40

*Research supported by the Mathematics Research Center, United States Army, Madison, Wisconsin, under Contract No. DA-31-124-ARO-D-462, and by the Office of Naval Research under Contract No. N-00014-67-A-0128-0004.

**Research supported by the Office of Naval Research under Contract No. N-00014-67-A-0128-0004.

SINGULAR PERTURBATIONS OF NON-LINEAR BOUNDARY VALUE PROBLEMS WITH TURNING POINTS

by

Fred W. Dorr*

&

Seymour V. Parter**

Technical Report #47

November 1968

SINGULAR PERTURBATIONS OF NON-LINEAR BOUNDARY VALUE PROBLEMS WITH TURNING POINTS

1. INTRODUCTION

This paper is concerned with the asymptotic behavior as $\epsilon \to 0+$ of solutions $u(t) = u(t, \epsilon)$ and $v(t) = v(t, \epsilon)$ to non-linear boundary value problems of the form

$$(1.1) \qquad \begin{cases} u'' = f(t, u, v) & (0 < t < 1) \\ u(0) = u(1) = 0 \end{cases}$$

$$(1.2) \qquad \begin{cases} \epsilon v'' + g(t, u, u')v' - c(t, u, u')v = 0 & (0 < t < 1) \\ v(0) = v_0, \quad v(1) = v_1 \end{cases}$$

We assume that $0 \le v_0 < v_1$, $c(t, u, u') \ge 0$, and $|f(t, u, v)| \le f_0(t, v)$.

We are particularly concerned with problems in which there is exactly one interior turning point for equation (1.2). That is, for each $\epsilon > 0$ there is a unique point $\alpha \in (0, 1)$ such that $g(\alpha, u(\alpha), u'(\alpha)) = 0$, and $g(t, u(t), u'(t))$ changes sign in a neighborhood of $t = \alpha$. In general α depends on ϵ, and is not known a-priori. This behavior occurs, for example, in the cases

$$(1.3) \qquad f(t, u, v) = \pm v, \quad g(t, u, u') = u'$$

These problems may be considered as one-dimensional analogs of the steady state Navier-Stokes equations in the form

$$
(1.4) \quad \begin{cases} \Delta\psi = -\omega & \text{in } G \\ \Delta\omega + R(\psi_x \omega_y - \psi_y \omega_x) = 0 & \text{in } G \\ \psi, \omega \text{ prescribed on } \partial G \end{cases} .
$$

where $R = \frac{1}{\epsilon}$ is the Reynolds number. Problem (1.4) has been studied numerically as $R \to +\infty$ by Greenspan [10]. With his choice of boundary conditions the non-linear partial differential equation always has an interior singular point ("stagnation point"), and the usual asymptotic analysis does not apply (see [15]). The asymptotic behavior of solutions to the Navier-Stokes equations has been studied by Batchelor [1-3] and others [4, 5, 8, 13, 19]. These authors, however, make substantial use of physical arguments as well as mathematical ones. In an effort to gain insight into such problems, we have therefore turned to the one-dimensional models (1.1) - (1.2).*

There is an extensive literature on singular perturbation and turning point problems for ordinary differential equations. A comprehensive bibliography is given in Wasow [22]. Specific examples of problems of the type we consider have been treated by Wasow [20, 21] and Cochran [6]. Macki [16] and Harris [11] have treated similar non-linear first order

---

*It is interesting to note that Batchelor's argument cannot be applied in these one-dimensional problems, since it would require that $\{t \mid u(t) = u_0\}$ be a connected set for any constant u_0.

systems in which one equation is reduced in order as $\epsilon \to 0+$.

In Section 2 we collect some preliminary results. Section 3 is devoted to the problems (1.3) and their generalizations. Problems with turning points at the ends of the interval are considered in Section 4. In Section 5 we study problems for which $c(t, u, u') \geq c_0 > 0$. In Section 6 we collect some remarks on further applications of the methods developed in this paper.

2. PRELIMINARY RESULTS

We first prove that the coupled boundary value problems (1.1) - (1.2) have solutions in $C^2[0, 1]$ for each fixed $\epsilon > 0$. Assume that v_0 and v_1 are two real numbers with $0 \le v_0 < v_1$. Define $S_1 = [0, 1] \times [0, v_1]$, and let $f_0(x, y) \in C(S_1)$ with $f_0(x, y) \ge 0$. Choose $M > 0$ so that $f_0(x, y) \le M$, and define

$$S_2 = [0, 1] \times [-\tfrac{1}{8}M, \tfrac{1}{8}M] \times [0, v_1]$$

and

$$S_3 = [0, 1] \times [-\tfrac{1}{8}M, \tfrac{1}{8}M] \times [-\tfrac{1}{2}M, \tfrac{1}{2}M].$$

Assume that:

$$(2.1)\quad \begin{cases} \text{(a)} & f(x, y, z) \in C(S_2) \text{ and} \\ & |f(x, y, z)| \le f_0(x, z) \text{ for } (x, y, z) \in S_2 \\ \text{(b)} & g(x, y, z) \in C(S_3) \\ \text{(c)} & c(x, y, z) \in C(S_3) \text{ and} \\ & c(x, y, z) \ge 0 \end{cases}$$

Under these conditions we have:

Theorem 1. For each fixed ϵ with $0 < \epsilon < \infty$, there are solutions $u(t) = u(t, \epsilon)$ and $v(t) = v(t, \epsilon)$ in $C^2[0, 1]$ to (1.1) - (1.2).

Proof. Choose constants $M_1, M_2, M_3 > 0$ so that $|f| \le M_1$, $|g| \le M_2$, and $|c| \le M_3$. Define $M_4 = \exp(\frac{1}{\epsilon}M_2)[v_1 - v_0 + \frac{1}{\epsilon}v_1 M_3 \exp(\frac{2M_2}{\epsilon})]$,

$M_5 = \exp(\frac{1}{\epsilon} M_2)\,[M_4 + \frac{1}{\epsilon} v_1 M_3 \exp(\frac{1}{\epsilon} M_2)]$, and $M_6 = \frac{1}{\epsilon}(M_2 M_5 + v_1 M_3)$.
Introduce a norm on $C^1[0, 1]$ by setting $\|u\| = \max(\|u\|_\infty, \|u'\|_\infty)$,
and define the following two sets:

$$K_1 = \{u \in C^2[0, 1] \mid u(0) = u(1) = 0, \quad \|u\|_\infty \le \frac{1}{8} M,$$
$$\|u'\|_\infty \le \frac{1}{2} M, \quad \|u''\|_\infty \le M_1\}$$

$$K_2 = \{v \in C^2[0, 1] \mid v(0) = v_0, \quad v(1) = v_1, \quad 0 \le v(t) \le v_1,$$
$$\|v'\|_\infty \le M_5, \quad \|v''\|_\infty \le M_6\}$$

If we define $K = K_1 \times K_2$ with a topology induced from that of $C^1[0,1] \times C^1[0, 1]$, then $\overline{K}$ is convex and compact by the Arzela-Ascoli Theorem [7, p. 266]. Define an operator $T: \overline{K} \to C^2[0, 1] \times C^2[0, 1]$ by $T(u, v) = (\overline{u}, \overline{v})$, where $\overline{u}(t)$, $\overline{v}(t)$ are solutions to the boundary value problems

$$(2.2) \qquad \begin{cases} \overline{u}'' = f(t, u, v) & (0 < t < 1) \\ \overline{u}(0) = \overline{u}(1) = 0 \end{cases}$$

$$(2.3) \qquad \begin{cases} \epsilon \overline{v}'' + g(t, u, u')\overline{v}' - c(t, u, u')\overline{v} = 0 & (0 < t < 1) \\ \overline{v}(0) = v_0, \quad \overline{v}(1) = v_1 \end{cases}$$

T is well-defined and continuous by the maximum principle for differential operators of the form appearing in (2.2) - (2.3) (see [18, p. 16]). Furthermore, it is easy to show that $T(\overline{K}) \subset \overline{K}$, so that T has a fixed point $(u, v) \in \overline{K}$ by the Schauder fixed point theorem [7, p. 456]. These functions $u(t)$, $v(t)$ are then in $C^2[0, 1]$ and satisfy (1.1) - (1.2).

Remark. It is clear that Theorem 1 can be applied to problems with $f(x, y, z) = f_0(x, z) f_1(y)$ where $f_1(y)$ is uniformly bounded. A slight modification of the proof shows that the result is also true for "mildly nonlinear" problems. Let $S_4 = [0, 1] \times (-\infty, \infty) \times [0, v_1]$, and assume that, instead of (2.1a), we have:

$$(2.4) \quad \begin{cases} \text{(a)} & f(x, y, z) \in C(S_4) \\ \text{(b)} & \dfrac{\partial f}{\partial y} \in C(S_4) \\ \text{(c)} & \dfrac{\partial f}{\partial y} \geq -\sigma > -\pi^2 \qquad ((x, y, z) \in S_4) \end{cases}$$

Let $M_1 = \max_{(x, z) \in S_5} |f(x, 0, z)|$ where $S_5 = [0, 1] \times [0, v_1]$. We replace (2.2) with

$$(2.5) \quad \begin{cases} \bar{u}'' = f(t, \bar{u}, v) & (0 < t < 1) \\ \bar{u}(0) = \bar{u}(1) = 0 \end{cases}$$

Lees [14] has shown that (2.5) has a unique solution $\bar{u}(t)$, and furthermore $\|\bar{u}\|_\infty \leq KM_1$, where $K = \pi/2(\pi^2 - \sigma)$. We then let $M_2 = \max_{S_6} |f(x, y, z)|$ where $S_6 = [0, 1] \times [-KM_1, KM_1] \times [0, v_1]$, so that $\|\bar{u}'\|_\infty \leq \frac{1}{2} M_2$ and $\|\bar{u}''\|_\infty \leq M_2$. The rest of the proof of Theorem 1 follows as before with the use of the results of [14].

