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DISCRETE MECHANICS

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DISCRETE MECHANICS

1. INTRODUCTION

The subject of mechanics is one of the oldest and most fully developed of the physical sciences, and yet its practical application to dynamical problems is severely restricted by the absence of general analytical methods for solving nonlinear differential equations. In this paper we will reformulate the basic concepts, laws and equations of Newtonian mechanics in a discrete fashion, thereby creating a physical system which will be compatible with modern high speed digital computer capabilities. Such an approach to physical problems has been of occasional interest to mathematicians throughout the centuries, but only with the development of the modern digital computer has it become a source of exceptional power (see, e.g., references 1-6, 8-11). The equations of our models, whether linear or nonlinear, will be easily solvable with computable error bounds [7] by means only of arithmetic processes, while the physically nonrealizable concept of infinity and the convergence proofs required for continuous models will become unnecessary.

Throughout, we will emphasize plane Newtonian mechanics, since the concepts, laws and equations to be developed extend in a natural way to n dimensions. And though we will modify the form of Newton's Second Law in Section 8, we will in the present paper assume the validity of the first and third laws.

2. IMPLICATIONS OF CONSTRAINTS ON MEASUREMENT

In the simplest of all possible worlds, experimental scientists use three basic measuring instruments, one which measures time and will be called a clock, one which measures weight and will be called a scale, and one which measures length and will be called a ruler. Each instrument has a limited accuracy and in practice is applied a finite number of times to produce a finite set of data. Often these data are combined by arithmetic processes to determine certain additional scalar quantities, like speed or work. At other times one may use the data to infer certain equations associated with the system under consideration and then proceed to solve the equations in some comprehensive way. In any event, a clock, scale and ruler will always be considered as part of a given experiment, and hence define a smallest meaningful time interval (which is positive), a smallest meaningful weight (which is positive), and a smallest meaningful length (which is positive). These smallest positive units are determined by the precision of the clock, scale and ruler and no smaller units are meaningful because they cannot be measured by the given instruments.

By necessity, then, the data of a given experiment consist of rational numbers which have a finite number of significant digits, and it is important to note that the arithmetic combination of such numbers is the primary function of the high speed digital computer [7].

3. FUNCTIONS

The concept of a function is fundamental in contemporary mathematical thought and is defined as follows.

Definition 3.1

Let R and S be two nonempty sets. If to each element x in R there corresponds by some rule f a unique element y in S, then one says that f is a function of x on R with values y in S, and one writes $y = f(x)$.

When the variable x in Definition 3.1 represents time, one usually replaces it with the letter t.

In order to incorporate the finite data and limited accuracy concepts of Section 2, we will, unless otherwise stated, take R and S in Definition 3.1 to be finite sets of rational numbers, in which case f will be called a discrete function. For discrete functions (as, indeed, for all functions), the set of points (x, y) in the real plane, where $x \in R$ and $y = f(x)$, is called the graph of f.

Example.

Let R be the set of $10^6 + 1$ rational numbers $x_k = k \cdot 10^{-6}$; $k = 0, 1, 2, \cdots, 10^6$. (These may be considered as $10^6 + 1$ numbers which are each accurate to six decimal places.) On R the equation $y = 2x^2$ defines a discrete function f whose graph consists of the $10^6 + 1$ points plotted in Figure 3.1.

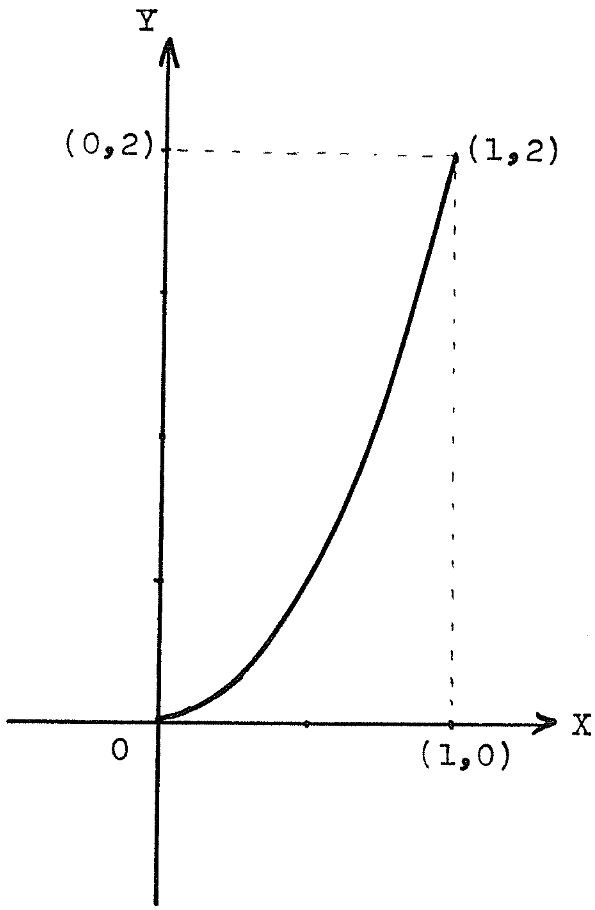


FIGURE 3.1

With regard to the above example, note that the graph of the discrete function appears to the naked eye to be identical with the graph of $y = 2x^2$, x real, $0 \leq x \leq 1$. The idea of "packing" a large, but finite, number of points sufficiently close so that each is physically indistinguishable from others nearby is, of course, the basis of television reception, and it reenforces the notion that finite sets can serve one's purposes just as well, if not better, than infinite ones [4].

Of special interest from the digital computer point of view are the rational, square root and trigonometric functions, which will be considered next.

A rational discrete function is one whose values are determined from the independent variable by a finite number of arithmetic operations (where, of course, division by zero is not an arithmetic operation).

If $f(x)$ is a discrete, nonnegative rational function, then the discrete square root of f, denoted by $F = [f]^{\frac{1}{2}}$, is defined as follows. Let x be any element in a finite set R of rational numbers, each of which has only a finite number of significant digits. Let f be defined on R. Then $[f(x)]^{\frac{1}{2}}$ is that rational number y which has the same number of significant digits as $f(x)$ and whose square best approximates $f(x)$. Of course the actual determination of a square root can be accomplished on a computer by means of the elementary square root algorithm found in almost all books on arithmetic.

With regard to the geometry of a right triangle, it will be assumed that, given measured lengths of any two sides, then the law of Pythagoras and the definition of the square root function can be applied to yield the length of the third side.

As regards trigonometric functions, a word must first be said about the measurement of angles. Since a protractor can be constructed with the aid of a ruler, it will be assumed for the present that the accuracy with which one can measure angles is that defined by the ruler of a given experiment. The trigonometric functions themselves will be treated in the usual elementary fashion as ratios of various segments of right triangles, since these definitions involve only rational functions and square roots.

Since the square root function is defined above only in terms of arithmetic operations, and since the trigonometric functions are all ratios, we assume that tables of these functions are available for our use.

4. PARTICLES, TIME AND MOTION

From a purely mathematical point of view, one can consider the terms particle, time and motion as undefined and then proceed to define other concepts in terms of these. Nevertheless, physically it is desirable to have some intuition about these rudiments and our present purpose is to develop such intuition.

For a given experiment, a particle of a given solid will be considered to be the smallest spherical portion of the solid whose diameter can be measured to the accuracy of the associated ruler and whose weight can be measured to the accuracy of the associated scale. A plane particle, which is the only kind with which we will deal, will be any great circle section of a particle. The centroid of a plane particle is defined to be the center of the associated great circle. By the position of a particle we will mean the position of its centroid. The mass of a particle is defined in the usual way in Newtonian mechanics. All plane figures are to be considered as compositions of packed particles and are said to be of uniform density only if each particle in its structure has the same mass.

