39

FINITE DIFFERENCE METHODS AND THE
EIGENVALUE PROBLEM FOR NON
SELF-ADJOINT STURM-LIOUVILLE OPERATORS

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Alfred Carasso

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1. INTRODUCTION

Many authors (e.g. [1], [6], [8], [9]) have studied the convergence of finite difference methods for self-adjoint Sturm-Liouville eigenvalue problems. In this report we are concerned with the non self-adjoint problem

$$(1.1)\qquad \mathcal{L}(u) \equiv -[a(x)\,u']' - b(x)u' + c(x)u = \lambda u, \quad 0 < x < 1$$
$$u(0) = u(1) = 0$$

where $a(x) \geq a_0 > 0$, $c(x) \geq 0$, and $b(x)$ are all smooth functions. This problem has an infinite sequence of positive and distinct eigenvalues

$$0 < \lambda_1 < \lambda_2 < \lambda_3 < \cdots$$

and a corresponding sequence of smooth eigenfunctions $u^1(x)$, $u^2(x)$, $u^3(x)$, ... which we assume normalized so that

$$(1.2)\qquad \int_0^1 |u^p|^2\,dx = 1. \qquad p = 1, 2, \cdots.$$

Of course, as is well known, the transformation

$$(1.3)\qquad u(x) = [\exp(-\tfrac{1}{2}\int_0^x \frac{b(t)}{a(t)}\,dt)]\; v(x)$$

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**Department of Mathematics, Michigan State University, East Lansing, Michigan.

puts (1.1) into the self adjoint form

$$(1.4)\quad \hat{\mathfrak{L}}[v] \equiv -(av')' + (c + \frac{1}{2}b' + \frac{1}{4}\frac{b^2}{a})v = \lambda v$$

$$v(0) = v(1) = 0$$

However, we consider the direct approximation of (1.1) by means of the finite difference equations

$$(1.5)\quad -\frac{\{a_{k+\frac{1}{2}}(w_{k+1} - w_k) - a_{k-\frac{1}{2}}(w_k - w_{k-1})\}}{\Delta x^2} - \frac{b_k(w_{k+1} - w_{k-1})}{2\Delta x}$$

$$+ c_k w_k = \Lambda w_k \qquad k = 1, 2, \cdots M$$

$$w_0 = w_{M+1} = 0$$

where M is a large positive integer, $\Delta x = \frac{1}{M+1}$ is the mesh spacing and the notation g_k is used for $g(k\Delta x)$. Equivalently, we may write (1.5) as the finite dimensional eigenvalue problem:

$$(1.6)\qquad LW = \Lambda W$$

where W is the M component vector $W = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_M \end{bmatrix}$ and L the $M \times M$ tridiagonal matrix

$$(1.7)\qquad L = \frac{1}{\Delta x^2}\begin{bmatrix} \alpha_1 & \beta_1 & & \\ \gamma_2 & \alpha_2 & \beta_2 & \\ & \ddots & \ddots & \ddots \\ & & \ddots & \ddots & \beta_{M-1} \\ & & & \gamma_M & \alpha_M \end{bmatrix}$$

Lemma 1

There exists a non singular, positive, diagonal matrix D such that $D^{-1} L D = \hat{L}$ is a real symmetric matrix. Moreover, $\|D\|_2$, $\|D^{-1}\|_2$ remain bounded as $M \to \infty$, $\Delta x \to 0$, $(M+1)\Delta x = 1$.

Proof: We construct such a matrix.