We note that the above remark allows us to prove the existence of solutions to the problem

$$(2.6) \qquad \begin{cases} u'' = v(1+u) \\ \epsilon v'' + g(t,u,u')\,v' = 0 \end{cases}$$

with the usual boundary conditions. The theorems we prove in the following sections generally require that $f(t,u,v) \neq 0$ if $v > 0$. For (2.6), if $\pi v_1 < 2(\pi^2 + v_0)$ we see that $|u| < 1$, and hence $v(1+u) > 0$ if $v > 0$.

In the following sections we need only be concerned with the asymptotic behavior of $v(t,\epsilon)$. For, assume that we have a sequence $\epsilon_n \to 0$ and a function $\bar{v}(t)$ such that $\lim_{n\to\infty} v(t,\epsilon_n) = \bar{v}(t)$ pointwise almost everywhere (a.e.) on $[0,1]$ (this then implies that $\bar{v}(t) \in L^\infty[0,1]$). Notice that $u(t,\epsilon)$ satisfies the integral equation

$$(2.7) \qquad \begin{cases} u(t,\epsilon) = (t-1)\displaystyle\int_0^t \tau f(\tau, u(\tau,\epsilon), v(\tau,\epsilon))\,d\tau \\ \qquad\qquad - t\displaystyle\int_t^1 (1-\tau) f(\tau, u(\tau,\epsilon), v(\tau,\epsilon))\,d\tau \end{cases}$$

Since $\{u(t,\epsilon_n)\}$, $\{u'(t,\epsilon_n)\}$, and $\{u''(t,\epsilon_n)\}$ are all uniformly bounded, by the Arzela-Ascoli Theorem there is a subsequence $\epsilon_{n(k)}$ of ϵ_n and a function $\bar{u}(t) \in C^1[0,1]$ such that

$$(2.8) \qquad \begin{cases} \lim\limits_{k\to\infty} u(t,\epsilon_{n(k)}) = \bar{u}(t) \\ \lim\limits_{k\to\infty} u'(t,\epsilon_{n(k)}) = \bar{u}'(t) \end{cases}$$

Furthermore, the convergence in (2.8) is uniform on $[0, 1]$. We can then let $\epsilon = \epsilon_{n(k)} \to 0$ in (2.7) to find that $\bar{u}(t)$ satisfies

$$(2.9) \qquad \begin{cases} \bar{u}(t) = (t-1) \int_0^t \tau f(\tau, \bar{u}(\tau), \bar{v}(\tau))\, d\tau \\ \qquad - t \int_t^1 (1-\tau) f(\tau, \bar{u}(\tau), \bar{v}(\tau))\, d\tau \end{cases}$$

In the event that the solution $\bar{u}(t)$ is unique, we can take $\epsilon_{n(k)} = \epsilon_n$ in (2.8). We also see from (2.9) that $\bar{u}(0) = \bar{u}(1) = 0$, so the <u>boundary values of u(t) are always preserved</u>.

Motivated by these remarks, we define S_0 to be the set of all $\bar{v}(t) \in L^1[0, 1]$ such that there exists a sequence $\epsilon_n \to 0$ with $\lim_{n \to \infty} v(t, \epsilon_n) = \bar{v}(t)$ pointwise a.e. on $[0, 1]$. Then we have:

<u>Theorem 2.</u> (a) S_0 is not empty.

(b) Assume that $g(x, y, z) \in C^1(S_3)$. For any $\bar{v}(t) \in S_0$, let $\bar{u}(t)$ be as in (2.8). Assume that there is an interval $(a, b) \subset (0, 1)$ with $g(t, \bar{u}(t), \bar{u}'(t)) \neq 0$ for $t \in (a, b)$. Then $\bar{v}(t) \in C^1(a, b)$, and

$$(2.10) \qquad g(t, \bar{u}, \bar{u}')\bar{v}' - c(t, \bar{u}, \bar{u}')\bar{v} = 0 \qquad (a < t < b)$$

<u>Proof.</u> It follows from the maximum principle (see [18, p. 7]) that, if there exists a $t_0 \in [0, 1)$ such that $v'(t_0, \epsilon) \geq 0$, then $v'(t, \epsilon) \geq 0$ for $t_0 \leq t \leq 1$. Let $t_0(\epsilon)$ be the smallest zero of $v'(t, \epsilon)$ if $v'(0, \epsilon) < 0$, and let $t_0(\epsilon) = 0$ if $v'(0, \epsilon) \geq 0$. We then have

$$v'(t,\epsilon)\quad \begin{cases} < 0 & 0 \le t < t_0(\epsilon) \\ \ge 0 & t_0(\epsilon) \le t \le 1 \end{cases}$$

If we let $V(h)$ be the total variation over $[0, 1]$ of an arbitrary function $h(t)$, it follows that $V(v(t,\epsilon)) = v_1 + v_0 - 2v(t_0(\epsilon), \epsilon)$. Since $v(t,\epsilon) \ge 0$, we have

(2.11) $$V(v(t,\epsilon)) \le v_1 + v_0$$

Then (a) follows from the Helly selection Theorem [17, p. 222], and in fact $\lim_{n\to\infty} v(t, \epsilon_n) = \overline{v}(t)$ for all $t \in [0, 1]$. To prove (b), define a differential operator E by

$$E\phi(t) = \overline{g}(t)\phi'(t) - \overline{c}(t)\phi(t) \qquad (a < t < b)$$

where $\overline{g}(t) = g(t, \overline{u}(t), \overline{u}'(t))$ and $\overline{c}(t) = c(t, \overline{u}(t), \overline{u}'(t))$. Let $\phi(t) \in C^{\infty}(a, b)$ be such that support $\phi(t) \subset [c, d]$ with $a < c < d < b$ (we then write $\phi(t) \in C_0^{\infty}(a, b)$). Using the inner product $(p, q) = \int_a^b p(t)\,q(t)\,dt$, it follows that

$$\begin{aligned} 0 &= (\phi, \epsilon v'' + gv' - cv) \\ &= \epsilon(\phi'', v) - ((g\phi)', v) - (c\phi, v) \end{aligned}$$

Since $\|u''\|_{\infty} \le M_1$, we have $|u'(x,\epsilon_n) - u'(y,\epsilon_n)| \le M_1 |x-y|$. Thus $|\overline{u}'(x) - \overline{u}'(y)| \le M_1 |x-y|$, so $\overline{u}'(t)$ satisfies a uniform Lipschitz condition on $[0,1]$, and hence is absolutely continuous. Thus we see that

$$(2.12) \qquad ((\overline{g}\phi)', \overline{v}) + (\overline{c}\phi, \overline{v}) = 0$$

The formal adjoint E^* of E is defined by $E^*\phi = (-(\overline{g}\phi)' - \overline{c}\phi)$. Thus (2.12) is equivalent to $(E^*\phi, \overline{v}) = 0$ for all $\phi \in C_0^\infty(a, b)$. A density argument then shows that $(E^*\phi, \overline{v}) = 0$ for all $\phi \in C_0^1(a, b)$. Let R' be any interval of the form $(a + \delta, b - \delta)$ with $\delta > 0$ but δ small. By a theorem of Friedrichs*, there is a sequence $v_n(t) \in C_0^1(a, b)$ such that:

$$(2.13) \qquad \lim_{n \to \infty} \| v_n - \overline{v} \|_{L^2(R')} = 0$$

$$(2.14) \qquad \lim_{n \to \infty} \| Ev_n \|_{L^2(R')} = 0$$

Equation (2.14) is the same as

$$(2.15) \qquad \lim_{n \to \infty} \| \overline{g}\, v'_n - \overline{c}\, v_n \|_{L^2(R')} = 0$$

Since $|\overline{g}(t)| \geq C_1 > 0$ on R', it follows from (2.13) - (2.15) that $\| v'_n \|_{L^2(R')} \leq C_2$. An elementary argument then shows that $\overline{v}(t)$ is absolutely continuous on R', and that (2.10) holds on R'. Since R' is arbitrary, this completes the proof of Theorem 2.

---

*We remark that the Friedrichs Theorem [9, p. 135] requires $\overline{u'} \in C^1(a, b)$. However, it is clear in [9], and in the proof of this result given by Hormander [12], that the C^1 requirement is needed only to permit integration by parts. Thus the absolute continuity of $\overline{u'}$ is sufficient to make the theorem applicable.