The concept of time which we will use can be described as follows. Let Δt be the smallest measureable time interval defined by a given clock. On a real number axis, mark off the time positions $t_k = k\Delta t$, $k = 0, 1, 2, \cdots, n$, where n is a fixed positive integer. The numbers t_0 and t_n may be considered to be initial and terminal times, respectively, of some physical event. The passage of time between t_0 and t_n may be thought of as follows. At first a small "packet" or "particle" of time containing Δt units of time is located at t_0. In Figure 4.1, this packet is denoted by circle C with diameter Δt and center at t_0. After Δt seconds, C is located with its center at t_1, after $2\Delta t$ seconds with its center at t_2, after $3\Delta t$ seconds with its center at t_3, and so on, until after $n\Delta t$ seconds the packet is located with its center at t_n. The passage of time between t_0 and t_n is conceived then only as the presence

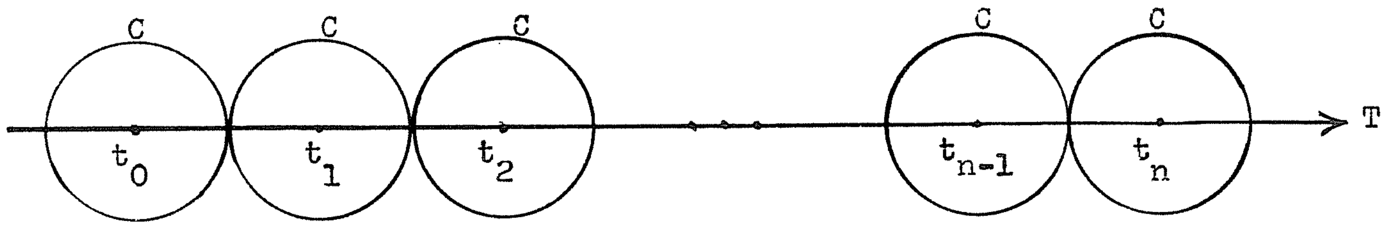


FIGURE 4.1

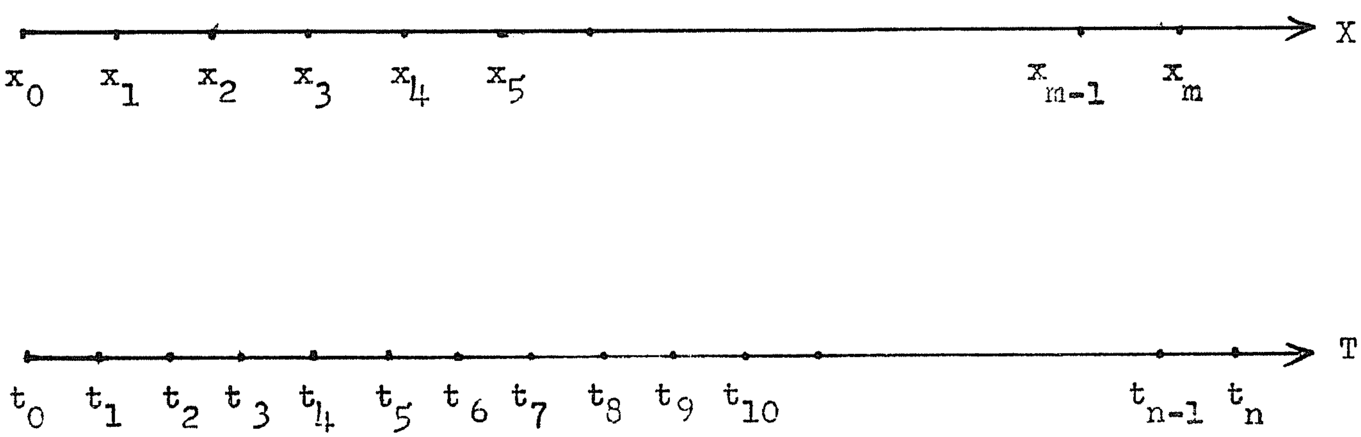


FIGURE 4.2

of time packet C with its center at $t_0, t_1, \ldots, t_n$ at the respective times $0, \Delta t, \ldots, n\Delta t$. How packet C got from t_0 to t_n can be idealized as follows. C remains at t_0 (i.e., its center remains at t_0) until Δt seconds have elapsed. It then makes an instantaneous jump, called a quantum jump, to t_1. It remains at t_1 until Δt additional seconds have elapsed, at which time it makes a second quantum jump and locates at t_2. After n such quantum jumps, the center of C appears at t_n.

That the above model of the passage of time is reasonable is a consequence of the limitation that no time interval of length smaller than Δt can be measured by the given clock.

Let us next examine how the above concept of time applies to the development of the concept of motion. Let Δx be the smallest measurable length defined by a given ruler and let Δt be the smallest measurable time interval defined by a given clock. On a real x-axis, mark off $x_k = k\Delta x$, $k = 0, 1, \ldots, m$, and on a real t-axis mark off $t_j = j\Delta t$, $j = 0, 1, \ldots, n$, as shown in Figure 4.2. For illustrative purposes, assume that a particle C has center P which is located at x_0 when $t = t_0$, at x_1 when $t = t_3$, at x_2 when $t = t_6$ and at x_3 when $t = t_{10}$. The motion of C from x_0 to x_3 is merely C's being with center P at x_0, x_1, x_2, x_3 at respective times t_0, t_3, t_6, t_{10}. Thus the motion of C from x_0 to x_3 is considered to be a sequence of four "stills". This concept of motion is physically realized in motion pictures, where the eyes observe motion from a sequence of stills. The question of how C actually relocated from x_0 to x_1, from x_1 to x_2 and finally from x_2 to x_3 can be described again in terms of quantum jumps. Also, from the physical viewpoint, numbers between x_0 and x_1, x_1 and x_2, and x_2 and x_3 are unrealistic because they cannot be measured by the given ruler.

Motion of physical systems which consist of many particles will be considered later.

5. ARC LENGTH

Let x_1, x_2, y_1 and y_2 be four rational numbers. If (x_1, y_1) and (x_2, y_2) are the centers of two plane particles, then the distance s between the particles is given by

$$s = [(x_2 - x_1)^2 + (y_2 - y_1)^2]^{\frac{1}{2}} . \tag{5.1}$$

On a computer, one may, however, have difficulty in calculating the quantities $(x_2 - x_1)^2$ and $(y_2 - y_1)^2$ in (5.1), since each computer has a limited word length and rounds off when this length is exceeded. To show the computational methods which one can actually use in practice to determine s, we next give a simple, illustrative example.

Example.

Suppose that the centers of two particles are given as (0.21, 0.13) and (0.53, 2.81), where each coordinate is known to be correct to only two decimal places, and that one wishes to calculate s. If the centers of the particles are (x_1, y_1) and (x_2, y_2), then one can say

$$0.205 \le x_1 \le 0.215$$

$$0.125 \le y_1 \le 0.135$$

$$0.525 \le x_2 \le 0.535$$

$$2.805 \le y_2 \le 2.815 \quad .$$

Thus,

$$x_1 \in [0.205,\ 0.215]$$

$$y_1 \in [0.125,\ 0.135]$$

$$x_2 \in [0.525,\ 0.535]$$

$$y_2 \in [2.805,\ 2.815]\ .$$

Substitution of these intervals for the corresponding variables in (5.1) and computing in interval arithmetic [4, 7] implies

(5.2) $$2.687 \le s \le 2.711$$

Choosing s to be the mid-value of the bounds in (5.2) implies that

$$s = 2.699 \ ,$$

which is in error by at most 0.012.

The length of a physical object, like a string, can be defined next as follows. Let $x_0, x_1, \ldots, x_n$ be an ordered set of distinct rational numbers. Let $y_i = f(x_i)$, $i = 0, 1, \ldots, n$, be a discrete function. The points (x_i, y_i), $i = 0, 1, \ldots, n$, may be the centers of all the particles which compose the string. Then the length s of the string, or, equivalently, the length s of discrete function f, is defined by

(5.3) $$s = \sum_{i=1}^{n} [(x_i - x_{i-1})^2 + (y_i - y_{i-1})^2]^{\frac{1}{2}}$$

6. AREA AND CENTROID

Let $x_0, x_1, \ldots, x_n$ be an ordered set of distinct rational numbers. Let $y_i = f(x_i)$, $i = 0, 1, 2, \cdots, n$ be a discrete, non-negative function. Again, the points (x_i, y_i), $i = 0, 1, \cdots, n$ may be considered to be the centers of the particles which compose a string. Then the area under the string, or, equivalently, the area under the discrete function f, is defined to have measure A given by

(6.1) $$A = \sum_{i=1}^{n} [f(x_{i-1})(x_i - x_{i-1})]$$

If the moment of a force is defined in the usual way, then the centroid of the area under f is defined to be the point $(\overline{x}, \overline{y})$ whose coordinates are given by

