SPLINE APPROXIMATION OF THE CAUCHY PROBLEM

$$\frac{\partial^{p+q} u}{\partial x^p \partial y^q} = f(x, y, u, \ldots . \frac{\partial^{i+j} u}{\partial x^i \partial y^j}, \ldots .)$$

To-yat Cheung

Computer Sciences Technical Report #29

August, 1968

CONTENTS

INTRODUCTION

Consider the Cauchy problem

$$\frac{\partial^{p+q} u}{\partial x^p \partial y^q} = f(x, y, u, \cdots \frac{\partial^{i+j} u}{\partial x^i \partial y^j}, \cdots), \quad x \in [0, a], \ y \in [0, b], \ i + j < p + q$$

with initial conditions

$$\frac{\partial^j u}{\partial y^j} = \phi^j(x), \quad x \in [0,a], \quad j = 0, 1, \ldots q-1 \ ,$$

$$\frac{\partial^i u}{\partial x^i} = \psi^i(y), \quad x \in [0, b], \quad i = 0, 1, \ldots p-1 \ .$$

The particular case when $p = q = 1$ has been investigated by many mathematicians. Zwirner [7] and Diaz [2] suggested the Euler-Cauchy polygon method. Moore [5] derived methods of Runge-Kutta type. Day [1] introduced a quadrature method.

The case for general p and q is much less studied. Walter [6] proved the existence and uniqueness of the solution through some existence theory of integral equations. In [4], Margolis showed the existence, uniqueness and convergence of successive approximations under various assumptions. In [3], a special case $p = q = n$ is considered. But all these are not computational methods.

In this report, a spline approximation method is suggested (Section 2). Convergence of the method is proved (Section 3). In Section 4, an a posterior error estimate and an a posterior error bound are derived. The instability property of the method is also discussed. Some of the numerical results are presented in Section 5.

SECTION 0. NOTATIONS

In most cases, the following rules will be used for defining notations:

1. Scalars are denoted by small letters.
2. Vectors are denoted either by capital letters (e.g. V) or by small letters with a bar below (e.g. $\underline{v}(x, y)$).
3. Matrices are denoted by capital letters.
4. Superscripts of functions denote derivatives (e.g. $u^{ij}(x, y) = \frac{\partial^{i+j}}{\partial x^i \partial y^j} u(x, y)$).

Without explicitly redefined, the following notations will be used throughout all sections:

1) $i \in P$ means $i = 0, 1, \ldots . p$

$i \in P^1$ means $i = 0, 1, \ldots . p - 1$

$j \in Q$ means $j = 0, 1, \ldots . q$

$j \in Q^1$ means $j = 0, 1, \ldots . q - 1$

$m \in M$ means $m = 1, 2, \ldots M$

$m \in M^1$ means $m = 1, 2, \ldots M + 1$

$m \in M_2$ means $m = 2, 3, \ldots M$

$n \in N$ means $n = 1, 2, \ldots N$

$n \in N^1$ means $n = 1, 2, \ldots N + 1$

$n \in N_2$ means $n = 2, 3, \ldots N$

2) R - the region: $0 \le x \le a, \quad 0 \le y \le b$

$\bar{\Delta}$ - a mesh over R: $\begin{cases} x_m = (m-1)h, & \text{where } m \in M^1, & Mh = a \\ y_n = (n-1)k, & \text{where } n \in N^1, & Nk = b \end{cases}$

Δ - a deleted mesh: $\begin{cases} x_m = (m-1)h, & \text{where } m \in M, & Mh = a \\ y_n = (n-1)k, & \text{where } n \in N, & Nk = b \end{cases}$.

R_{mn} - a subregion: $x_m \le x < x_{m+1}, \quad y_n \le y < y_{n+1}$.

$C^{p,q}[R]$ - The class of functions whose derivatives up to order p in x and order q in y exist and are continuous.

3) (p) - spline - a one-dimensional spline of degree p .