Remarks. (1) Let g, $\bar{u}$, $\bar{v}$, and (a, b) satisfy the conditions of Theorem 2(b). Then for any $t_0 \in (a, b)$ we have

$$(2.16) \qquad \bar{v}(t) = \bar{v}(t_0) \exp\left(\int_{t_0}^{t} \frac{\bar{c}(\tau)}{\bar{g}(\tau)}\, d\tau\right) \qquad (a < t < b)$$

Assume that:

$$(2.17) \quad \begin{cases} \text{(a)} & \bar{g}(t) > 0 \quad \text{for} \quad t \in (a, b) \\ \text{(b)} & \lim_{t \to b^-} \bar{g}(t) = 0 \\ \text{(c)} & \bar{c}(t) \geq c_0 > 0 \quad \text{in some interval } [b - \delta, b],\ \delta > 0 \end{cases}$$

We can then conclude that $\bar{v}(t) = 0$ for $t \in (a, b)$. For, suppose we fix $t_0 \in (a, b)$. It is easy to show that $\bar{g}(t)$ satisfies a uniform Lipschitz condition on [0, 1], and this fact together with (2.17) implies that $\lim_{t \to b^-} \int_{t_0}^{t} \frac{\bar{c}(\tau)}{\bar{g}(\tau)}\, d\tau = +\infty$. Since $\bar{v}(t)$ is bounded, we have $\bar{v}(t_0) = 0$. A similar argument shows that $\bar{v}(t) = 0$ for $t \in (a, b)$ if the following hold:

$$(2.18) \quad \begin{cases} \text{(a)} & \bar{g}(t) < 0 \quad \text{for} \quad t \in (a, b) \\ \text{(b)} & \lim_{t \to a^+} \bar{g}(t) = 0 \\ \text{(c)} & \bar{c}(t) \geq c_0 > 0 \quad \text{in some interval } [a, a+\delta],\ \delta > 0 \end{cases}$$

(2) If we have more precise information about the form of f(x, y, z) and g(x, y, z), we can make a stronger statement. Assume that g, $\bar{u}$, and $\bar{v}$ satisfy the conditions of Theorem 2(b), and that

in addition we have:

$$(2.19)\quad \begin{cases} \text{(a)} & f(x,y,z) = z\, f_1(x,y), \text{ and } |f_1(x,y)| \geq \sigma > 0 \\ \text{(b)} & g(x,y,z) = z\, g_1(x,y), \text{ and if } g_1(x,y) = 0 \text{ then } y = 0 \\ \text{(c)} & c(x,y,z) \geq c_0 > 0 \end{cases}$$

Also assume that:

$$(2.20)\quad \begin{cases} \text{(a)} & \text{There exists an } \alpha \in (0,1) \text{ such that } \bar{u}'(\alpha) = 0 \\ \text{(b)} & \text{There exists an interval } I = (\alpha - \delta, \alpha) \text{ or } I = (\alpha, \alpha + \delta) \\ & \text{for } \delta > 0 \text{ such that } \bar{u}'(t) \neq 0 \text{ if } t \in I \end{cases}$$

We then conclude that $\bar{v}(t) = 0$ for $t \in I$. For, suppose $\bar{v}(t) \not\equiv 0$ for $t \in I$. Then we must have $\bar{u}(t) \not\equiv 0$ for $t \in I$, and hence by the convexity (or concavity, depending upon the sign of $f_1(x,y)$) of $\bar{u}(t)$ we have $\bar{u}(t) \neq 0$ for $t \in I$. Thus $\bar{g}(t) \neq 0$ for $t \in I$. Suppose $I = (\alpha - \delta, \alpha)$, and let $t_0 \in I$ and $t \in (t_0, \alpha)$. Then

$$(2.21)\quad \begin{cases} \bar{u}'' = \bar{v}\, f_1(t, \bar{u}) & (t \in I) \\ \bar{u}' \;\; g_1(t,\bar{u})\, \bar{v}' = \bar{c}\, \bar{v} & (t \in I) \end{cases}$$

Thus $\bar{u}(t)$ satisfies

$$(2.22)\quad \bar{u}''(\tau) - F(\tau)\bar{u}'(\tau) = 0 \qquad (t_0 < \tau < t)$$

where $F(t) = f_1(t, \bar{u}(t))\, g_1(t, \bar{u}(t))\, \bar{v}'(t)\, (\bar{c}(t))^{-1}$. We can integrate (2.22) to obtain

$$(2.23)\quad \bar{u}'(t) = \bar{u}'(t_0) \exp\left(\int_{t_0}^{t} F(\tau)\, d\tau\right)$$

Now $\overline{u}'(t) \in C[0, 1]$ and $F \in L^1(I)$, so we let $t \to \alpha$ in (2.23) and find that $\overline{u}'(t_0) = 0$. Since $t_0 \in I$ is arbitrary we have $\overline{u}'(t) = 0$ for $t \in I$. The proof for $I = (\alpha, \alpha + \delta)$ follows in the same fashion, and this completes the proof that $\overline{v}(t) = 0$ for $t \in I$.

(3) The results of the following sections will typically show that $\overline{v}(t) = C$ for $t \in (a, b)$, where C is a constant and $(a, b) \subset (0, 1)$. If $C > 0$ it is also easy to show that $\lim_{n \to \infty} v'(t, \epsilon_n) = 0$ for $t \in (a, b)$. Furthermore, in both cases the convergence will be uniform on any interval $[c, d]$ with $a < c < d < b$.

3. PROBLEMS WITH INTERIOR TURNING POINTS

The problems we consider in this section are motivated by the special cases $g(t, u, u') = u'$ and $f(t, u, v) = \pm v$. It will be seen that $u'(t, \epsilon)$ has exactly one zero in $[0,1]$, and this occurs at $t = \alpha(\epsilon) \in (0, 1)$. Furthermore, $\pm u'(t, \epsilon) < 0$ for $0 \le t < \alpha(\epsilon)$, and $\pm u'(t, \epsilon) > 0$ for $\alpha(\epsilon) < t \le 1$. Thus equations (1.1) - (1.2) have exactly one (unknown) turning point at $\alpha(\epsilon)$.

Theorem 3. Let $u(t) = u(t, \epsilon)$ and $v(t) = v(t, \epsilon)$ be solutions in $C^2[0, 1]$ to

$$(3.1) \qquad \begin{cases} u''(t) = v(t) & (0 < t < 1) \\ u(0) = u(1) = 0 & \\ \epsilon v''(t) + u'(t)\, v'(t) = 0 & (0 < t < 1) \\ v(0) = v_0, \quad v(1) = v_1 & \end{cases}$$

Let $d = v_0/v_1$, and $\alpha = (1 - \sqrt{d})/(1 - d)$. Then

$$(3.2) \qquad \lim_{\epsilon \to 0} v(t, \epsilon) = \begin{cases} v_0 & 0 \le t < \alpha \\ v_1 & \alpha < t \le 1 \end{cases}$$

Proof. It follows from the maximum principle that $v(t) > 0$ for $0 < t < 1$, so that $u'(0) < 0$ and $u'(1) > 0$. This proves the existence of $\alpha = \alpha(\epsilon) \in (0, 1)$ such that $u'(\alpha(\epsilon), \epsilon) = 0$. The uniqueness of $\alpha(\epsilon)$ follows because $u''(t) = v(t) > 0$ for $0 < t < 1$. Furthermore, $u''(\alpha) > 0$, so that the unique minimum of $u(t)$ occurs at $t = \alpha$. We now state a preliminary lemma:

Lemma 1. There is a sequence $\epsilon_n \to 0$ such that:

(a) $\{u(t, \epsilon_n)\}$ converges uniformly to $\overline{u}(t) \in C^1[0, 1]$

(b) $\{u'(t, \epsilon_n)\}$ converges uniformly to $\overline{u}'(t) \in C[0, 1]$

(c) $\{\alpha(\epsilon_n)\}$ converges to $\alpha \in [0, 1]$

(d) $\{v(t, \epsilon_n)\}$ converges to $\overline{v}(t) \in L^1[0, 1]$, and $\overline{v}(t)$ is monotone non-decreasing.

Statements (a) and (b) follow from the Arzela-Ascoli Theorem, (c) follows from the Bolzano-Weierstrass Theorem, and (d) follows from the Helly selection Theorem.