Let $$D = \begin{bmatrix} d_1 & & & \\ & d_2 & & \\ & & \ddots & \\ & & & d_M \end{bmatrix}$$ where $d_j \neq 0$ $j = 1 \cdots M$ and

$d_1 = 1$ and let $\hat{L} = D^{-1} L D = (\hat{\ell}_{ij})$

Since we require $\hat{L} = \hat{L}^T$ we must have

$$d_i^{-1} \ell_{ij} d_j = d_j^{-1} \ell_{ji} d_i \qquad \text{where } L = (\ell_{ij})$$

Further, since $\ell_{ij} = 0$ for $j > i+1$, $j < i-1$, the d_j's must be determined so that

$$d_i^2 = \frac{\ell_{i,i-1}}{\ell_{i-1,i}} d_{i-1}^2 \qquad i = 2, \cdots M$$

Starting from $d_1 = 1$, we may solve recursively to obtain

$$d_i^2 = \prod_{k=1}^{i-1} \left(\frac{\gamma_{k+1}}{\beta_k}\right) \qquad i = 2, \cdots M$$

and, since γ_k, $\beta_k < 0$ for sufficiently small Δx, $d_i^2 > 0$ if Δx is small enough.

With D constructed as above, we have

$$(2.1) \quad \hat{L} = \frac{1}{\Delta x^2} \begin{bmatrix} \alpha_1 & -(\gamma_2\beta_1)^{\frac{1}{2}} & & \bigcirc \\ -(\gamma_2\beta_1)^{\frac{1}{2}} & \ddots & \ddots & \\ & \ddots & \ddots & \ddots \\ \bigcirc & & -(\gamma_M\beta_{M-1})^{\frac{1}{2}} & \alpha_M \quad -(\gamma_M\beta_{M-1})^{\frac{1}{2}} \end{bmatrix}$$

and we must show that $\|D\|_2$, $\|D^{-1}\|_2$ remain bounded as $M \to \infty$.

Let $Q_i = \prod_{k=1}^{i-1} (1 - \frac{b_{k+1}\Delta x}{2a_{k+\frac{1}{2}}})$ and $P_i = \prod_{k=1}^{i-1} (1 + \frac{b_k \Delta x}{2a_{k+\frac{1}{2}}})$

then $d_i^2 = \frac{Q_i}{P_i}$. Now for sufficiently small Δx,

$$\log (1 - \frac{b_{k+1}\Delta x}{2a_{k+\frac{1}{2}}}) = - \frac{b_{k+1}\Delta x}{2a_{k+\frac{1}{2}}} + O(\Delta x^2)$$

so that

$$\log Q_i = -\Delta x \sum_{k=1}^{i-1} \frac{b_k}{2a_{k+\frac{1}{2}}} + \Delta x \sum_{k=1}^{i-1} O(\Delta x)$$

Hence,

$$\lim_{\substack{\Delta x \to 0,\ i \to \infty \\ i\Delta x = \bar{x}}} [\log Q_i] = -\frac{1}{2} \int_0^{\bar{x}} \frac{b(t)}{a(t)} dt$$

Similarly,

$$\lim_{\substack{\Delta x \to 0,\ i \to \infty \\ i\Delta x = \bar{x}}} [\log Q_i] = \frac{1}{2} \int_0^{\bar{x}} \frac{b(t)}{a(t)}\, dt$$

Consequently, $\lim d_i = e^{-\frac{1}{2} \int_0^{\bar{x}} \frac{b(t)}{a(t)} dt} \leq K_0 < \infty$ which shows both $\|D\|_2$, $\|D^{-1}\|_2$ remain bounded as $\Delta x \to 0$, $M \to \infty$, $(M+1)\Delta x = 1$.

Lemma 2

For Δx sufficiently small, the eigenvalues of L are strictly positive and they remain bounded away from zero as $M \to \infty$, $\Delta x \to 0$, $(M+1)\Delta x = 1$.

Proof: For Δx sufficiently small, γ_k, $\beta_k < 0$. Hence if $L = (\ell_{ij})$ and $\Omega_i = \sum_{j \neq i} |\ell_{ij}|$, then

$$\Omega_i = \frac{a_{i+\frac{1}{2}} + a_{i-\frac{1}{2}}}{\Delta x^2}$$

and $\ell_{ii} = \dfrac{a_{i+\frac{1}{2}} + a_{i-\frac{1}{2}}}{\Delta x^2} + c_i \geq \Omega_i$ since $c_i \geq 0$.