(6.2) $$\overline{x}A = \tfrac{1}{2} \sum_{i=1}^{n} [f(x_{i-1})(x_i^2 - x_{i-1}^2)]$$

(6.3) $$\overline{y}A = \tfrac{1}{2} \sum_{i=1}^{n} \{[f(x_{i-1})]^2 (x_i - x_{i-1})\} .$$

7. VELOCITY AND ACCELERATION

Let $t_0, t_1, \ldots, t_n$ be a linearly ordered set of distinct rational numbers and suppose that a man driving an automobile in a fixed direction measures and records the distance x_i he has traveled at time t_i. Then at time t_i, the driver will have recorded $x_0, x_1, \cdots, x_i$, but not, say,

x_{i+1}, x_{i+2} and x_{i+3}, since these distances are still to be traversed. The driver has knowledge only of the present and of the past. Then assuming that v_0 is known (since all problems can be formulated so that v_0 is zero), it is reasonable to define $v_i = v(t_i)$, $i = 1, 2, \cdots, n$, implicitly by

$$(7.1) \qquad \frac{x_k - x_{k-1}}{t_k - t_{k-1}} = \frac{v_k + v_{k-1}}{2}, \quad k = 1, 2, \ldots, n .$$

However, if for any set $w_0, w_1, w_2, \ldots, w_n$, one defines the difference Δ by

$$\Delta w_k = w_k - w_{k-1}, \quad k = 1, 2, \ldots, n$$

then (7.1) is equivalent to the linear difference equation

$$(7.2) \qquad v_k = 2 \frac{\Delta x_k}{\Delta t_k} - v_{k-1}, \quad k = 1, 2, \ldots, n ,$$

which can be rewritten explicitly as

$$(7.3) \qquad v_n = 2 \sum_{j=0}^{n-1} \left[(-1)^j \frac{\Delta x_{n-j}}{\Delta t_{n-j}} \right] + (-1)^n v_0, \quad n \geq 1 .$$

Acceleration $a_k \equiv a(t_k)$, however, is usually not known initially, that is, at t_0, so that it is reasonable to define it by

$$(7.4) \qquad a_k = \frac{\Delta v_k}{\Delta t_k}, \quad k = 1, 2, \ldots, n ,$$

which, from (7.3) and (7.4), implies

(7.5a)
$$a_1 = \frac{2}{\Delta t_1}\left[\frac{\Delta x_1}{\Delta t_1} - v_0\right]$$

(7.5b)
$$a_n = \frac{2}{\Delta t_n}\left\{\frac{\Delta x_n}{\Delta t_n} + 2\sum_{j=1}^{n-1}\left[(-1)^j \frac{\Delta x_{n-j}}{\Delta t_{n-j}}\right] + (-1)^n v_0\right\}, \quad n \geq 2.$$

In certain later discussions, it will be of interest to allow all the time intervals Δt_k to be equal, that is,

$$\Delta t_k \equiv \Delta t \ , \qquad k = 1, 2, 3, \ldots, n \ .$$

In such cases, formulas (7.3) and (7.5) reduce to

(7.6a)
$$v_1 = \frac{2}{\Delta t}(x_1 - x_0) - v_0$$

(7.6b)
$$v_n = \frac{2}{\Delta t}\left\{x_n + (-1)^n x_0 + 2\sum_{j=1}^{n-1}\left[(-1)^j (x_{n-j})\right]\right\} + (-1)^n v_0, \quad n \geq 2$$

and

(7.7a)
$$a_1 = \frac{2}{(\Delta t)^2}\left[x_1 - x_0 - v_0 \Delta t\right]$$

(7.7b)
$$a_2 = \frac{2}{(\Delta t)^2}\left[x_2 - 3x_1 + 2x_0 + v_0 \Delta t\right]$$

(7.7c)
$$a_n = \frac{2}{(\Delta t)^2}\left\{x_n - 3x_{n-1} + 2(-1)^n x_0 + 4\sum_{j=2}^{n-1}\left[(-1)^j x_{n-j}\right] + (-1)^n v_0 \Delta t\right\}, \quad n \geq 3$$

Formulas (7.3), (7.5), (7.6) and (7.7) are taken then to be the formulas for the velocity and acceleration of a particle moving in a fixed direction. The values x_i are the coordinates of the centroid of the particle at time t_i .

Two dimensional motion of a particle can be developed now in a natural way simply by delineating clearly between quantities which are vectors and quantities which are scalars. This is done as follows. Let $t_0, t_1, \ldots, t_n$ be a linearly ordered set of rational numbers. At time t_i let a particle be located at (x_i, y_i), $i = 0, 1, \ldots, n$. At time t_n the particle's x component of velocity $v_{x,n}$ and its y component of velocity $v_{y,n}$ are defined to be the scalars

$$(7.8)\qquad v_{x,n} = 2\sum_{j=0}^{n-1}\left[(-1)^j\frac{\Delta x_{n-j}}{\Delta t_{n-j}}\right] + (-1)^n v_{x,0}\,, \qquad n \geq 1$$

$$(7.9)\qquad v_{y,n} = 2\sum_{j=0}^{n-1}\left[(-1)^j\frac{\Delta y_{n-j}}{\Delta t_{n-j}}\right] + (-1)^n v_{y,0}\,, \qquad n \geq 1\,.$$

The velocity of the particle at time t_n is defined to be the vector

$$(7.10)\qquad \vec{v}_n = (v_{x,n}, v_{y,n}), \qquad n \geq 1.$$

In the special case $\Delta t_k \equiv \Delta t$, $k = 1, 2, \ldots, n$, (7.8) and (7.9) reduce to

$$(7.11a)\qquad v_{x,1} = \frac{2}{\Delta t}[x_1 - x_0] - v_{x,0}$$

$$(7.11b)\qquad v_{x,n} = \frac{2}{\Delta t}\{x_n + (-1)^n x_0 + 2\sum_{j=1}^{n-1}[(-1)^j(x_{n-j})]\} + (-1)^n v_{x,0}, \quad n \geq 2$$

$$(7.12a)\qquad v_{y,1} = \frac{2}{\Delta t}[y_1 - y_0] - v_{y,0}$$

(7.12b) $$v_{y,n} = \frac{2}{\Delta t} \{y_n + (-1)^n y_0 + 2 \sum_{j=1}^{n-1} [(-1)^j (y_{n-j})]\} + (-1)^n v_{y,0}, \quad n \geq 2.$$

The particle's x-component of acceleration $a_{x,n}$ and its y component of acceleration $a_{y,n}$ are defined to be the scalars

(7.13a) $$a_{x,1} = \frac{2}{\Delta t_1} \left[\frac{\Delta x_1}{\Delta t_1} - v_{x,0}\right]$$

(7.13b) $$a_{x,n} = \frac{2}{\Delta t_n} \left\{\frac{\Delta x_n}{\Delta t_n} + 2 \sum_{j=1}^{n-1} \left[(-1)^j \frac{\Delta x_{n-j}}{\Delta t_{n-j}}\right] + (-1)^n v_{x,0}\right\}, \quad n \geq 2$$

(7.14a) $$a_{y,1} = \frac{2}{\Delta t_1} \left[\frac{\Delta y_1}{\Delta t_1} - v_{y,0}\right]$$

(7.14b) $$a_{y,n} = \frac{2}{\Delta t_n} \left\{\frac{\Delta y_n}{\Delta t_n} + 2 \sum_{j=1}^{n-1} \left[(-1)^j \frac{\Delta y_{n-j}}{\Delta t_{n-j}}\right] + (-1)^n v_{y,0}\right\}, \quad n \geq 2.$$

The acceleration of the particle at time t_n is defined to be the vector

(7.15) $$\vec{a}_n = (a_{x,n}, a_{y,n}), \quad n \geq 1.$$

In the special case $\Delta t_k \equiv \Delta t$, $k = 1, 2, \ldots, n$, (7.13) and (7.14) reduce to

