\mathfrak{R}$

In the present paper we have tried to place as few restrictions upon $\mathfrak{R}$ as possible. Many of the proofs could have been considerably shortened if more assumptions about $\mathfrak{R}$ had been made. In most practical problems, $\mathfrak{R}$ is a Jordan domain with piecewise analytic boundary, and the reader may have wondered whether these was any practical need to consider non-Jordan domains. In this appendix we give two examples of non-Jordan domains $\mathfrak{R}$ arising from practical problems.

The first example is the bending of a thin circular plate of unit radius which is clamped at its edges and displaced a unit amount at its center. If w denotes the normal displacement then (Love [19, p. 488]),

$$\nabla^4 w = w_{xxxx} + 2w_{xxyy} + w_{yyyy} = 0, \ (x,y) \in \mathfrak{R}_7, \tag{B. 1}$$

$$\left.\begin{aligned} & w = \frac{\partial w}{\partial r} = 0, \text{ for } r = 1, \\ & w = 1, \text{ for } r = 0, \end{aligned}\right\} \tag{B. 2}$$

where

$$r^2 = x^2 + y^2 .$$

The domain $\mathfrak{R}_7$ is the punctured disk,

$$\mathfrak{R}_7 = \{(x,y);\ 0 < x^2 + y^2 < 1\}, \tag{B. 3}$$

shown in Figure B. 1.

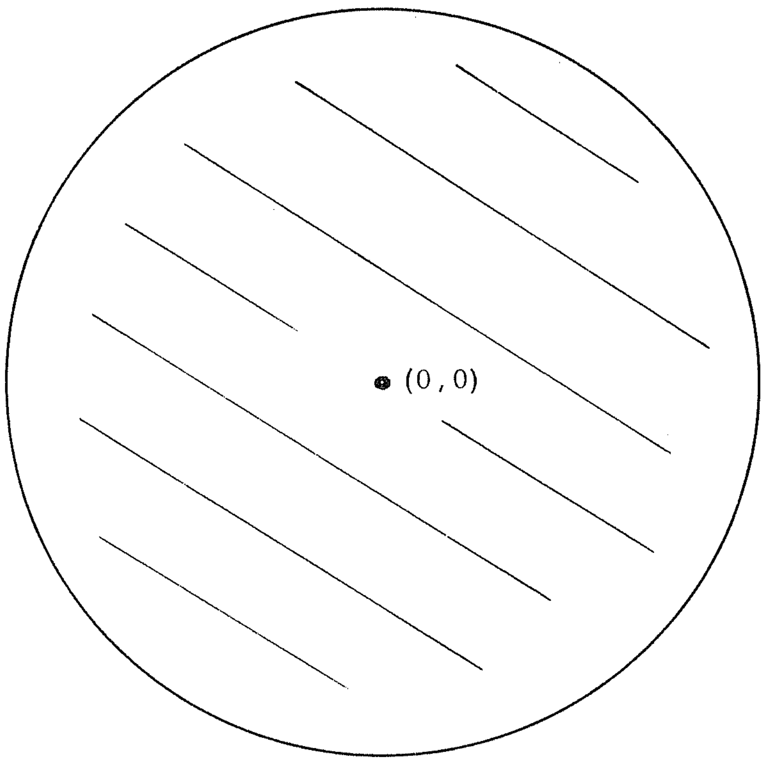


Figure B.1

The punctured disk $\mathfrak{R}_7$.

The solution of this problem is (Love [19, p. 490]),

$$w = 1 - r^2 + 2r^2 \log r. \tag{B. 4}$$

It is worth pointing out that domains with isolated boundary points (such as the point (0,0) for $\mathfrak{R}_7$) do not occur when the governing differential equation is a second order equation. This is because, for second order equations, boundary conditions imposed at isolated points can be ignored (Petrovsky [26, p. 180]). However, for equations of order four or higher, boundary values imposed at isolated points cannot be ignored (Sobolev [28, p. 105]).

The second example is

$$\nabla^2\Phi = \Phi_{xx} + \Phi_{yy} = 0, \ (x,y) \in \mathfrak{R}_8, \tag{B. 5}$$

$$\left.\begin{array}{l} \Phi = \Phi_1, \ (x,y) \in \partial_1 \mathfrak{R}_8, \\ \Phi = \Phi_2, \ (x,y) \in \partial_2 \mathfrak{R}_8. \end{array}\right\} \tag{B. 6}$$

Here, $\mathfrak{R}_8$ is the domain shown in Figure B. 2, $\partial_2 \mathfrak{R}_8 = GH \cup G'H'$, and $\partial_1 \mathfrak{R}_8 = \partial \mathfrak{R}_8 - \partial_2 \mathfrak{R}_8$; note that $OO' \subset \mathfrak{R}_8$.