By Gershgorin's theorem, ([7]), the eignevalues of L lie in the union of the discs

$$|z - \ell_{ii}| \leq \Omega_i$$

in the complex plane. Hence if Λ is an eigenvalue of L , then $\Lambda \geq 0$ since Λ is real.

Now let ℓ_h be the finite difference operator corresponding to $-L$ i.e.

$$[\ell_h v]_k \equiv -\left[\frac{(a_{k+\frac{1}{2}} + a_{k-\frac{1}{2}}) + c_k \Delta x^2}{\Delta x^2}\right] v_k + \left[\frac{a_{k+\frac{1}{2}} + \frac{b_k \Delta x}{2}}{\Delta x^2}\right] v_{k+1}$$

$$+ \left[\frac{a_{k-\frac{1}{2}} - \frac{b_k \Delta x}{2}}{\Delta x^2}\right] v_{k-1}$$

Then, for sufficiently small Δx, ℓ_h is of positive type and so satisfies the discrete maximum principle (See [3]). Consequently if $w(k\Delta x)$, $k = 0, 1, \ldots M+1$ is an arbitrary real valued mesh function, there exists positive constraints K and δ such that if $0 < \Delta x < \delta$,

(2.2) $$\|w\|_\infty \equiv \underset{k}{\text{Max}} |w_k| \le \text{Max}\,\{|w_0|, |w_{M+1}|\} + K\,\|(\ell_h w)\|_\infty$$

Now let $V = \{v_k\}_{k=1}^{M}$ be an eigenvector of L corresponding to Λ. We may assume V to be real. Defining $v_0 = v_{M+1} = 0$, $LV = \Lambda V$ is equivalent to

(2.3) $$[\ell_h v]_k = -\Lambda v_k \qquad K = 1, \ldots M .$$

Hence, using (2.2) and the fact that $\Lambda \ge 0$,

$$\|v\|_\infty \le K\,\|(\ell_h v)\|_\infty = \Lambda K\,\|v\|_\infty$$

i.e. $$\Lambda \ge \frac{1}{K} > 0$$ Q.E.D.

<u>Corollary</u>

Let Γ be the $M \times M$ matrix given by

$$\Gamma = \begin{bmatrix} +1 & & & & \bigcirc \\ & -1 & & & \\ & & +1 & & \\ & & & \ddots & \\ \bigcirc & & & & (-)^{M-1} \end{bmatrix}$$

then $\Gamma^{-1} \hat{L} \Gamma$ is an oscillation matrix.

<u>Proof</u>: $\Gamma^{-1} \hat{L} \Gamma$ is a positive definite real symmetric matrix with positive elements along the first super and sub diagonals. The proof now follows from a theorem of Gantmacher and Krein [[4], p. 103].

3. CONVERGENCE OF THE CHARACTERISTIC PAIRS OF L

Let $0 < \Lambda_1 < \Lambda_2 < \cdots < \Lambda_M$ be the eigenvalues of L. Fix a positive integer p and let $V^p(\Delta x)$ be the eigenvector corresponding $\Lambda_p(\Delta x)$, normalized so that $\|V^p\|_2 = 1$. Let $\tilde{V}^p$ be the continuous piecewise linear function, vanishing at $x = 0, 1$, and which, in the interior of $[0, 1]$, is obtained from V^p by linear interpolation. Consider the families $\{\Lambda_p(\Delta x)\}$, $\{\tilde{V}^p(\Delta x)\}$ as the mesh size $\Delta x \to 0$.

A direct proof of convergence of $\tilde{V}^p$ to $u^p(x)$ and Λ_p to λ_p may be given, which is based on the compactness of the family $\{\tilde{V}^p(\Delta x)\}$ in $C[0, 1]$. Such an approach was used by PARTER in [9], (See also [2]); but this method does not yield estimates on the rates of convergence. Nevertheless we will make use of the fact that $\Lambda_p \rightarrow \lambda_p$ together with lemma 1 above to obtain these estimates. The argument given below is a modification of that given by GARY in [6] for the self-adjoint case.