(7.16a) $$a_{x,1} = \frac{2}{(\Delta t)^2} [x_1 - x_0 - v_{x,0} \Delta t]$$

(7.16b) $$a_{x,2} = \frac{2}{(\Delta t)^2} [x_2 - 3x_1 + 2x_0 + v_{x,0} \Delta t]$$

$$(7.16c) \qquad a_{x,n} = \frac{2}{(\Delta t)^2} \left\{ x_n - 3x_{n-1} + 2(-1)^n x_0 + 4 \sum_{j=2}^{n-1} [(-1)^j x_{n-j}] + (-1)^n v_{x,0} \Delta t \right\}, \quad n \geq 3$$

$$(7.17a) \qquad a_{y,1} = \frac{2}{(\Delta t)^2} [y_1 - y_0 - v_{y,0} \Delta t]$$

$$(7.17b) \qquad a_{y,2} = \frac{2}{(\Delta t)^2} [y_2 - 3y_1 + 2y_0 + v_{y,0} \Delta t]$$

$$(7.17c) \qquad a_{y,n} = \frac{2}{(\Delta t)^2} \left\{ y_n - 3y_{n-1} + 2(-1)^n y_0 + 4 \sum_{j=2}^{n-1} [(-1)^j y_{n-j}] + (-1)^n v_{y,0} \Delta t \right\}, \quad n \geq 3 .$$

Of course in the special case when the motion of a particle is in a fixed direction, say that of the x-axis, then (7.8), (7.11), (7.13) and (7.16) reduce to (7.3), (7.6), (7.5) and (7.7), respectively, while (7.9) and (7.14) imply $v_{y,n} \equiv a_{y,n} \equiv 0, \quad n \geq 1$.

8. <u>THE LAW OF MOTION</u>

In the last section, it was shown, among other things, how one could determine a particle's acceleration from a knowledge of its position at $t_0, t_1, \ldots, t_n$. In this and in the next sections, we consider the problem of determining a particle's position from a knowledge of its acceleration at times $t_1, t_2, \ldots, t_n$.

Let $t_0, t_1, \ldots, t_n$ be a linearly ordered set of rational numbers. At time t_i, let a particle of mass m be located at (x_i, y_i), $i = 0, 1, 2, \ldots, n$, and let it be acted upon by a force $\vec{F}$. Then the motion of the particle is assumed to be governed by the vector equation

$$(8.1) \qquad \vec{F}(t_{i-1}, x_{i-1}, y_{i-1}, v_{x,i-1}, v_{y,i-1}) = m\,\vec{a}(t_i)\,, \quad i = 1, 2, \ldots, n.$$

Difference equation (8.1) is a generalized Newton's equation. The components of $\vec{F}$ can be given either in tabular form from experimental measurements or in the form of a mathematical expression, and in both cases the equation can be handled with facility on a high speed digital computer.

9. DAMPED MOTION IN A NONLINEAR FORCE FIELD

In order to illustrate the application of (8.1), we consider now a prototype problem of nonlinear mechanics, and only because interest has focused on $\vec{F}$ in the form of a mathematical expression will we do the same. Consider, then, as shown in Figure 9.1, a particle of unit mass which is constrained to move with its center Q on the x-axis. A displacement of the particle such that the directed distance OQ is x_i is opposed by a field force of magnitude $\sin x_i$ and a viscous damping force of magnitude αv_i, where α is a positive constant. Then the equation of motion, from (8.1), is

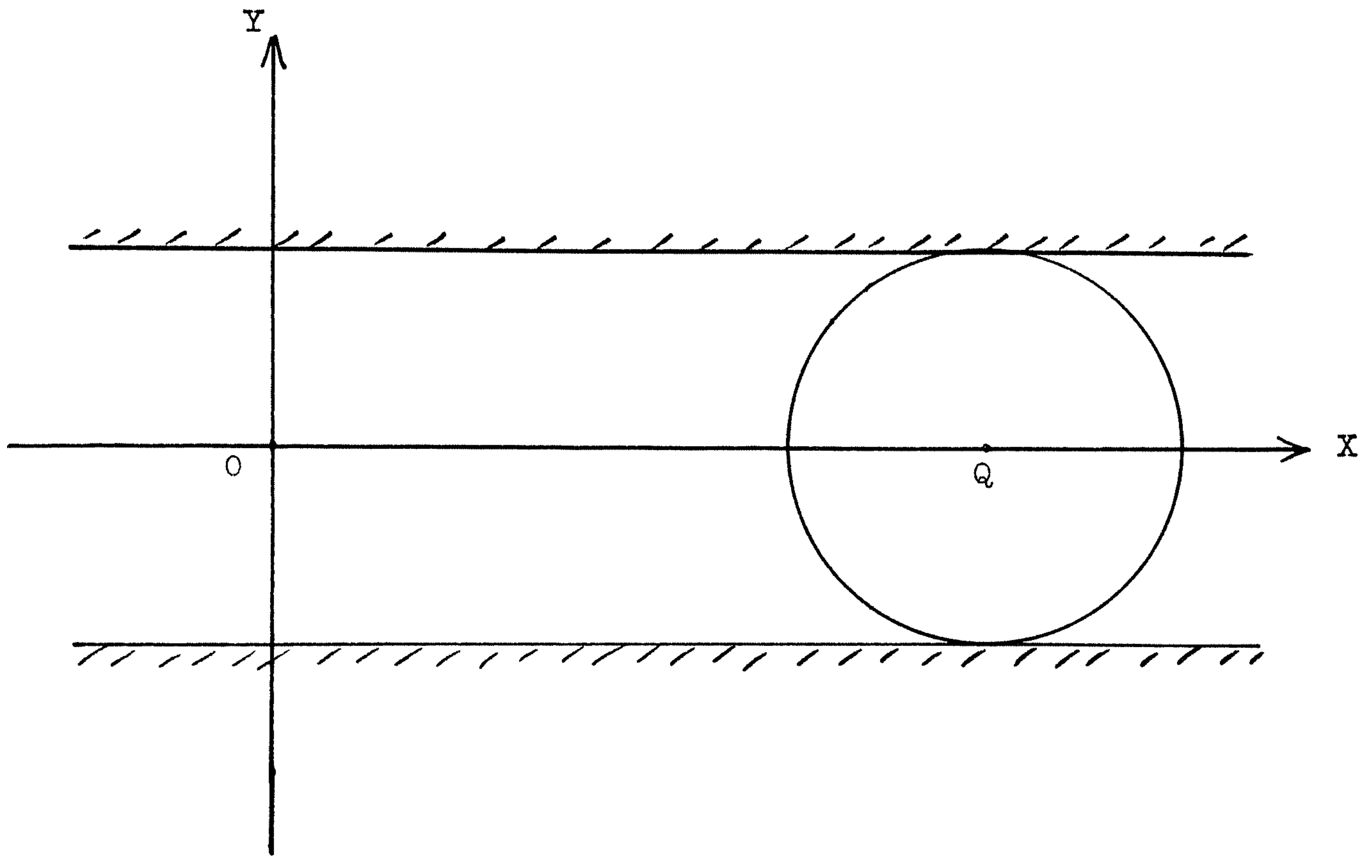


FIGURE 9.1

(9.1) $$a_i + \text{restoring force} = 0 ,$$

which, takes the particular form

(9.2) $$a_i + \alpha v_{i-1} + \sin x_{i-1} = 0 , \quad i = 1, 2, \ldots .$$

For constant Δt, one can rewrite (9.2) by means of (7.6) and (7.7) as

(9.3a) $$x_1 = x_0 + v_0 \Delta t - \frac{(\Delta t)^2}{2} [\alpha v_0 + \sin x_0]$$

(9.3b) $$x_2 = 3x_1 - 2x_0 - v_0 \Delta t - \frac{(\Delta t)^2}{2} \{\alpha [\frac{2}{\Delta t} (x_1 - x_0) - v_0] + \sin x_1\}$$

(9.3c) $$x_n = (3 - \alpha\Delta t)x_{n-1} + (-1)^{n-1}(2 - \alpha \Delta t)x_0 + (2 \alpha \Delta t - 4) \sum_{j=2}^{n-1} [(-1)^j x_{n-j}] + (-1)^{n-1} (1 - \frac{\alpha \Delta t}{2}) v_0 \Delta t - \frac{(\Delta t)^2}{2} \sin x_{n-1}, \quad n \geq 3 .$$