This example is a simplified model of an electrostatic electron microscope (Grivet [14, p. 5]). Electrons leave the hot cathode C, and are accelerated into the microscope because the microscope is at a high potential. After striking the object to be viewed, OO', the electrons are focussed by the lens GH $\cup$ G'H' $\cup$ EDBF $\cup$ E'D'B'F' and an image is formed on the fluorescent screen SS'. Φ is the electrostatic potential field in the microscope.

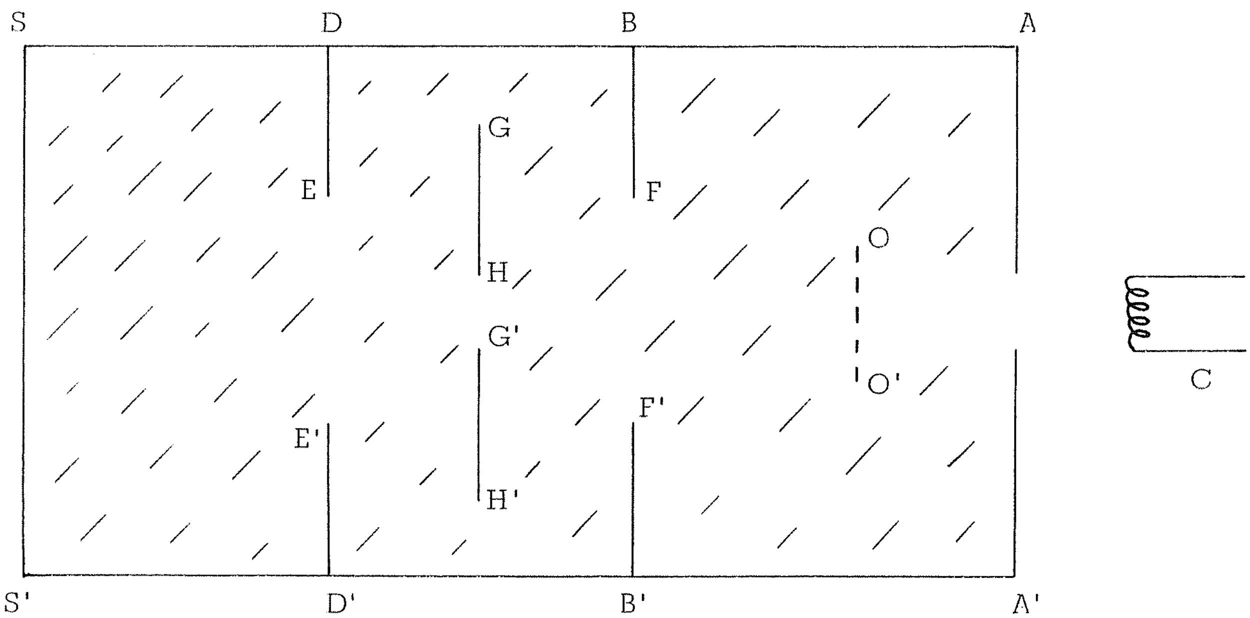


Figure B.2

The domain $\mathscr{W}_8$.

The most interesting feature of $\mathfrak{R}_8$ is that $\partial\mathfrak{R}_8$ contains the arcs GH and G'H'. Similar regions occur in crack problems in the theory of elasticity.

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13. ABSTRACT

When the method of finite differences is used to approximately solve a boundary value problem for an elliptic partial differential equation over a two-dimensional domain $\mathcal{R}$ with boundary $\partial\mathcal{R}$, the first step is to choose a set of netpoints, $\mathcal{N}$. Next, a system of algebraic equations connecting the values of the approximate solutions at the netpoints is set up. Finally, the system of algebraic equations is solved.

Usually, $\mathcal{N}$ is taken to be the set of points belonging to a rectangular grid, together with the points of intersection of gridlines with $\partial\mathcal{R}$.

When a computer is used, one or more of the following assumptions are often made in order to simplify the programming:

1. All the points of $\mathcal{N}$ are gridpoints.
2. The "interior netpoints" are "gridconnected".
3. The number of "irregular" netpoints is much smaller than the number of "regular" netpoints.

In the present paper these three assumptions are analyzed.

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