Assume that $\overline{u}(t) \not\equiv 0$. Then $-u(\alpha(\epsilon_n), \epsilon_n) = \|u(\cdot, \epsilon_n)\|_\infty \to \|\overline{u}\|_\infty$, so $\overline{u}(\alpha) = -\|\overline{u}\|_\infty$. Since $\|\overline{u}\|_\infty > 0$ and $\overline{u}(0) = \overline{u}(1) = 0$, we have $0 < \alpha < 1$. We now show that $\overline{u}'(t) = 0$ for $0 < t < 1$ if and only if $t = \alpha$. Since $u'(\alpha(\epsilon_n), \epsilon_n) = 0$, we have $\overline{u}'(\alpha) = 0$. Conversely, suppose $\overline{u}'(t_0) = 0$ for some $t_0 \in [0, \alpha)$. Since $u'(t, \epsilon)$ is monotone increasing, $\overline{u}'(t)$ is monotone non-decreasing. Thus $\overline{u}'(t) = 0$ for $t_0 \le t \le \alpha$. Let $\phi(t) \in C^1[0, 1]$ satisfy $\phi(0) = 0$ and support $\phi \subset (t_0, \alpha)$. Then

$$\begin{aligned}\int_0^1 v(t, \epsilon_n)\, \phi(t)\, dt &= \int_0^1 u''(t, \epsilon_n)\, \phi(t)\, dt \\ &= -\int_0^1 u'(t, \epsilon_n)\, \phi'(t)\, dt \\ &= \int_\alpha^{t_0} u'(t, \epsilon_n)\, \phi'(t)\, dt\end{aligned}$$

Thus

$$\int_0^1 \overline{v}(t)\,\phi(t)\,dt = \int_\alpha^{t_0} \overline{u}'(t)\,\phi'(t)\,dt = 0$$

Since $\phi(t)$ is arbitrary, $\overline{v}(t) = 0$ a.e. for $t \in [t_0, \alpha]$. But $\overline{v}(t)$ is monotone non-decreasing, so $\overline{v}(t) = 0$ a.e. for $t \in [0, \alpha]$. It is easy to show that

$$(3.3) \qquad u(\alpha(\epsilon_n), \epsilon_n) = -\int_0^{\alpha(\epsilon_n)} t\, v(t, \epsilon_n)\, dt$$

Thus

$$(3.4) \qquad \overline{u}(\alpha) = -\|\overline{u}\|_\infty = -\int_0^{\alpha} t\, \overline{v}(t)\, dt = 0$$

This contradicts $\overline{u}(t) \not\equiv 0$, and shows that $\overline{u}'(t) \neq 0$ for $0 \le t < \alpha$. The case in which $\overline{u}'(t) = 0$ for $\alpha < t \le 1$ leads to a similar contradiction.

We now show that, under the assumption that $\overline{u}(t) \not\equiv 0$, we have

$$(3.5) \qquad \overline{v}(t) = \begin{cases} v_0 & 0 \le t < \alpha \\ v_1 & \alpha < t \le 1 \end{cases}$$

Let $t \in (0, \alpha)$, and set $\overline{t} = \frac{1}{2}(t + \alpha)$. We then have $\overline{u}'(\tau) \le \overline{u}'(\overline{t}) < 0$ for $\tau \in [0, \overline{t}]$, so that for $n \ge N$

$$(3.6) \qquad u'(\tau, \epsilon_n) \le C < 0 \qquad \text{for} \quad \tau \in [0, \overline{t}]$$

and the constant C is independent of n. Define $\phi(\tau, \epsilon)$ by

$$(3.7) \qquad \phi(\tau,\epsilon) = (v(\tau, \epsilon) - v_0) \exp\left(\frac{C}{\epsilon}(\tau - \overline{t})\right)$$

Then $\phi(\tau) = \phi(\tau,\epsilon)$ satisfies the differential equation

$$(3.8)\qquad \begin{cases} \epsilon\phi''(\tau) + (u'(\tau) - 2C)\phi'(\tau) + \frac{C}{\epsilon}(C - u'(\tau))\phi(\tau) = 0 & (0 < \tau < \bar{t}) \\ \phi(0) = 0, \quad \phi(\bar{t}) = v(\bar{t}) - v_0 \end{cases}$$

Since $\frac{C}{\epsilon_n}(C - u'(\tau, \epsilon_n)) \le 0$ for $0 \le \tau \le \bar{t}$, we have $0 \le \phi(\tau) \le (v(\bar{t}) - v_0) \le (v_1 - v_0)$. But $0 < t < \bar{t}$ and $C < 0$, so by letting $\epsilon = \epsilon_n \to 0$ in (3.7) we see that $\bar{v}(t) = v_0$ for $0 \le t < \alpha$. To show that $\bar{v}(t) = v_1$ for $\alpha < t \le 1$, we use a similar argument with the comparison function

$$(3.9)\qquad \phi(\tau, \epsilon) = (v_1 - v(\tau, \epsilon)) \exp\left(\frac{C}{\epsilon}(\tau - \bar{t})\right)$$

where $\bar{t} = \frac{1}{2}(t + \alpha)$, and $u'(\tau, \epsilon_n) \ge C > 0$ for $\bar{t} \le \tau \le 1$ and $n \ge N$.

In the case that $v_0 = 0$, we assume that $\bar{u}(t) \not\equiv 0$ and get a contradiction. For, the above argument then shows that $\bar{v}(t) = 0$ for $0 \le t < \alpha$, and by (3.4) we would then have $\|\bar{u}\|_\infty = 0$. Thus $\bar{u}(t) \equiv 0$, so that $\bar{v}(t) \equiv 0$ for $0 \le t < 1$. If $\lim_{\epsilon \to 0} v(t, \epsilon) = 0$ for $0 \le t < 1$ is false, there is a point $t_0 \in (0, 1)$ and a sequence $\epsilon_n \to 0$ so that $\lim_{n \to \infty} v(t_0, \epsilon_n) \ne 0$. By taking a subsequence so that the conclusions of Lemma 1 hold, we repeat the above argument to find that $\lim_{n \to \infty} v(t_0, \epsilon_n) = 0$. This contradiction proves that $\lim_{\epsilon \to 0} v(t, \epsilon) = 0$ for $0 \le t < 1$.

If $v_0 > 0$ we have $\bar{v}(t) \ge v_0 > 0$ so that $\bar{u}(t) \not\equiv 0$. Thus to complete the proof of the Theorem we need only determine the value of α. To do this, we compute the general solution $\bar{u}(t)$:

$$\overline{u}(t) = \begin{cases} \frac{1}{2} v_0 t(t - \alpha + \frac{2\overline{u}(\alpha)}{\alpha v_0}) & 0 \leq t \leq \alpha \\ \frac{1}{2} v_1 (t - 1)(t - \alpha + \frac{2\overline{u}(\alpha)}{(\alpha - 1) v_1}) & \alpha \leq t \leq 1 \end{cases} \tag{3.10}$$

Since $\overline{u}(t) \in C^1[0, 1]$ and $\overline{u}'(\alpha) = 0$, we derive the additional conditions

$$\alpha^2 v_0 = -2\,\overline{u}(\alpha) = (\alpha - 1)^2 v_1 \tag{3.11}$$

Solving (3.11) for α, we find that $\alpha = (1 \pm \sqrt{d})/(1 - d)$ with $d = v_0/v_1$. Since $d \in (0, 1)$ and $\alpha \in (0, 1)$, we see that $\alpha = (1 - \sqrt{d})/(1 - d)$.

The method of proof used in this Theorem easily leads to the following generalizations. We assume in both Theorems that $u(t) = u(t, \epsilon)$ and $v(t) = v(t, \epsilon)$ are solutions in $C^2[0, 1]$ to (1.1) - (1.2).

Theorem 4. Assume that:

(a) If $z > 0$ then $f(x, y, z) > 0$

(b) $f(x, y, 0) = 0$

(c) If $z \leq 0$ then $g(x, y, z) \leq 0$

(d) If $x \in (0, 1)$, $y \leq 0$, and $g(x, y, z) = 0$, then $z = 0$

(e) $v_0 = 0$

Then $\lim_{\epsilon \to 0} v(t, \epsilon) = 0$ for $0 \leq t < 1$.

Theorem 5. Assume that:

(a) $f(x, y, z)$ is independent of y (write

$$f(x, y, z) = f_0(x, z))$$

(b) If $z > 0$ then $f_0(x, z) > 0$

(c) $f_0(x, 0) = 0$

(d) $z\, g(x, y, z) \geq 0$

(e) If $x \in (0, 1)$, $y \leq 0$, and $g(x, y, z) = 0$, then $z = 0$

(f) $c(x, y, z) \equiv 0$

(g) $v_0 > 0$

In addition, assume that there is a unique solution $\alpha \in (0, 1)$ to

$$\int_0^{\alpha} t\, f_0(t, v_0)\, dt = \int_{\alpha}^{1} (1 - t)\, f_0(t, v_1)\, dt \tag{3.12}$$

Then

$$\lim_{\epsilon \to 0} v(t, \epsilon) = \begin{cases} v_0 & 0 \leq t < \alpha \\ v_1 & \alpha < t \leq 1 \end{cases} \tag{3.13}$$

Remark. It is clear that we can relax the restrictions on $f(x, y, z)$ and α, at the expense of the uniqueness of the limit function $\overline{v}(t)$. The obvious generalization of Theorem 5 in that case provides a characterization of S_0 for this problem.

In the preceding example with $f(t, u, v) = v$, we have seen that, for $v_0 > 0$, the limit function $\overline{v}(t)$ retains both boundary conditions. In the next case, $f(t, u, v) = -v$, the limit function loses both boundary conditions:

<u>Theorem 6.</u> Let $u(t) = u(t, \epsilon)$ and $v(t) = v(t, \epsilon)$ be solutions in $C^2[0, 1]$ to

$$(3.14)\quad \begin{cases} u''(t) = -v(t) & (0 < t < 1) \\ u(0) = u(1) = 0 & \\ \epsilon v''(t) + u'(t)\, v'(t) = 0 & (0 < t < 1) \\ v(0) = v_0, \quad v(1) = v_1 & \end{cases}$$

Then $\lim_{\epsilon \to 0} v(t, \epsilon) = \frac{1}{2}(v_0 + v_1)$ for $0 < t < 1$.