Suppose now that the particle is displaced to a point Q which has coordinate x_0, is held rigid (so that $v_0 = 0$), and is then released. We wish to trace the subsequent motion of the particle and consider first the particular set of parameters $\alpha = 0.3$, $\Delta t = 0.01$, $x_0 = \frac{\pi}{4}$. For $T = 150$, the solution (9.3) was generated on the UNIVAC 1108 in 33 seconds. That portion of the solution between $T = 0$ and $T = 25$ is shown graphically in Figure 9.2 and exhibits strong damping and peak values of 0.754615, -0.480136, 0.297837, -0.185740, 0.116061, -0.072585, 0.045407 and -0.028421 which occur at times 0, 3.27, 6.48, 9.66, 12.84, 16.01, 19.19, and 22.36, respectively. The time required for the particle to

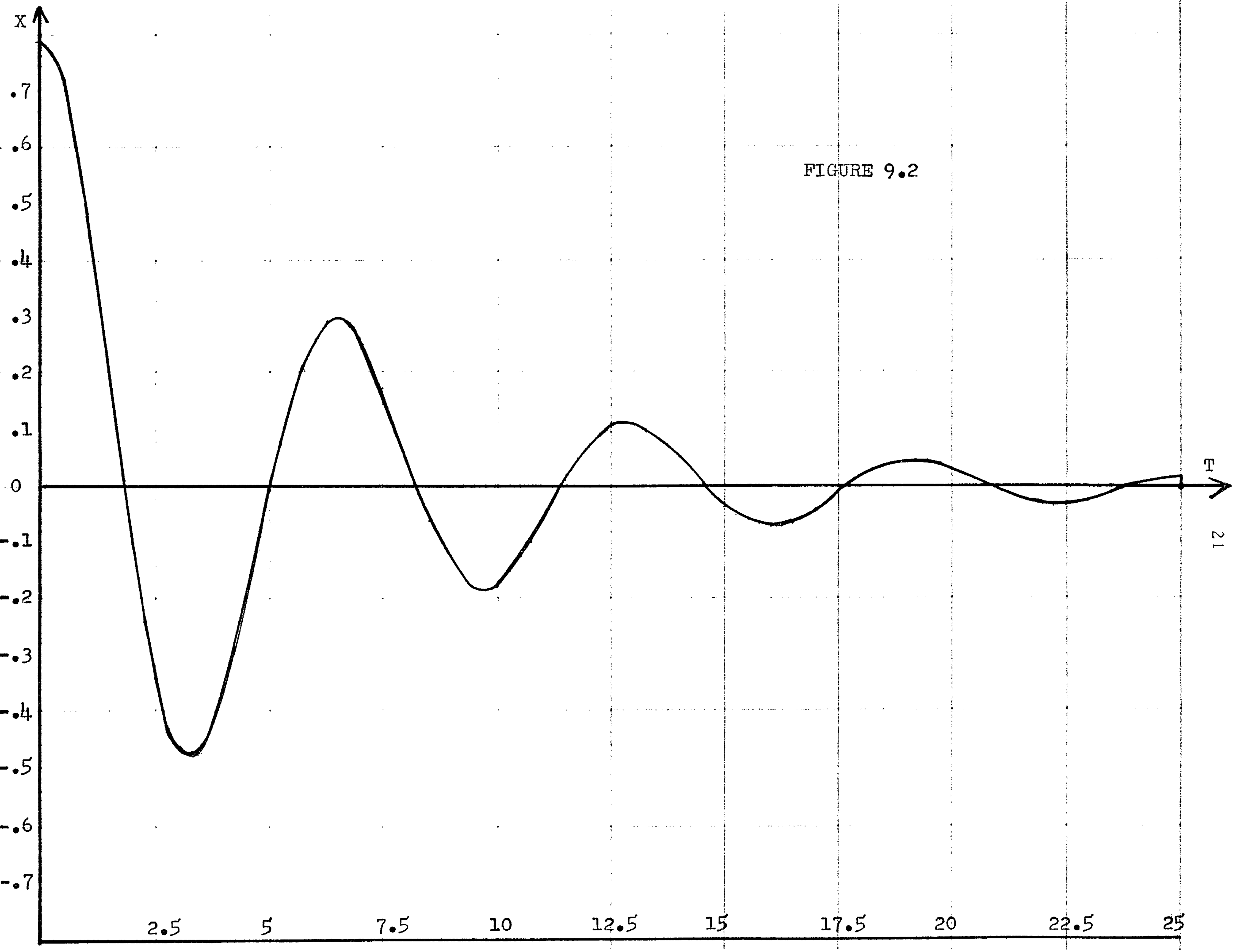


FIGURE 9.2

travel from one peak value to the next decreased monotonically until $T = 66.44$, at which time it was located at $x = -0.000039$ and had relocated to this value from $x = 0.000049$ in 1.73 time units. Though damped motion continued from $T = 66.44$ to $T = 81.09$, the time required between the attainment of successive peak values behaved erratically. However, from $T = 81.09$ until $T = 150$, damping ceased and the motion became completely periodic, one period of which is shown graphically in Figure 9.3.

A large variety of other examples were run with various combinations of input parameters chosen from

$$x_0 = 0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \pi, \frac{5\pi}{4},$$

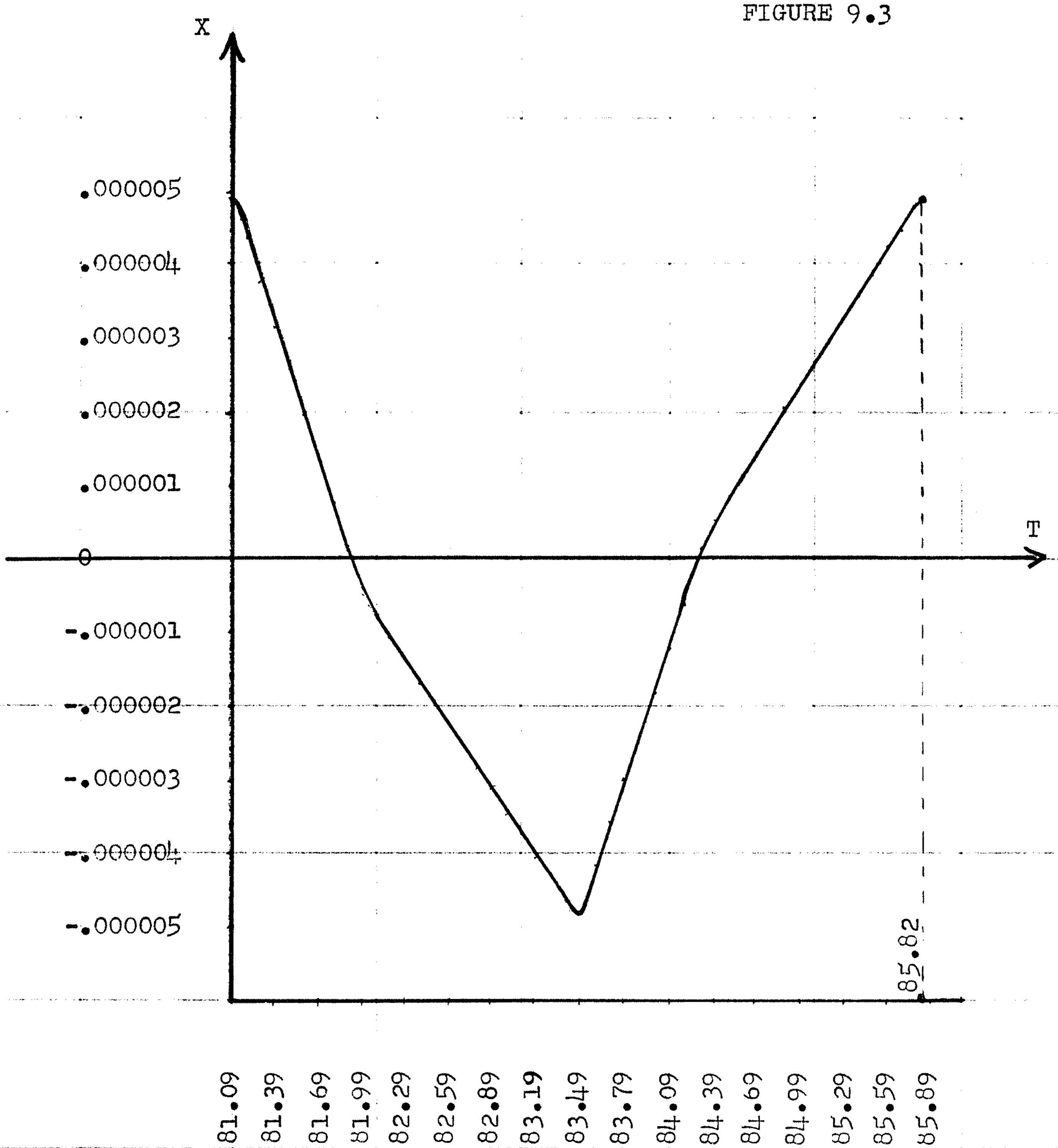
$$\Delta t = .0001, .001, .01, .02, .05, .1, 1$$

$$\alpha = 0, .0004, .0005, .001, .009, .01, .02, .024, .025, .04, .05, .06, .1, .2, .3, .49, 1.$$

In all cases damped motion resulted for $\Delta t \leq 2\alpha$, while sequence (9.3) usually diverged for $\Delta t > 2\alpha$. In all cases when Δt converged to 2α from above, the rate of divergence of (9.3) decreased monotonically with Δt when the solution diverged.