Theorem 1

Let Λ_p, V^p be characteristic pairs of L with $\|V^p\|_2 = 1$. Let D be the diagonal matrix of Lemma 1. Let u^p be an eigenfunction of $\mathcal{L}$ corresponding to λ_p and let U^p be the M vector obtained from u^p by mesh-point evaluation. Assume $u^p(x)$ normalized so that

(3.1) $$\|D^{-1} U^p\|_2 = \|D^{-1} V^p\|_2$$

then as $\Delta x \rightarrow 0$, we have

(3.2) $$|\lambda_p - \Lambda_p| \leq K \Delta x^2$$

(3.3) $$\|U^p - V^p\|_2 \leq K_1 \Delta x^2$$

where K, K_1 are positive constants defining only a p.

Proof: Because the difference scheme in (1.5) is properly centered and we assume sufficient smoothness of u^p and the coefficients of $\mathcal{L}$, we have at the mesh points,

(3.4) $$\mathcal{L}[u^p] = L U^p + \tau = \lambda_p U^p$$

where τ is the "truncation" error and

(3.5) $\quad \|\tau\|_2 \le K(p)\,\Delta x^2$ where K is a constant.

Let $\hat{L} = D^{-1} L\ D$ have orthonormal eigenvectors $X^1, X^2, \ldots X^M$ and write U^p as a linear combination of the DX^j's:

(3.6) $$U^p = \sum_{j=1}^{M} \sigma_j\ DX^j$$

so that

$$LU^p = \sum_{j=1}^{M} \sigma_j\ LDX^j = \sum_{j=1}^{M} \sigma_j \Lambda_j\ DX^j$$

then,

$$\tau = (\lambda_p - L)U^p = \sum_{j=1}^{M} \sigma_j\ (\lambda_p - \Lambda_j)\ DX^j$$

and

(3.7) $$\sum_{j=1}^{M} \sigma_j^2\ |\lambda_p - \Lambda_j|^2 = \|D^{-1}\ \tau\|_2^2 \le \|D^{-1}\|_2^2\ \|\tau\|_2^2$$

$$\le K_1(p)\ \Delta x^4 \text{ where } K_1$$

is a constant.

Now, the eigenvalues of L are distinct and converge to the corresponding distinct eigenvalues of $\mathfrak{L}$. It follows that

(3.8) $$\inf_{j \ne p} \{|\lambda_p - \Lambda_j|\} \ge \omega_0 > 0 \text{ for all sufficiently small } \Delta x .$$

Hence, on using (3.7),

(3.9) $$\sum_{j \ne p} \sigma_j^2 \le K_1\, \Delta x^4$$

and

$$(3.10) \qquad \sigma_p^2 = \| D^{-1} U^p \|_2^2 + O(\Delta x^4) \geq \omega_1 > 0$$

for all sufficiently small Δx.

Thus

$$(3.11) \qquad |\lambda_p - \Lambda_p| \leq K_2(p) \Delta x^2$$

Since $V^p = \beta D X^p$ for some β and $\|X^p\|_2 = 1$ we have

$$|\beta| = \| D^{-1} V^p \|_2$$

On taking square roots in (3.10), we have

$$\sigma_p = \| D^{-1} U^p \|_2 + O(\Delta x^4)$$

and we may assume that σ_p and β have the same sign; hence using (3.1),

$$(3.12) \qquad (\sigma_p - \beta) = O(\Delta x^4)$$

Writing $U^p - V^p = \sum_{j \neq p} \sigma_j D X^j + (\sigma_p - \beta) D X^p$

we have

$$(3.13) \qquad \| D^{-1} (U^p - V^p) \|_2^2 = \sum_{j \neq p} \sigma_j^2 + (\sigma_p - \beta)^2 = O(\Delta x^4)$$

i.e.