(p,q) - spline - a two-dimensional spline which is a (p)-spline in x and a (q)-spline in y .

sp-0, sp-1, sp-2 - spline relations

$s^{ij}(x, y)$, $s^{ij}_{mn}(x, y)$ - the (i, j)-derivative of a spline function

$\underline{s}(x, y) = \text{col}\,(s^{00}(x, y), s^{10}(x, y), \cdots s^{ij}(x, y), \cdots s^{pq}(x, y))$

s^{ij}_{mn} = spline coefficients representing the (i, j)-derivative at the grid point (x_m, y_n)

$\underline{s}_{mn} = \text{col}\,(s^{00}_{mn}, s^{10}_{mn}, \cdots s^{ij}_{mn}, \cdots s^{pq}_{mn})$

$\underline{r}^{i}_{mn} = \text{row}\,(s^{i0}_{mn}, s^{i1}_{mn}, \cdots s^{ij}_{mn}, \cdots s^{i,q-1}_{mn})$

$\underline{c}^{j}_{mn} = \text{col}\,(s^{0j}_{mn}, s^{1j}_{mn}, \cdots s^{ij}_{mn}, \cdots s^{p-1,j}_{mn})$

$\Phi^{i}_{s}(x) = \text{row}\,(s^{i0}(x, 0), s^{i1}(x, 0), \cdots s^{i,q-1}(x, 0))$

$\Psi^{j}_{s}(y) = \text{col}\,(s^{0j}(0, y), s^{1j}(0, y), \cdots s^{p-1,j}(0, y))$

$$\Phi_s = \begin{vmatrix} s_{11}^{00} & s_{11}^{01} & \cdots & s_{11}^{0,q-1} \\ s_{11}^{10} & s_{11}^{11} & \cdots & s_{11}^{1,q-1} \\ \vdots & \vdots & & \vdots \\ s_{11}^{p-1,0} & s_{11}^{p-1,1} & \cdots & s_{11}^{p-1,q-1} \end{vmatrix} = \Psi_s$$

4) $D_x^i = \dfrac{\partial^i}{\partial x^i}, \quad D_y^j = \dfrac{\partial^j}{\partial y^j}$.

5) $\varphi^j(x), \psi^i(y),\ j \in Q^1,\ i \in P^1$ -- initial function of Problem C .

$$\varphi_m^j = \varphi^j(x_m), \quad \psi_n^j = \psi^j(y_n)$$

$$\Phi^i(x) = \text{row}\ (D_x^i\, \varphi^0(x),\ D_x^i\, \varphi^1(x), \cdots D_x^i\, \varphi^{q-1}(x)\ )$$

$$\Psi^j(y) = \text{col}\ (D_y^j\, \psi^0(y),\ D_y^j\, \psi^1(y), \cdots D_y^j\, \psi^{p-1}(y)\ )$$

$$\Phi = \begin{bmatrix} \varphi^0(0) & \varphi^1(0) & \cdots & \varphi^{q-1}(0) \\ D_x^1\ \varphi^0(0) & D_x^1\ \varphi^1(0) & \cdots & D_x^1\ \varphi^{q-1}(0) \\ \vdots & \vdots & & \vdots \\ D_x^{p-1}\varphi^0(0) & D_x^{p-1}\varphi^1(0) & \cdots & D^{p-1}\,\varphi^{q-1}(0) \end{bmatrix}$$

$$\Psi = \begin{bmatrix} \psi^0(0) & D_y^1\,\psi^0(0) & \cdots & D_y^{q-1}\ \psi^0(0) \\ \psi^1(0) & D_y^1\,\psi^1(0) & \cdots & D_y^{q-1}\ \psi^1(0) \\ \vdots & \vdots & & \vdots \\ \psi^{p-1}(0) & D_y^1\,\psi^{p-1}(0) & \cdots & D_y^{q-1}\ \psi^{p-1}(0) \end{bmatrix}$$