It is of interest to note that our solution of (9.2) in this section coupled with the ideas of Section 13 enable the interested reader to solve fully nonlinear pendulum problems.

FIGURE 9.3


X
T
.000005
.000004
.000003
.000002
.000001
0
-.000001
-.000002
-.000003
-.000004
-.000005
85.82
81.09
81.39
81.69
81.99
82.29
82.59
82.89
83.19
83.49
83.79
84.09
84.39
84.69
84.99
85.29
85.59
85.89

10. WORK (I)

Let $t_0, t_1, \ldots, t_n$ be a linearly ordered set of rational numbers. At time t_i, $i = 1, 2, \ldots, n$, let a particle be located at point (x_i, y_i) which is on the straight line through $A(x_0, y_0)$ and $B(x_n, y_n)$, one possible arrangement of which is shown in Figure 10.1. Let $f(t_i), i = 0, 1, \ldots, n$, represent the component in the direction $\overrightarrow{AB}$ of a force $\vec{F}$ applied to the particle. Then the work W done by $\vec{F}$ in moving the particle from A to B is defined to be

$$(10.1) \qquad W = \sum_{i=1}^{n} f(t_{i-1}) \Delta s_i \, ,$$

where $\Delta s_i \equiv s_i - s_{i-1}$ is the directed distance from (x_{i-1}, y_{i-1}) to (x_i, y_i).

11. ENERGY

With regard to the concept of work described in Section 10, it will be conveneient in this section to let a_i and v_i represent the acceleration and velocity, respectively, at (x_i, y_i), in the direction $\overrightarrow{AB}$. Then, from (8.1) and (10.1),

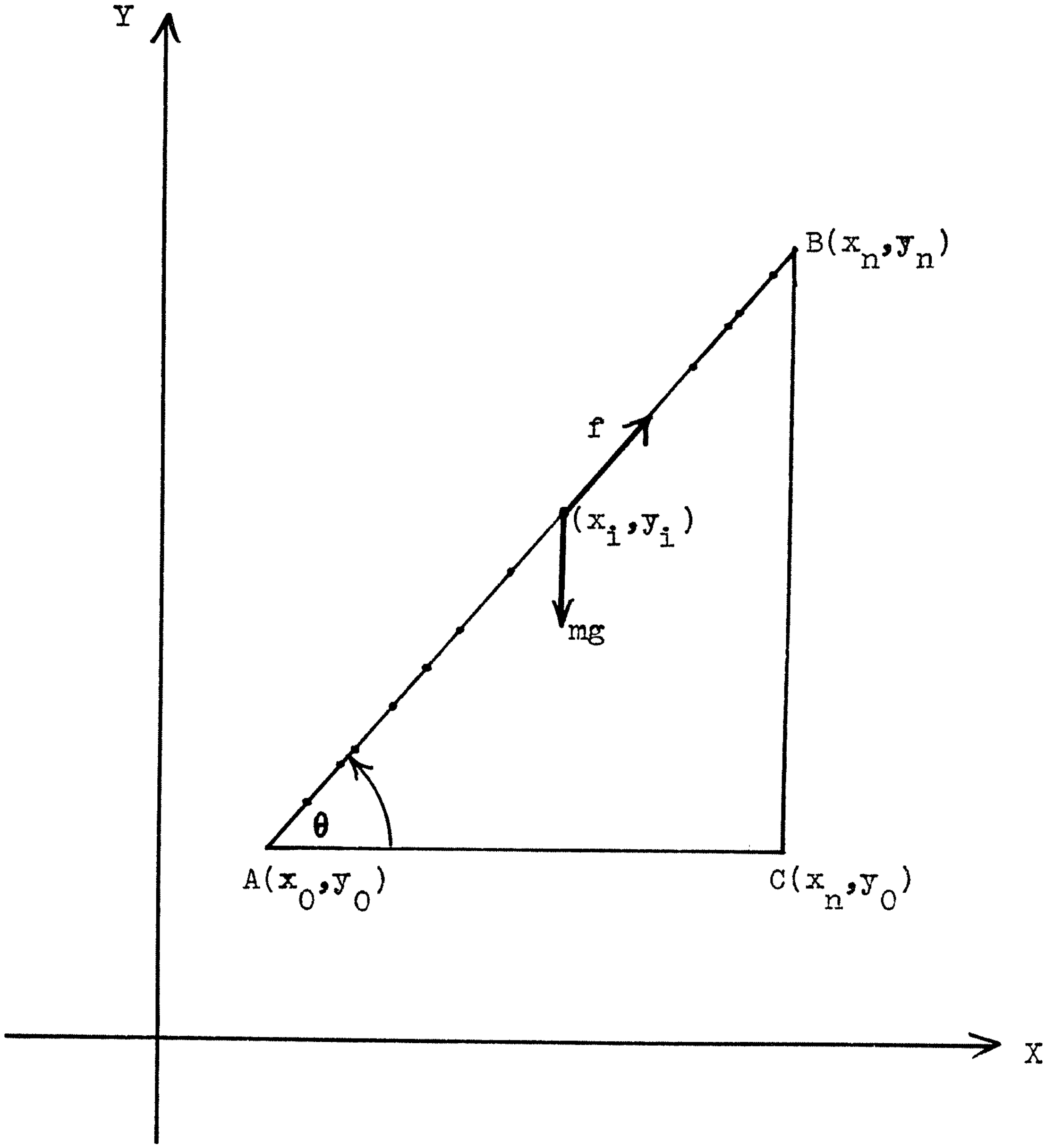


FIGURE 10.1

$$W = \sum_{i=1}^{n} f(t_{i-1})\Delta s_i$$

$$= m \sum_{i=1}^{n} a_i \Delta s_i$$

$$= m \sum_{i=1}^{n} \left[\left(\frac{v_i - v_{i-1}}{\Delta t_i}\right) \left(s_i - s_{i-1}\right)\right]$$

$$= m \sum_{i=1}^{n} \left[\left(v_i - v_{i-1}\right) \left(\frac{v_i + v_{i-1}}{2}\right)\right]$$

$$= \frac{m}{2} \sum_{i=1}^{n} [v_i^2 - v_{i-1}^2].$$

Thus,

(11.1) $$W = \frac{mv_n^2}{2} - \frac{mv_0^2}{2} .$$

The quantity

(11.2) $$K_i = \frac{1}{2} mv_i^2$$

is defined to be the kinetic energy of the particle at time t_i . Equation (11.1) states that the work done in moving the particle from A to B is the difference of the kinetic energies at B and at A .

Neglecting friction, the force necessary to move a particle of mass m only along the vertical component of $\overrightarrow{AB}$ must be equal to the weight mg of the particle. Hence, the amount of work done along the vertical component of the motion is

$$W = mg(y_n - y_0)$$

$$= mg y_n - mg y_0 .$$

The value

$$V_i = mgy_i \tag{11.3}$$

is called the gravitational potential energy of the particle at the point (x_i, y_i).