The proof of this Theorem proceeds with a sequence of Lemmas. We first note that there is a unique $\alpha(\epsilon) \in (0, 1)$ such that $u'(\alpha(\epsilon), \epsilon) = 0$. If we let ϵ_n, $\overline{u}(t)$, and $\overline{v}(t)$ be as in Lemma 1, we then have:

<u>Lemma 2.</u> $\overline{u}(t) \not\equiv 0$

<u>Proof.</u> This is trivial if $v_0 > 0$, since then $\overline{v}(t) \geq v_0 > 0$. Thus we consider the case $v_0 = 0$. We have the integral representation

$$(3.15)\qquad v(t, \epsilon) = v_1 \left(\int_0^t U(\tau, \epsilon)\, d\tau\right)\left(\int_0^1 U(\tau, \epsilon)\, d\tau\right)^{-1}$$

where $U(\tau, \epsilon) = \exp\left(-\frac{1}{\epsilon} u(\tau, \epsilon)\right)$. Let $F(t) = u(t) - u(1 - t)$, so that $F(\frac{1}{2}) = F(1) = 0$ and $F''(t) = v(1 - t) - v(t)$. Since $v(t)$ is monotone increasing, $F''(t) \leq 0$ for $\frac{1}{2} \leq t \leq 1$. Thus

$$(3.16)\qquad u(t) \geq u(1 - t) \qquad \text{for } \frac{1}{2} \leq t \leq 1$$

Inserting this bound in (3.15) immediately gives $v(\frac{1}{2}, \epsilon) \geq \frac{1}{2} v_1$. Thus $\overline{v}(\frac{1}{2}) \geq \frac{1}{2} v_1$, and so $\overline{u}(t) \not\equiv 0$.

Define $w(t, \epsilon)$ by

(3.17) $$w(t, \epsilon) = \frac{v(1-t, \epsilon) - v(t, \epsilon)}{v_1 - v_0}$$

It is easy to show that

(3.18) $$w(t, \epsilon) = \left(\int_t^{1-t} U(\tau, \epsilon)\, d\tau\right)\left(\int_0^1 U(\tau, \epsilon)\, d\tau\right)^{-1}$$

Lemma 3. There is a constant $C > 0$ such that, for $0 < \epsilon \le 1$, we have

(3.19) $$0 < \left(\int_0^1 U(t, \epsilon)\, dt\right)^{-1} \le \frac{C}{\epsilon}$$

Proof. It is easy to show that $u(t, \epsilon) \le \frac{1}{2} v_1 (t - t^2)$, so that $U(t, \epsilon) \ge \exp\left(-\frac{v_1}{2\epsilon}(t - t^2)\right)$. Thus

$$\begin{aligned}
\int_0^1 U(t, \epsilon)\, dt &\ge \int_0^1 \exp\left(-\frac{v_1}{2\epsilon}(t - t^2)\right) dt \\
&= 2\int_0^{\frac{1}{2}} \exp\left(-\frac{v_1}{2\epsilon}(t - t^2)\right) dt \\
&\ge 2\int_0^{\frac{1}{2}} \exp\left(-\frac{v_1}{2\epsilon} t\right) dt \\
&= \left(\frac{4\epsilon}{v_1}\right)\left(1 - \exp\left(-\frac{v_1}{4\epsilon}\right)\right) \\
&\ge \frac{\epsilon}{C}
\end{aligned}$$

for $C = \left(\frac{4}{v_1}\left(1 - \exp\left(-\frac{v_1}{4}\right)\right)\right)^{-1}$.

Lemma 4. (a) $\lim_{n \to \infty} w(t, \epsilon_n) = 0$ for $0 < t < 1$.

(b) $\overline{v}(t) = \frac{1}{2}(v_0 + v_1)$ for $0 < t < 1$.

Proof. Since $w(t, \epsilon) = -w(1 - t, \epsilon)$, it suffices to prove (a) for $t \in (0, \frac{1}{2}]$. It follows from the convexity of $u(t, \epsilon)$ that $\bar{u}(t) = 0$ if and only if $t = 0$ or $t = 1$. Let $t \in (0, \frac{1}{2}]$ be fixed. Since $\bar{u}(\tau) \neq 0$ for $\tau \in [t, 1 - t]$, there is a constant $C_1 = C_1(t)$ such that

$$\bar{u}(\tau) \geq C_1 > 0 \qquad \text{for} \quad \tau \in [t, 1 - t] \tag{3.20}$$

But $\{u(\tau, \epsilon_n)\}$ converges uniformly to $\bar{u}(\tau)$, so that there exists a constant $C_2 = C_2(t)$ and an integer N such that for $n \geq N$ we have

$$u(\tau, \epsilon_n) \geq C_2 > 0 \quad \text{for} \quad \tau \in [t, 1 - t] \tag{3.21}$$

Thus for $n \geq N$

$$0 \leq w(t, \epsilon_n) \leq \frac{C(1 - 2t)}{\epsilon_n} \exp\left(-\frac{1}{\epsilon_n} C_2\right) \tag{3.22}$$

This proves (a), which immediately implies that $\bar{v}(t) = \text{constant}$ for $t \in (0, 1)$. Denote this constant by C, and notice that $\bar{u}(t) = \frac{1}{2}C(t - t^2)$. Since $u(0, \epsilon) = u(1, \epsilon)$ we see that $v'(0, \epsilon) = v'(1, \epsilon)$. Thus

$$v_1 u'(1, \epsilon) = v_0 u'(0, \epsilon) - \int_0^1 v^2(t, \epsilon)\, dt \tag{3.23}$$

Letting $\epsilon = \epsilon_n \to 0$ in (3.23), we find that C satisfies the equation $C(2C - v_0 - v_1) = 0$. Since $\bar{u}(t) \not\equiv 0$ we have $C > 0$, so that $C = \frac{1}{2}(v_0 + v_1)$. The remainder of the proof of Theorem 6 follows in the same fashion as the proof of Theorem 3 .

Remark. Since $\|\bar{u}(t)\|_\infty = \bar{u}(\alpha)$ and $\bar{u}(t) = \frac{1}{4}(v_0 + v_1)(t - t^2)$, we see that $\lim_{\epsilon \to 0} \alpha(\epsilon) = \frac{1}{2}$.

We state the following generalization of Theorem 6. The proof differs only in details, and hence is omitted:

Theorem 7. Let $u(t) = u(t, \epsilon)$ and $v(t) = v(t, \epsilon)$ be solutions in $C^2[0, 1]$ to (1.1) - (1.2) with $c(x, y, z) \equiv 0$. Assume that:

(a) If $z > 0$ then $f(x, y, z) < 0$

(b) $f(x, y, 0) = 0$

(c) $z\, g(x, y, z) \geq 0$

(d) If $x \in (0, 1)$, $y \geq 0$, and $g(x, y, z) = 0$, then $z = 0$.

Let $\bar{v}(t) \in S_0$. Then there is a constant $C \in [v_0, v_1]$ such that $\bar{v}(t) = C$ for $0 < t < 1$. If, in addition to the above, we have $g(x, y, z) \in C^1(S_3)$ and

(3.24) $$\lim_{n \to \infty} \epsilon_n (v'(0, \epsilon_n) - v'(1, \epsilon_n)) = 0$$

then C satisfies

(3.25) $$v_1\, g(1, 0, \bar{u}'(1)) = v_0\, g(0, 0, \bar{u}'(0)) + C \int_0^1 \left[\frac{\partial g}{\partial x} + \bar{u}' \frac{\partial g}{\partial y} + f(t, \bar{u}, C) \frac{\partial g}{\partial z}\right] dt$$

The partial derivatives of g in (3.25) are evaluated at $(t, \bar{u}(t), \bar{u}'(t))$, and $\bar{u}(t)$ satisfies

$$(3.26)\qquad \begin{cases} \bar{u}''(t) = f(t, \bar{u}(t), C) & (0 < t < 1) \\ \bar{u}(0) = \bar{u}(1) = 0 \end{cases}$$

Remark. If (3.24) holds for all sequences $\epsilon_n \to 0$, and if C is unique, then $\lim_{\epsilon \to 0} v(t, \epsilon) = C$ for $0 < t < 1$.

The verification of condition (3.24) is not always an easy matter. In the case of Theorem 6 we had $v'(0, \epsilon) = v'(1, \epsilon)$, so that the question did not arise. We now give a class of examples for which (3.24) can be verified. We do not treat the example in its greatest generality, but it is easily seen that the method of proof may be applied to more general problems.

Example. Let $f(t, u, v) = -v$, $c(t, u, u') \equiv 0$, and $g(t, u, u') = (u')^{2K+1}$ for some fixed integer $K \geq 1$. We will show that (3.24) is satisfied.