$$(3.14) \qquad \| U^p - V^p \|_2^2 \leq \| D \|_2^2 \, \| D^{-1} (U^p - V^p) \|_2^2 \leq K_3(p) \Delta x^4 \qquad \text{Q.E.D.}$$

Notice that the above inequality also implies uniform convergence at the rate of $O(\Delta x)^{3/2}$

4. PROOF OF THEOREM 2

Lemma 3

Let $0 < \Lambda_1 < \cdots < \Lambda_M$ be the eigenvalues of L. Then there exists a positive integer j_0, independent of M, such that for $j_0 \le j \le M$ we have

(4.1) $$K_1 j^2 \pi^2 \le \Lambda_j \le K_2 j^2 \pi^2 \qquad K_1, K_2 \text{ positive constants.}$$

Proof: In the self adjoint case, this result may be found in Bückner [1]. In the present more general case see [2].

Proof of Theorem 2.

Let $W^j = \begin{bmatrix} w_1^j \\ \vdots \\ w_M^j \end{bmatrix}$ be an eigenvector of L corresponding to Λ_j. Then W^j satisfies the difference equations.

(4.2) $$\begin{cases} -\left[2 + \dfrac{(c_k - \Lambda_j)\Delta x^2}{\omega_k}\right] w_k^j + \left[\dfrac{a_{k+\frac{1}{2}} + \frac{b_k \Delta x}{2}}{\omega_k}\right] w_{k+1}^j \\ \quad + \left[\dfrac{a_{k-\frac{1}{2}} - \frac{b_k \Delta x}{2}}{\omega_k}\right] w_{k-1}^j = 0 \qquad k = 1, \cdots M \\ \text{where } w_0^j = w_{M+1}^j = 0 \text{ and } \omega_k = \frac{1}{2}(a_{k+\frac{1}{2}} + a_{k-\frac{1}{2}}) \end{cases}$$

Let $\tilde{\alpha}_k = -[2 + \frac{(c_k - \Lambda_j)\Delta x^2}{\omega_k}]$ $\tilde{\beta}_k = [\frac{a_{k+\frac{1}{2}} + \frac{1}{2} b_k \Delta x}{\omega_k}]$

$$\tilde{\gamma}_k = [\frac{a_{k-\frac{1}{2}} - \frac{1}{2} b_k \Delta x}{\omega_k}]$$

and let A be the tridiagonal $M \times M$ matrix

$$(4.3) \qquad A = \begin{bmatrix} \tilde{\alpha}_1 & \tilde{\beta}_1 & & \bigcirc \\ \tilde{\gamma}_2 & \ddots & \ddots & \\ & \ddots & \ddots & \tilde{\beta}_{M-1} \\ \bigcirc & & \tilde{\gamma}_M & \tilde{\alpha}_M \end{bmatrix}$$

Then we may write (4.2) as

$$(4.4) \qquad AW^j = 0 \qquad \text{or equivalently}$$

$$(4.5) \qquad (P^{-1}AP)\, P^{-1}W^j = 0 \qquad \text{if } P \text{ is any non singular matrix.}$$

Choose P to be the diagonal matrix

$$(4.6) \qquad P = \begin{bmatrix} p_1 & & \bigcirc \\ & \ddots & \\ \bigcirc & & p_M \end{bmatrix}$$

where $p_1 = 1$ and $p_i^2 = \prod_{k=1}^{i-1} \left(\frac{\tilde{\gamma}_{k+1}}{\tilde{\beta}_k}\right) \quad i = 2, \cdots M$.

For all sufficiently small Δx, $p_i^2 > 0$ and as in Lemma 1, P symmetrizes A .