If now a particle of mass m is moved from A to B as described in Section 10, then, neglecting friction

$$f(t_{i-1}) = m \cdot a_i + mg \sin \theta , \tag{11.4}$$

where θ is the angle $\overrightarrow{AB}$ makes with the positive x-axis, as shown in Figure 10.1. Then

$$\begin{aligned} f(t_{i-1})\Delta s_i &= ma_i \Delta s_i + mg \sin \theta \, \Delta s_i \\ &= \frac{m}{2}(v_i^2 - v_{i-1}^2) + mg \, \Delta y_i \quad . \end{aligned}$$

Hence,

$$\begin{aligned} \sum_{i=1}^{n} [f(t_{i-1})\Delta s_i] &= \frac{m}{2} v_n^2 - \frac{m}{2} v_0^2 + mgy_n - mgy_0 \\ &= K_n - K_0 + V_n - V_0 . \end{aligned}$$

In the special case when the external force acting on a particle is zero, so that $f \equiv 0$, the last result implies that

$$K_n + V_n = K_0 + V_0 \quad , \tag{11.5}$$

which is called the principle of conservation of energy.

12. MOMENTUM

Suppose two particles move with their centers on an x-axis, as shown in Figure 12.1. Let the first particle have center A, mass m_A and velocity $v(A)$, and let the second particle have center B, mass m_B and velocity $v(B)$. We assume that the relative motion is such that the two bodies collide. During the collision, let the force of the first body on the second be F_A, and that of the second on the first be F_B. These forces are equal and opposite, but are non-constant. Both are small at first, increase to a maximum, and then decrease to zero, at which time the bodies separate. Hence, at time t_i during the period of contact

(12.1) $$F_A(t_{i-1}) = m_A\, a_i(A)\,,$$

so that

$$\sum_{i=1}^{n} [F_A(t_{i-1})\Delta t_i] = \sum_{i=1}^{n} m_A\, a_i(A)\,\Delta t_i$$

$$= m_A \sum_{i=1}^{n} [v_i(A) - v_{i-1}(A)]\,,$$

and

(12.2) $$\sum_{i=1}^{n} [F_A(t_{i-1})\Delta t_i] = m_A\,[v_n(A) - v_0(A)].$$

Similarly,

(12.3) $$\sum_{i=1}^{n} [F_B(t_{i-1})\Delta t_i] = m_B[v_n(B) - v_0(B)]\quad.$$

If the impulse I of a force F during the time of contact is defined to be

$$(12.4) \qquad I = \sum_{i=1}^{n} [F(t_{i-1})\Delta t_i]$$

and if the momentum M at time t_i of a particle of mass m is defined by

$$(12.5) \qquad M = m \cdot v_i \ ,$$

then (12.2) and (12.3) imply that the work done by either particle during the collision is equal to its change in momentum. Moreover, since at each time t_i, $F_A = -F_B$, substitution into (12.2) and (12.3) implies

$$m_A [v_n(A) - v_0(A)] = -m_B [v_n(B) - v_0(B)] \ ,$$

or, equivalently, that

$$(12.6) \qquad m_A v_0(A) + m_B v_0(B) = m_A v_n(A) + m_B v_n(B) \ ,$$

which is called the conservation law of linear momentum.

13. ANGULAR VELOCITY AND ACCELERATION

Thus far we have treated only the motion of a particle. If one wishes to study the motion of larger configurations, then, as is well known, that motion can be described completely by the rotations of the configuration and by the motion of its centroid. The motion of a centroid is that of a particle whose mass is that of the larger body and whose centroid is that of the larger body. Thus, our attention in the remainder of this paper will be directed to fundamental concepts and laws relating to the

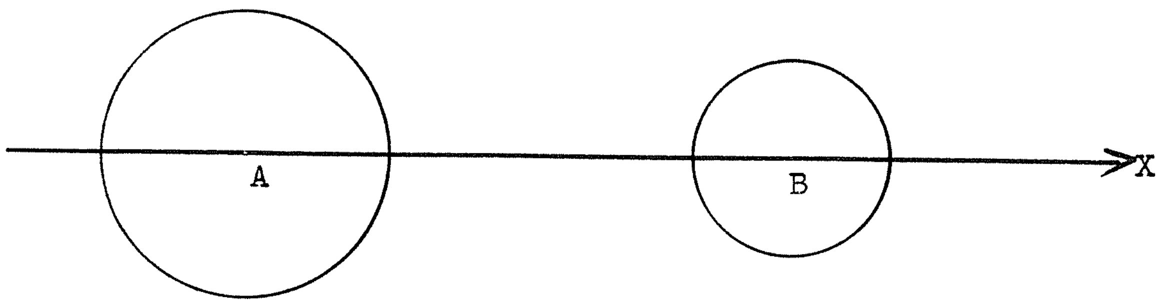


FIGURE 12.1

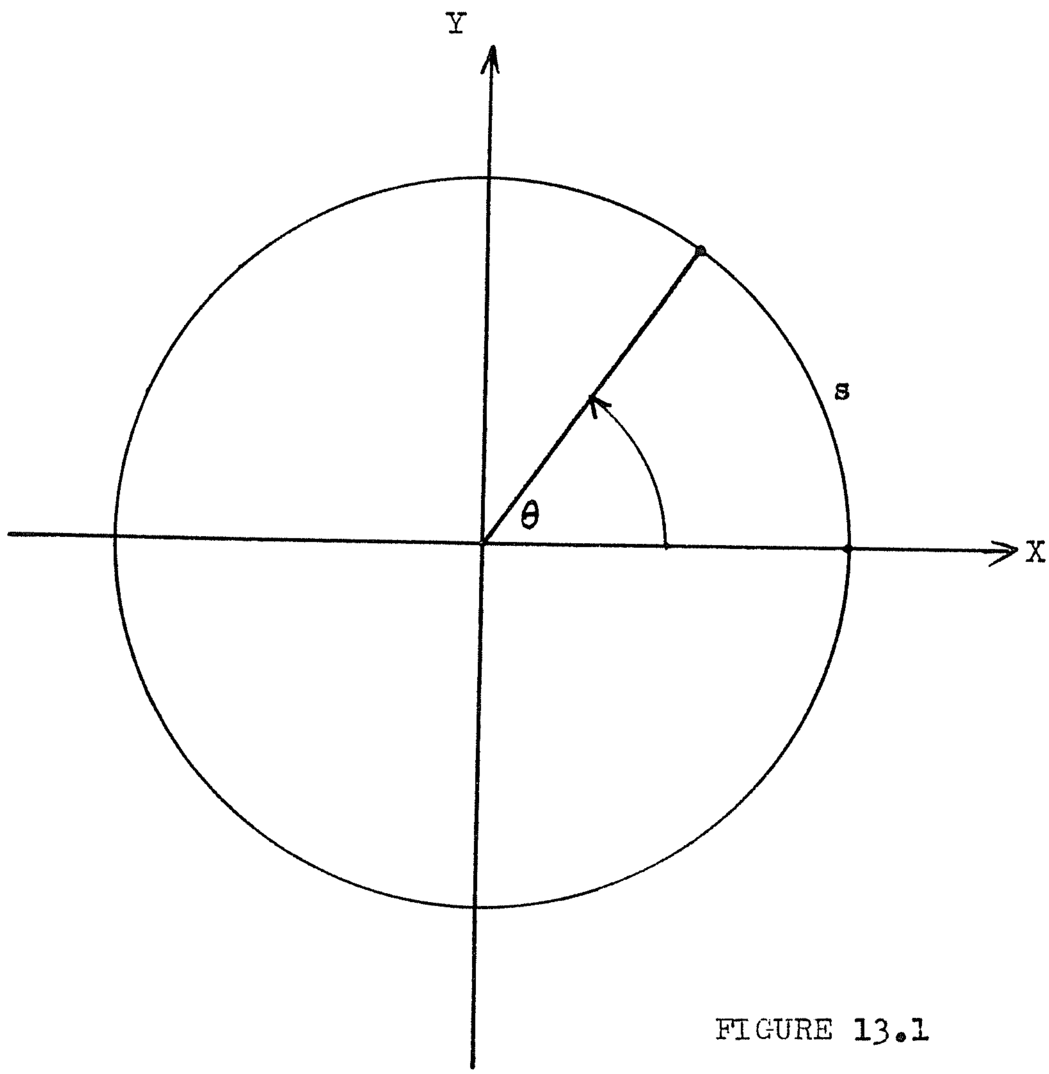


FIGURE 13.1

rotation of a large configuration.