Let $\bar{v}(t) \in S_0$, and let $C \in [v_0, v_1]$ be such that $\bar{v}(t) = C$ for $t \in (0, 1)$. It then follows that $\bar{u}(t) = \frac{1}{2} C(t - t^2)$. Using integration by parts, it is easy to show that

$$\lim_{n \to \infty} \epsilon_n (v'(0, \epsilon_n) - v'(1, \epsilon_n)) = (\tfrac{1}{2} C)^{2K+1} (2C - v_0 - v_1)$$

If $C = 0$ then (3.24) is clearly satisfied. Thus we assume that $C > 0$, and hence $\bar{u}(t) \not\equiv 0$.

Let $G(t, \epsilon) = \int_0^t g(\tau, u(\tau, \epsilon), u'(\tau, \epsilon))\, d\tau$, so that

$$(3.27)\qquad v(t, \epsilon) = v_0 + (v_1 - v_0) \left(\int_0^t G_0(\tau, \epsilon)\, d\tau\right) \left(\int_0^1 G_0(\tau, \epsilon)\, d\tau\right)^{-1}$$

where $G_0(\tau, \epsilon) = \exp\left(-\frac{1}{\epsilon} G(\tau, \epsilon)\right)$. For these examples we have

$$(3.28) \qquad \begin{cases} G'(t, \epsilon) = (u'(t, \epsilon))^{2K+1} \\ G''(t, \epsilon) = -(2K+1)\, v(t, \epsilon)\, (u'(t, \epsilon))^{2K} \end{cases}$$

Thus $G''(t, \epsilon) \le 0$ for $0 \le t \le 1$. Using integration by parts, it is easy to show that

$$(3.29) \qquad G(1, \epsilon) = - \sum_{j=1}^{K} C_{j-1}\, I_j$$

where

$$(3.30) \qquad \begin{cases} I_j = \int_0^1 v' \; v^{j-1} \; u^{j+1} \; (u')^{2(K-j)} dt \\ C_j = \left(\frac{j+1}{j+2}\right) 2^{j+1} \prod_{n=0}^{j} \left(\frac{K-n}{n+1}\right) \end{cases}$$

Thus $G(1, \epsilon) < 0$, and so from (3.27) we see that $v'(1, \epsilon) > v'(0, \epsilon)$. Thus for $K \ge 1$ condition (3.24) is not satisfied trivially, as it is for $K = 0$.

It is easy to see that there is a unique $\gamma = \gamma(\epsilon) \in (\alpha(\epsilon), 1)$ such that

$$G(t) \begin{cases} > 0 & 0 < t < \gamma \\ = 0 & t = 0, \gamma \\ < 0 & \gamma < t \le 1 \end{cases}$$

We now want to prove that $G(1, \epsilon) = 0(\epsilon^2)$ as $\epsilon \to 0$ (for the sake of simplicity, we have deleted the subscript on ϵ_n). Since $u(t, \epsilon)$, $u'(t, \epsilon)$, and $v(t, \epsilon)$ are uniformly bounded, we have $|G(1, \epsilon)| \le C\, J(\epsilon)$, where

(3.31) $$J = J(\epsilon) = \int_0^1 v'(t,\epsilon)\, u^2(t,\epsilon)\, dt$$

Let $J = \sum_{j=1}^{4} J_j$, where the regions of integration of the integrals J_j are $(0,\sqrt{\epsilon})$, $(\sqrt{\epsilon}, \gamma(\epsilon) - \sqrt{\epsilon})$, $(\gamma(\epsilon) - \sqrt{\epsilon}, \gamma(\epsilon))$, and $(\gamma(\epsilon), 1)$. Since $\overline{u}(t) \not\equiv 0$, we see that $\lim_{\epsilon \to 0} \gamma(\epsilon) = 1$. Thus the regions of integration are well defined for $0 < \epsilon \leq \epsilon_0$. Let $C > 0$ be a generic constant. By the same method used in the proof of Lemma 3, we see that

(3.32) $$0 < v'(t,\epsilon) \leq \left(\frac{C}{\epsilon}\right) \exp\left(-\frac{1}{\epsilon} G(t,\epsilon)\right)$$

If $C_0 > 0$ is such that $\overline{v}(t) = C_0$ for $0 < t < 1$, then $\overline{u}'(0) = \frac{1}{2} C_0 > 0$. Thus for ϵ small enough

(3.33) $$G(t,\epsilon) \geq Ct \qquad (0 \leq t \leq \sqrt{\epsilon})$$

Using the bound $0 \leq u(t,\epsilon) \leq Ct$, we find that

(3.34) $$0 < J_1 \leq C\,\epsilon^2$$

For the integral J_2, by using the convexity of $G(t,\epsilon)$ and the fact that $\overline{u}(t) \not\equiv 0$, it is easy to show that

(3.35) $$G(t,\epsilon) \geq C\sqrt{\epsilon} \qquad (\sqrt{\epsilon} \leq t \leq \gamma(\epsilon) - \sqrt{\epsilon})$$

Thus for ϵ small enough

(3.36) $$0 < J_2 \leq \frac{C}{\epsilon} \exp\left(-\frac{C}{\sqrt{\epsilon}}\right)$$

The technique used for J_1, together with the estimate $u(t,\epsilon) \le C(1-t)$, can be used to show that

$$(3.37) \qquad 0 < J_3 \le C\left[(1-\gamma)^2 + \epsilon(1-\gamma) + \epsilon^2\right]$$

Finally, we note that J_4 can be bounded in the following way:

$$\begin{aligned} J_4 &= \int_\gamma^1 u^2(t)\, v'(t)\, dt \\ &= C\Big(\int_\gamma^1 u^2(t)\, G_0(t,\epsilon)\, dt\Big)\Big(\int_0^1 G_0(t,\epsilon)\, dt\Big)^{-1} \\ &\le C\Big(\int_\gamma^1 u^2(t)\, G_0(t,\epsilon)\, dt\Big)\Big(\int_\gamma^1 G_0(t,\epsilon)\, dt\Big)^{-1} \\ &\le C \max_{\gamma \le t \le 1} u^2(t) \\ &= C\, u^2(\gamma) \end{aligned}$$

Thus

$$(3.38) \qquad 0 < J_4 \le C(1-\gamma)^2$$

From the mean value theorem, we have $G(1,\epsilon) = (1-\gamma(\epsilon))\, G'(\tau,\epsilon)$, where $\gamma(\epsilon) < \tau < 1$. Since $\gamma(\epsilon) \to 1$ and $G'(1,\epsilon) \to (\overline{u}'(1))^{2K+1} < 0$, there exist constants $C_1, C_2 > 0$ so that, for ϵ small enough, we have:

$$(3.39) \qquad C_1\,|G(1,\epsilon)| \le |1-\gamma(\epsilon)| \le C_2\,|G(1,\epsilon)|$$

Combining all of the previous estimates, we have shown that:

$$
\begin{aligned}
|G(1,\epsilon)| &\leq C \; J(\epsilon) \\
&\leq C\Big[\epsilon^2 + \frac{1}{\epsilon}\exp\Big(-\frac{C}{\sqrt{\epsilon}}\Big) \\
&\qquad + (1-\gamma)^2 + \epsilon(1-\gamma)\Big] \\
&\leq C\,\epsilon^2 + C|G(1,\epsilon)| \;((1-\gamma)+\epsilon)
\end{aligned}
$$

Thus

$$|G(1,\epsilon)| \leq C\,\epsilon^2 + \frac{1}{2}|G(1,\epsilon)|$$

for $0 < \epsilon \leq \epsilon_1$, and hence

$$|G(1,\epsilon)| \leq C\,\epsilon^2 \tag{3.40}$$

The estimate (3.40) immediately shows that (3.24) is satisfied.

It is interesting to note that the integration in (3.25) can easily be performed. Theorem 7 then shows the following:

(a) If $v_0 > 0$, then $\lim_{\epsilon \to 0} v(t,\epsilon) = \frac{1}{2}(v_0 + v_1)$ for $0 < t < 1$

(b) If $v_0 = 0$ and $\overline{v}(t) \in S_0$ we have $\overline{v}(t) = C_0$ for $0 < t < 1$, and either $C_0 = 0$ or $C_0 = \frac{1}{2} v_1$.

We observe that, in the case $v_0 = 0$ and $K \geq 1$, although we have not eliminated the possibility of having a sequence $\{u(t,\epsilon_n)\}$ that converges uniformly to 0 , in that case the <u>rate</u> of convergence can not be too rapid. Indeed, it can be shown that

$$\lim_{n \to \infty} \frac{1}{\epsilon_n} u(t,\epsilon_n) = \infty \qquad (0 < t < 1) \tag{3.41}$$

4. PROBLEMS WITH TURNING POINTS AT THE ENDS OF THE INTERVAL

The problems we consider in this section are motivated by the special cases $g(t, u, u') = u$ and $f(t, u, v) = \pm v$. In these examples we no longer have an interior turning point, since $|u(t, \epsilon)| > 0$ for $t \in (0, 1)$. However, we have $u(t, \epsilon) = 0$ and $|u'(t, \epsilon)| > 0$ for $t = 0$ and $t = 1$, so there are turning points at each end of the interval. The asymptotic behavior is greatly simplified in this case: exactly one boundary condition is lost, and the one retained is determined by the sign of $u(t, \epsilon)$ for $t \in (0, 1)$. Because the proofs are essentially the same, we state the theorems in a general form and include the specific cases as examples. The functions $u(t) = u(t, \epsilon)$ and $v(t) = v(t, \epsilon)$ are solutions in $C^2[0, 1]$ to (1.1) - (1.2) with $c(x, y, z) \equiv 0$.