Let $\sigma_k = (\tilde{\gamma}_{k+1} \tilde{\beta}_k)^{\frac{1}{2}}$, then

(4.7) $$P^{-1} AP = \begin{bmatrix} \tilde{\alpha}_1 & \sigma_1 & & \bigcirc \\ \sigma_1 & \ddots & \ddots & \\ & \ddots & \ddots & \sigma_{M-1} \\ \bigcirc & & \sigma_{M-1} & \tilde{\alpha}_M \end{bmatrix}$$

Observe that by the mean value theorem

(4.8) $$\omega_k \omega_{k+1} = (a_{k+\frac{1}{2}})^2 \; [1 + O(\Delta x^2)] \quad \text{as} \quad \Delta x \to 0$$

Also if $b(x) \in C^1[0,1]$,

(4.9) $$(\tilde{\gamma}_{k+1} \tilde{\beta}_k) = \frac{[(a_{k+\frac{1}{2}})^2 + a_{k+\frac{1}{2}} \frac{(b_k - b_{k+1})\Delta x}{2} - \frac{b_k b_{k+1} \Delta x^2}{4}]}{\omega_k \omega_{k+1}}$$

$$= \frac{(a_{k+\frac{1}{2}})^2 \; [1 + O(\Delta x^2)]}{(a_{k+\frac{1}{2}})^2 \, [1 + O(\Delta x^2)]} \qquad \text{as} \quad \Delta x \to 0$$

Hence,

(4.10) $$\sigma_k = (\tilde{\gamma}_{k+1} \tilde{\beta}_k)^{\frac{1}{2}} = 1 + O(\Delta x^2) \quad \text{as} \quad \Delta x \to 0$$

Let $V = P^{-1} W^j$ and write the system (4.5) as

$$- \left[2 + \frac{(c_k - \Lambda_j)\,\Delta x^2}{\omega_k} \right] v_k + \sigma_k\, v_{k+1} + \sigma_k\, v_{k-1} = 0 \tag{4.11}$$

$$v_0 = v_{M+1} = 0 \qquad K = 1, \cdots M$$

Let K_1 and K_2 be the constants in Lemma 3 and define

$$\beta_j^2 = \frac{\Lambda_j}{K_2} \tag{4.12}$$

Let $y(x) = \operatorname{Sin} \beta_j x$. Then $y_k = y(k\Delta x)$ satisfies the difference equations.

$$- [2 - \mu_j \Delta x^2]\, y_k + y_{k+1} + y_{k-1} = 0 \qquad k = 1, 2, \ldots . \tag{4.13}$$

where

$$\mu_j = \frac{4}{\Delta x^2} \operatorname{Sin}^2 \frac{\beta_j \Delta x}{2} \tag{4.14}$$

The distance between successive zeros of $y(x)$ is $\dfrac{\pi}{\beta_j} = \dfrac{K_2 \pi^2}{\Lambda_j} \geq \dfrac{1}{j}$

for j large enough by Lemma 3.

Let $v(x)$ be the piecewise linear function corresponding to "graph" of vector $V = P^{-1} W^j$. Define the auxiliary function $z(x)$ by

$$z(x) = \frac{y(x)}{v(x)} \quad \text{whenever } v(x) \neq 0$$

We proceed to estimate the distance between successive nodes of $v(x)$ by investigating the difference equation satisfied by $z(x)$.

We may assume that $\delta_{Max}(V) > 3\Delta x$. For if $\delta_{Max}(V) \le 3\Delta x$, then in particular, $\delta_{Max}(V) \le \frac{3}{M+1} < \frac{3}{j} \le 3\pi \frac{K_2}{\Lambda_j}$ for all sufficiently large j. If $\delta_{Max} > 3\Delta x$, then there exists a set N of consecutive mesh points, containing at least three members on which $v(x)$ is strictly positive (or strictly negative). Let N' be N minus the 2 end points of N. Since $z_k = \frac{y_k}{v_k}$ for $k \in N'$,

$$(4.17) \qquad [\ell_h z]_k \equiv - \left[\frac{(2-\mu_j \Delta x^2)\sigma_k}{2 + \frac{(c_k - \Lambda_j)\Delta x^2}{\omega_k}} (v_{k+1} + v_{k-1}) \right] z_k$$

$$+ v_{k+1} z_{k+1} + v_{k-1} z_{k-1} = 0 \qquad K \in N' .$$

<u>We now show that for all sufficiently large j, the difference operator</u> ℓ_h (or $-\ell_h$ if v is strictly negative) <u>occurring in (4.17) is of positive type, and hence satisfies the discrete maximum principle:</u>