Let C be a circle whose center is 0 and whose radius is r (see Figure 13.1). Let a central angle θ be subtended from an arc of length s . Then the radian measure of θ is defined by

$$s = r\theta \ . \tag{13.1}$$

Of course, sine the length s in practice can be measured only to an accuracy defined by a given ruler, the measurement of θ is also limited to an accuracy defined by the ruler and by r . Intuitively speaking, the length s of a circular section can be measured by a straight ruler by considering the curve to be a string, and by cutting it and laying it straight to be measured. Note that, interestingly enough, with a given ruler the larger one can take r the more accurately one can measure θ .

Next, let $t_0, t_1, \ldots, t_n$ be a linearly ordered set of rational numbers. Let a particle be in motion on a circle of radius r and at time t_i let the angle subtended by the arc from (r, 0) to the centroid of the particle be θ_i . Then the angular velocity $\omega(t_i) \equiv \omega_i$, $i = 1, 2, \ldots, n$, is defined implicitly by

$$\frac{\Delta\theta_i}{\Delta t_i} = \frac{\omega_i + \omega_{i-1}}{2} \ , \quad i = 1, 2, \ldots, n \ . \tag{13.2}$$

By analogy with (7.1) and (7.3), then

$$\omega_n = 2 \sum_{j=0}^{n-1} \left[(-1)^j \frac{\Delta\theta_{n-j}}{\Delta t_{n-j}}\right] + (-1)^n \omega_0 \ , \quad n \geq 1 \ . \tag{13.3}$$

The angular acceleration $\alpha(t_i) \equiv \alpha_i$, $i = 1, 2, \ldots, n$ is defined explicitly by

(13.4) $$\alpha_i = \frac{\Delta\omega_i}{\Delta t_i}, \quad i = 1, 2, \ldots, n,$$

and by analogy with (7.4) and (7.5) one has

(13.5a) $$\alpha_1 = \frac{2}{\Delta t_1}\left[\frac{\Delta\theta_1}{\Delta t_1} - \omega_0\right]$$

(13.5b) $$\alpha_n = \frac{2}{\Delta t_n}\left\{\frac{\Delta\theta_n}{\Delta t_n} + 2\sum_{j=1}^{n-1}\left[(-1)^j \frac{\Delta\theta_{n-j}}{\Delta t_{n-j}}\right] + (-1)^n \omega_0\right\}$$

In the case when all the Δt_i are equal to Δt, the formulas (13.3) and (13.5) become

(13.6a) $$\omega_1 = \frac{2}{\Delta t}\,[\theta_1 - \theta_0] - \omega_0$$

(13.6b) $$\omega_n = \frac{2}{\Delta t}\,\{\theta_n + (-1)^n \theta_0 + 2\sum_{j=1}^{n-1}[(-1)^j \theta_{n-j}]\} + (-1)^n \omega_0, \quad n \geq 2$$

and

(13.7a) $$\alpha_1 = \frac{2}{(\Delta t)^2}\,[\theta_1 - \theta_0 - \omega_0 \Delta t]$$

(13.7b) $$\alpha_2 = \frac{2}{(\Delta t)^2}\,[\theta_2 - 3\theta_1 + 2\theta_0 + \omega_0 \Delta t]$$

(13.7c) $$\alpha_n = \frac{2}{(\Delta t)^2}\,\{\theta_n - 3\theta_{n-1} + 2(-1)^n \theta_0 + 4\sum_{j=2}^{n-1}[(-1)^j \theta_{n-j}] + (-1)^n \omega_0 \Delta t\}, \quad n \geq 3.$$

Note that from (7.3), (7.5), (13.1) and (13.5) one has

(13.8) $$v_k = r\omega_k \; , \quad k = 1, 2, \ldots, n$$

(13.9) $$a_k = r\alpha_k \; , \quad k = 1, 2, \ldots, n$$

14. KINETIC ENERGY OF ROTATION

Consider now a rigid system of particles which is rotating about the origin 0 of the xy-coordinate system. The particles will be denoted by P_j, $j = 1, 2, \ldots, q$.

By means of a given ruler, we can measure more accurately the angular velocity of those particles which are further from 0 than those which are nearer. However, it seems reasonable to assume at present that each particle has the same angular velocity, so that we define the angular velocity of each P_j to be that of any one of the particles whose distance from 0 is maximal.

For particle P_j, let m_j denote its mass and r_j its distance from 0. If P_j's angular velocity is ω_i at time t_i, then the kinetic energy $k_{i,j}$ of P_j at time t_i is given by

(14.1) $$k_{i,j} = \frac{1}{2} m_j v_i^2 = \frac{1}{2} m_j r_j^2 \omega_i^2 \; .$$

From (14.1), the kinetic energy $K(t_i) \equiv K_i$ of the system at time t_i is

(14.2) $$K_i = \sum_{j=1}^{q} \frac{1}{2} m_j r_j^2 \omega_i^2 = \frac{\omega_i^2}{2} \sum_{j=1}^{q} m_j r_j^2 \; .$$

Defining the moment of intertia I of the system by

$$I = \sum_{j=1}^{q} m_j r_j^2 \tag{14.3}$$

implies that the rotational kinetic energy K_i in (14.2) can be given by

$$K_i = \frac{1}{2} I \omega_i^2 . \tag{14.4}$$

15. WORK (II)

In the notation of Section 14, at time t_i let force $F_{i,j}$ act upon P_j. Then the moment $M_{i,j}$ of $F_{i,j}$ is

$$M_{i,j} = d_{i,j} F_{i,j} , \tag{15.1}$$

where $d_{i,j}$ is the perpendicular distance from 0 to the line of action of $F_{i,j}$.

Now, at time t_{i-1}, let P_j have traveled s_{i-1}, subtend angle θ_{i-1}, and be acted upon by force $F_{i-1,j}$, whose line of action is perpendicular to the radius to P_j. At time t_i, let P_i have traveled s_i and subtend angle θ_i. Then the work $W_{i,j}$ done in the time interval Δt_i on P_j is

$$W_{i,j} = (F_{i-1,j}) \Delta s_i . \tag{15.2}$$

Hence

$$W_{i,j} = (F_{i-1,j}) r_j \Delta\theta_i = M_{i-1,j} \Delta\theta_i . \tag{15.3}$$

The total work W_i done on the system during the period Δt_i is

$$(15.4) \qquad W_i = \sum_{j=1}^{q} [M_{i-1,j} \, \Delta\theta_i] ,$$

while the total work W done on the system from the initial time t_0 until final time t_n is

$$(15.5) \qquad W = \sum_{i=1}^{n} \sum_{j=1}^{q} [M_{i-1,j} \, \Delta\theta_i] .$$

Since the work done is equal to the change in kinetic energy, it follows from (14.4) and (15.4) that

$$(15.6) \qquad \sum_{j=1}^{q} [M_{i-1,j} \, \Delta\theta_i] = \frac{1}{2} I (\omega_i^2 - \omega_{i-1}^2) .$$

But (15.6) implies

$$\sum_{j=1}^{q} M_{i-1,j} = I \cdot \frac{\Delta\omega_i}{\Delta t_i} \cdot \frac{\Delta t_i}{\Delta\theta_i} \cdot \frac{\omega_i + \omega_{i-1}}{2} = I \alpha_i ,$$

so that

$$(15.7) \qquad \sum_{j=1}^{q} M_{i-1,j} = I\alpha_i ,$$

which is the rotational analogue of (8.1).

The sum $\sum_{j=1}^{q} M_{i-1,j}$ is called the torque of the given system at time t_{i-1}.

16. ANGULAR MOMENTUM

The angular momentum L_i of a given system at time t_i is defined by

$$L_i = I\omega_i .$$

Hence,

$$\sum_{j=1}^{q} M_{i-1,j} = \frac{\Delta L_i}{\Delta t_i} ,$$

so that

$$\sum_{j=1}^{q} [M_{i-1,j} \Delta t_i] = \Delta L_i$$

and

(16.1) $$\sum_{i=1}^{n} \sum_{j=1}^{q} [M_{i-1,j} \Delta t_i] = L_n - L_0 .$$

Thus, if the torque of the system is zero at each time $t_0, t_1, \ldots, t_n$, then from (16.1)

(16.2) $$L_n = L_0 ,$$

which is called the law of conservation of angular momentum.

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