Theorem 8. Assume that:

(a) If $z > 0$ then $f(x, y, z) > 0$

(b) If $y \leq 0$ then $g(x, y, z) \leq 0$

(c) If $x \in (0, 1)$, $y \leq 0$, and $g(x, y, z) = 0$, then $y = 0$.

Then $\lim_{\epsilon \to 0} v(t, \epsilon) = v_0$ for $0 \leq t < 1$.

Example.

$$\begin{cases} u'' = v \\ \epsilon v'' + uv' = 0 \end{cases}$$

Proof. We write $v(t, \epsilon)$ in the form

$$v(t, \epsilon) = v_0 + (v_1 - v_0)\left(\int_0^t G_0(\tau, \epsilon)d\tau\right)\left(\int_0^1 G_0(\tau, \epsilon)d\tau\right)^{-1} \tag{4.1}$$

where

$$\begin{cases} G(t, \epsilon) = \int_0^t g(\tau, u(\tau, \epsilon), u'(\tau, \epsilon))\,d\tau \\ G_0(t, \epsilon) = \exp\left(-\frac{1}{\epsilon} G(t, \epsilon)\right) \end{cases} \tag{4.2}$$

It is easy to see that $u(t, \epsilon) \le 0$, $G(t, \epsilon) \le 0$, and $G'(t, \epsilon) \le 0$. If $\overline{u}(t) \not\equiv 0$, (4.1) can be used directly to show that $\lim_{\epsilon \to 0} v(t, \epsilon) = v_0$ for $0 \le t < 1$. We omit the details. If $v_0 > 0$ we automatically have $\overline{u}(t) \not\equiv 0$. If $v_0 = 0$, we assume $\overline{u}(t) \not\equiv 0$. The above argument then shows that $\overline{v}(t) = 0$ for $0 \le t < 1$, which cannot happen if $\overline{u}(t) \not\equiv 0$. We thus have $\overline{u}(t) \equiv 0$, and hence $\overline{v}(t) = 0$ for $0 \le t < 1$.

Theorem 9. Assume that:

(a) If $z > 0$ then $f(x, y, z) < 0$

(b) If $y \ge 0$ then $g(x, y, z) \ge 0$

(c) If $x \in (0, 1)$, $y \ge 0$, and $g(x, y, z) = 0$, then $y = 0$.

Then $\lim_{\epsilon \to 0} v(t, \epsilon) = v_1$ for $0 < t \le 1$.

Example

$$\begin{cases} u'' = -v \\ \epsilon v'' + uv' = 0 \end{cases}$$

Proof. The proof is essentially the same as that of Theorem 8, and the details are again omitted. We remark that we always have $\overline{u}(t) \not\equiv 0$. For $v''(t) = [-\frac{1}{\epsilon} g(t, u, u')v'] \leq 0$, and hence

$$v(t, \epsilon) \geq v_0 + (v_1 - v_0)t \qquad (0 \leq t \leq 1) \tag{4.3}$$

5. PROBLEMS WITH $c(x, y, z) \geq c_0 > 0$

In this section we consider problems for which the following conditions hold:

$$(5.1) \quad \begin{cases} (a) & c(x, y, z) \geq c_0 > 0 \\ (b) & g(x, y, z) \in C^1(S_3) \end{cases}$$

These are included in a separate section because the asymptotic behavior is not determined by the nature of the turning points. Rather, it depends upon the fact that the reduced equations have no non-trivial solutions.

Let $u(t) = u(t, \epsilon)$ and $v(t) = v(t, \epsilon)$ be solutions in $C^2[0, 1]$ to (1.1) - (1.2). The theorems in this section give sufficient conditions for the following conclusion:

$$(A) \quad \lim_{\epsilon \to 0} v(t, \epsilon) = 0 \quad \text{for} \quad 0 < t < 1$$

Theorem 10. Assume that:

(a) If $z \neq 0$ then $f(x, y, z) \neq 0$

(b) $g(x, y, z) = 0$ if and only if $y = 0$

Then (A) holds.

Examples. Theorem 10 can be applied to:

$$(5.2) \quad \begin{cases} u'' = v \\ \epsilon v'' + uv' - cv = 0 \end{cases}$$

$$(5.3)\qquad \begin{cases} u'' = -v \\ \epsilon v'' + uv' - cv = 0 \end{cases}$$

The proof of Theorem 10 follows directly from the remarks following Theorem 2.

<u>Remark</u>. It is clear that condition (b) of Theorem 10 can be weakened to one of the following (assuming $\overline{u}(t) \not\equiv 0$):

$(b)_1$ $\overline{g}(0) = 0$ and $\overline{g}(t) < 0$ for $0 < t < 1$

or $(b)_2$ $\overline{g}(1) = 0$ and $\overline{g}(t) > 0$ for $0 < t < 1$

Condition (5.1a) can also be weakened in this case, since we require $c(x, y, z) \geq c_0 > 0$ only in a neighborhood of 0 or 1 (depending upon whether $(b)_1$ or $(b)_2$ holds). For the sake of simplicity, we do not elaborate on this. However, the generality of $(b)_1 - (b)_2$ allows us to consider cases in which the singularity in the problem is caused by the behavior of $g(t, u, u')$ as a function of t at $t = 0$ (or $t = 1$), instead of the behavior of $u(t)$ at $t = 0$ (or $t = 1$).

For example, suppose $g(x, y, z) = g_1(x)\, g_2(x, y, z)$ where $g(x, y, z)$ satisfies:

$$(5.4)\qquad \begin{cases} \text{(a)}\ \ g_1(x) = 0 \ \text{ for } \ 0 \leq x < 1 \ \text{ if and only if } \ x = 0 \\ \text{(b)}\ \ \text{If } \ g_2(x, y, z) = 0 \ \text{ then } \ y = 0 \\ \text{(c)}\ \ g(x, y, z) \leq 0 \end{cases}$$

We remark that the following can be substituted for (5.4 b-c):

$$(5.5)\quad \begin{cases} \text{(a) If } z \ge 0 \text{ then } f(x,y,z) \ge 0 \\ \text{(b) If } y \le 0 \text{ then } g(x,y,z) \le 0 \\ \text{(c) If } x \in (0,1),\ y \le 0, \text{ and } g(x,y,z) = 0, \text{ then } y = 0 \end{cases}$$

Then condition $(b)_1$ is satisfied, and so (b) holds. An example for which the above conditions are satisfied is:

$$(5.6)\quad \begin{cases} u''(t) = v(t) \\ \epsilon v''(t) + t(e^{u(t)} - 2)\, v'(t) - c(t,u,u')\, v(t) = 0 \end{cases}$$

A similar result holds if $(b)_2$ is to be satisfied. In that case, conditions (5.4) - (5.5) are replaced by:

$$(5.7)\quad \begin{cases} \text{(a) } g_1(x) = 0 \text{ for } 0 < x \le 1 \text{ if and only if } x = 1 \\ \text{(b) If } g_2(x,y,z) = 0 \text{ then } y = 0 \\ \text{(c) } g(x,y,z) \ge 0 \end{cases}$$

$$(5.8)\quad \begin{cases} \text{(a) If } z \ge 0 \text{ then } f(x,y,z) \le 0 \\ \text{(b) If } y \ge 0 \text{ then } g(x,y,z) \ge 0 \\ \text{(c) If } x \in (0,1),\ y \ge 0, \text{ and } g(x,y,z) = 0, \text{ then } y = 0 \end{cases}$$

An example for which these conditions are satisfied is:

$$(5.9)\quad \begin{cases} u''(t) = -v(t) \\ \epsilon v''(t) + (1-t)\,(e^{u(t)} - \frac{1}{2})v'(t) - c(t,u,u')\, v(t) = 0 \end{cases}$$

Theorem 11. Assume that:

(a) If $z > 0$ then $f(x, y, z) < 0$

(b) $f(x, y, 0) = 0$

(c) If $z \geq 0$ then $g(x, y, z) \geq 0$

(d) $g(x, y, 0) = 0$

(e) If $x \in (0, 1)$, $y \geq 0$, and $g(x, y, z) = 0$, then $z = 0$

Then (A) holds.

Example. Theorem 11 can be applied to:

$$(5.10) \qquad \begin{cases} u'' = -v \\ \epsilon v'' + u'v' - cv = 0 \end{cases}$$

Proof. Assume that $\overline{u}(t) \not\equiv 0$. Then, by taking a subsequence if necessary, we have $\lim_{n \to \infty} \alpha(\epsilon_n) = \alpha \in (0, 1)$. By applying the remarks following Theorem 2 on the interval $(0, \alpha)$, we conclude that $\overline{v}(t) = 0$ for $0 < t < \alpha$. But

$$\begin{aligned} \|\overline{u}\|_\infty &= |\overline{u}(\alpha)| \\ &= \left| \int_0^\alpha t f(t, \overline{u}(t), \overline{v}(t)) \, dt \right| \\ &= 0 \end{aligned}$$

This contradiction implies that $\overline{u}(t) \equiv 0$, which proves that (A) holds.