It is sufficient to show that if j is sufficiently large,

$$(4.18) \qquad \frac{[2-\mu_j \Delta x^2]\sigma_k}{2 + \frac{(c_k - \Lambda_j)\Delta x^2}{\omega_j}} \ge 1 \qquad \text{if } K \in N' .$$

From (4.14) we have $\mu_j \le \frac{\Lambda_j}{K_2} \le \frac{\Lambda_j}{2a_1}$ if K_2 is chosen so that $K_2 \ge 2a_1$, where a_1 is an upper bound for $a(x)$ on $[0,1]$. Hence,

$$(4.19) \qquad (2-\mu_j \Delta x^2)\sigma_k = 2 - \mu_j \Delta x^2 + O(\Delta x^2)$$

since $\mu_j \Delta x^2 \le 4$ and $\sigma_k = 1 + O(\Delta x^2)$.

Now,

$$2 - \mu_j \Delta x^2 + O(\Delta x^2) \ge 2 - \frac{\Lambda_j \Delta x^2}{K_2} + O(\Delta x^2)$$

$$\ge 2 - \frac{\Lambda_j \Delta x^2}{2\omega_k} + O(\Delta x^2)$$

$$= 2 + \frac{(c_k - \Lambda_j)\Delta x^2}{\omega_k} + \frac{(\Lambda_j - 2c_k)\Delta x^2}{2\omega_k} + O(\Delta x^2)$$

i.e.

(4.20) $\quad (2 - \mu_j \Delta x^2)\, \sigma_k \ge 2 + \underline{(c_k - \Lambda_j)}\Delta x^2$ if j is sufficiently large,

since we assume $c(x)$ is bounded.

Furthermore $2 + \frac{(c_k - \Lambda_j)\Delta x^2}{\omega_k}$ is positive for $K \in N'$ since v_k, v_{k+1}, v_{k-1} have the same sign, on using (4.11). Thus (4.18) is satisfied.

Suppose now that $z(x)$ has two zeroes in the interval spanned by N . At any mesh point lying between the two zeroes we must have $z(x) = 0$ by the maximum principle. Since $z(x) = 0$ if and only if $y(x) = 0$, this means that the distance between successive zeroes of $y(x)$ is $\le \Delta x = \frac{1}{M+1}$. However, as already noted, this distance is $\ge \frac{1}{j}$ and $j \le M$.

Thus $y(x)$ has at most one zero in the interval spanned by N. Hence the maximum distance between successive modes of $v(x)$ must be less than or equal to $\frac{\pi}{\beta_j} + 2\Delta x$.

Since $\Lambda_j = O(\frac{1}{\Delta x^2})$, we have

(4.21) $$\delta_{Max}(V) \leq K(\Lambda_j)^{-\frac{1}{2}}$$

A similar estimate is valid for the eigenvector W^j of L since $W^j = PV$ and P is a positive diagonal matrix. Q.E.D.

5. REMARKS

(a) It seems plausible that one also has an estimate

(4.22) $\delta_{Min} \geq K_0(\Lambda_j)^{-\frac{1}{2}}$ for j large for the minimum distance between the nodes of W^j.

(b) The estimate in Theorem 2 may be combined with Lemma 3 to show that if the eigenvectors $\{V^p\}$ of L are normalized so that $\|V^p\|_2 = 1$, then

(4.23) $\|V^p\|_\infty \leq K_1 \, p^{\frac{1}{2}}$ for all sufficiently large p. (See [2]). Such an estimate was obtained by Bückner in the self-adjoint case using an elementary device ([1]). Combined with Lemma 3, (4.23) shows that

$$\sum_p \frac{\|V^p\|_\infty}{\Lambda_p} \text{ remains bounded as } M \to \infty .$$

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