<u>Theorem 12.</u> Assume that:

(a) $f(x, y, z) = z\, f_1(x, y)$, and $|f_1(x, y)| \geq \sigma > 0$

(b) $g(x, y, z) = z\, g_1(x, y)$, and if $g_1(x, y) = 0$ for $x \in (0, 1)$ then $y = 0$.

Then (A) holds.

<u>Examples</u>. Theorem 12 can be applied to:

$$(5.11) \qquad \begin{cases} u'' = v \\ \epsilon v'' + u'v' - cv = 0 \end{cases}$$

$$(5.12) \qquad \begin{cases} u'' = -v \\ \epsilon v'' + u'v' - cv = 0 \end{cases}$$

The proof of Theorem 12 follows in the same way as that of Theorem 11.

6. OTHER PROBLEMS

In this section we give several examples of problems not covered by previous theorems. The proofs follow directly with the use of techniques already developed, and hence are omitted. The first two Theorems deal with problems in which $g(t, u(t), u'(t))$ is allowed to have a zero, but the function does not change sign in a neighborhood of this zero.

Let $u(t) = u(t, \epsilon)$ and $v(t) = v(t, \epsilon)$ be solutions in $C^2[0, 1]$ to (1.1) - (1.2) with $c(x, y, z) \equiv 0$.

Theorem 13. Assume that:

(a) If $z \neq 0$ then $f(x, y, z) \neq 0$

(b) $f(x, y, 0) = 0$

(c) $g(x, y, z) \geq 0$

(d) If $g(x, y, z) = 0$ for $x \in (0, 1)$, then either $y = 0$ or $z = 0$

Then $\lim_{\epsilon \to 0} v(t, \epsilon) = v_1$ for $0 < t \leq 1$.

Examples. Theorem 13 can be applied to:

$$\begin{cases} u'' = v \\ \epsilon v'' + (u')^2 v' = 0 \end{cases}$$

$$\begin{cases} u'' = -v \\ \epsilon v'' + (u')^2 v' = 0 \end{cases}$$

$$\begin{cases} u'' = v \\ \epsilon v'' + u^2 v' = 0 \end{cases}$$

$$\begin{cases} u'' = -v \\ \epsilon v'' + u^2 v' = 0 \end{cases}$$

Theorem 14. Assume that:

(a) If $z \neq 0$ then $f(x, y, z) \neq 0$

(b) $f(x, y, 0) = 0$

(c) $g(x, y, z) \leq 0$

(d) If $g(x, y, z) = 0$ for $x \in (0, 1)$, then either $y = 0$ or $z = 0$

Then $\lim_{\epsilon \to 0} v(t, \epsilon) = v_0$ for $0 \leq t < 1$.

Examples. Theorem 14 can be applied to:

$$\begin{cases} u'' = v \\ \epsilon v'' - (u')^2 v' = 0 \end{cases}$$

$$\begin{cases} u'' = -v \\ \epsilon v'' - (u')^2 v' = 0 \end{cases}$$

$$\begin{cases} u'' = v \\ \epsilon v'' - u^2 v' = 0 \end{cases}$$

$$\begin{cases} u'' = -v \\ \epsilon v'' - u^2 v' = 0 \end{cases}$$

Remark. The conclusion of Theorem 14 remains valid for the general case $c(x, y, z) \geq 0$ if $v_0 = 0$. The method of proof follows that of Theorem 3.

The next theorem gives an example in which the equation for $u(t, \epsilon)$ is modified. The proof that solutions exist to this problem follows in the same way as the proof of Theorem 1.

Theorem 15. Fix a constant $C \geq 0$, and let $u(t) = u(t, \epsilon)$ and $v(t) = v(t, \epsilon)$ be solutions in $C^2[0, 1]$ to:

$$(6.1) \quad \begin{cases} u''(t) + C u'(t) = -v(t) & (0 < t < 1) \\ u(0) = u(1) = 0 & \\ \epsilon v''(t) + u'(t) v'(t) = 0 & (0 < t < 1) \\ v(0) = v_0, \quad v(1) = v_1 & \end{cases}$$

Then we have:

(a) If $v_0 > 0$, then $\lim_{\epsilon \to 0} v(t, \epsilon) = \overline{C}$ for $0 < t < 1$, where

(6.2) $$\overline{C} = (C(e^C - 1))^{-1}\,[v_0(1 - e^C + C e^C) - v_1(1 - e^C + C)]$$

(b) If $v_0 = C = 0$, then $\lim_{\epsilon \to 0} v(t, \epsilon) = \overline{C}$ for $0 < t < 1$, where $\overline{C} = \frac{1}{2} v_1$.

(c) If $v_0 = 0$ and $C > 0$, we let $\overline{v}(t) \in S_0$. Then there is a constant $\overline{C}$ such that $\overline{v}(t) = \overline{C}$ for $0 < t < 1$, and either $\overline{C} = 0$ or $\overline{C} = (C(e^C - 1))^{-1}\,[v_1(e^C - 1 - C)]$.

REFERENCES

1. Batchelor, G. K. "Heat transfer by free convection across a closed cavity between vertical boundaries at different temperatures," Quart. Appl. Math., 12 (1954) 209-233. MR 16-299.

2. Batchelor, G. K. "On steady laminar flow with closed streamlines at large Reynolds number," J. Fluid Mech., 1 (1956) 177-190. MR 18-840.

3. Batchelor, G. K. "A proposal concerning laminar wakes behind bluff bodies at large Reynolds number," J. Fluid Mech., 1 (1956) 388-398. MR 18-842.

4. Burggraf, Odus R. "Analytical and numerical studies of the structure of steady separated flows," J. Fluid Mech., 24 (1966) 113-151.

5. Carrier, G. F. "Boundary layer problems in applied mechanics," Advances in Applied Mechanics, 3 (1953) 1-19. MR 15-959.

6. Cochran, James Alan. "Problems in singular perturbation theory." Ph.D. Dissertation, Stanford University, 1962.

7. Dunford, Nelson and Schwartz, Jacob T. Linear Operators, Part I. Interscience, New York, 1966. MR 22-8302.

8. Feynman, R. P. and Lagerstrom, P. A. "Remarks on high Reynolds number flows in finite domains." IX International Congress of Applied Mechanics, Vol. 3, pp. 342-343. University of Brussells, Brussells, 1957.

9. Friedrichs, K. O. "The identity of weak and strong extensions of differential operators," Trans. Amer. Math. Soc., 55 (1944) 132-151. MR 5-188.

10. Greenspan, Donald. "Numerical studies of two dimensional, steady state Navier-Stokes equations for arbitrary Reynolds number." Technical Report No. 9, Computer Sciences Department, University of Wisconsin at Madison, 1967.

11. Harris, W. A., Jr. "Singular perturbations of a boundary value problem for a nonlinear system of differential equations," Duke Math. J., 29 (1962) 429-445. MR 31-452.

12. Hormander, Lars. "Weak and strong extensions of differential operators," Comm. Pure Appl. Math., 14 (1961) 371-379. MR 24-A2750.

13. Lagerstrom, P. A. and Cole, J. D. "Examples illustrating expansion procedures for the Navier-Stokes equations," J. Rational Mech. Anal., 4 (1955) 817-882. MR 17-1021.

14. Lees, Milton. "Discrete methods for nonlinear two-point boundary value problems." Numerical Solution of Partial Differential Equations, pp. 59-72. Academic Press, New York, 1966. MR 34-2196.

15. Levinson, Norman. "The first boundary value problem for $\epsilon \Delta u + A(x, y)u_x + B(x, y)u_y + C(x, y) u = D(x, y)$ for small ϵ," Ann. of Math., 51 (1950) 428-445. MR 11-439.

16. Macki, Jack W. "Singular perturbations of a boundary value problem for a system of nonlinear ordinary differential equations," Arch. Rational Mech. Anal., 24 (1967) 219-232. MR 34-7908.

17. Natanson, I. P. Theory of Functions of a Real Variable. Ungar, New York, 1955. MR 16-804.

18. Protter, Murray H. and Weinberger, Hans F. Maximum Principles in Differential Equations. Prentice-Hall, Englewood Cliffs, N. J., 1967.

19. Taylor, G. I. "Experiments with rotating fluids," Proc. Roy. Soc. London. Ser. A., 100 (1921) 114-121.

20. Wasow, Wolfgang. "On boundary layer problems in the theory of ordinary differential equations." Ph.D. Dissertation, New York University, 1942.

21. Wasow, Wolfgang. "The complex asymptotic theory of a fourth order differential equation of hydrodynamics," Ann. of Math., 49 (1948) 852-871. MR 10-377.

22. Wasow, Wolfgang. Asymptotic Expansions for Ordinary Differential Equations. Interscience, New York, 1965. MR 34-3041.