6) $X(x) = \text{row}\ (1, x, \frac{x^2}{2}, \cdots \frac{x^{p-1}}{(p-1)!})$

$$X^i(x) = D_x^i X(x) = \begin{cases} \text{row}\ (0, 0, \cdots 1, x, \cdots \frac{x^{p-i-1}}{(p-i-1)!}), & i \in P^1 \\ \underline{0} & i \geq p \end{cases}$$

$$Y(y) = \text{col}\ (1, y, \frac{y^2}{2}, \cdots \frac{y^{q-1}}{(q-1)!})$$

$$Y^j(y) = D_y^j Y(y) = \begin{cases} \text{col}\ (0, 0, \cdots 1, y, \cdots \frac{y^{q-j-1}}{(q-j-1)!}), & j \in Q^1 \\ \underline{0} & j \geq q \end{cases}$$

7) $(\underline{u}, \underline{v})$ - inner product of two vectors $\underline{u}$ and $\underline{v}$

8) $$I^{ij}[x, y, k(x, y, \underline{v}(x, y))] = \int_0^y \int_0^{\eta_{q-j-1}} \cdots \int_0^{\eta_1} \int_0^x \int_0^{\xi_{p-i-1}} \cdots \int_0^{\xi_1}$$

$k(\alpha, \beta, \underline{v}(\alpha, \beta))\, d\xi\, d\eta$ where $d\xi = d\alpha\, d\xi_1 \cdots d\xi_{p-i-1}$, $d\eta = d\beta\, d\eta_1 \cdots d\eta_{q-j-1}$

$$G^{ij}[x, y, \underline{r}(x), \underline{c}(y), V] = (X^i(x), \underline{c}(y)) + (\underline{r}(x), Y^j(y)) - X^i(x)\, V Y^j(y)$$

where $\underline{r}(x)$ is a q-vector, $\underline{c}(y)$ is a p-vector, V is a $p \times q$ constant matrix.

9) $\|\underline{v}\| = \max_i |v^i|$, where $\underline{v} = \text{col}\ (v^1, v^2, \cdots v^i, \cdots v^p)$

$\|V\| = \max_i \sum_j |v^{ij}|$, where $V = (v^{ij})$ is a matrix.

10) $\Longrightarrow$ means 'converges uniformly to'

SECTION 1. MATHEMATICAL PRELIMINARIES

1.1 Some Aspects in Functional Analysis.

Definitions

1) R - a bounded closed domain in the real space E^m.

2) $S[R]$- a sequence of real vector functions $\underline{s}_\mu(\underline{x}) = (s_\mu^1(\underline{x}), \cdots s_\mu^p(\underline{x}))$

3) $C[R]$- the set of all real vector functions $\underline{u}(\underline{x}) = (u^1(\underline{x}), \cdots u^p(\underline{x}))$ with continuous components defined on R.

4) $S[R]$- is said to form an Arzelà sequence if it satisfies the following property:

Given $\epsilon > 0$, there exist μ_ϵ and $\delta_\epsilon > 0$ such that, if $\mu > \mu_\epsilon$ and $\|\underline{x}_1 - \underline{x}_2\| < \delta_\epsilon$, $\underline{x}_1, \underline{x}_2 \in R$, then $\|\underline{s}_\mu(\underline{x}_1) - \underline{s}_\mu(x_2)\| < \epsilon$.

Theorem 1.1 (Arzelà)

Let $S[R]$ and $C[R]$ be defined as above. If $S[R]$ is uniformly bounded and forms an Arzelà sequence, then it contains a subsequence which converges uniformly to an element in $C[R]$.

Proof. See appendix of [5].

For later application, we should notice one fact: A vector $\underline{s}_\mu(\underline{x})$ in $S[R]$ may have discontinuous components. However, any discontinuity should become small as $\mu \to \infty$.

Example 1.1 Let $R = [0, a]$ and $s_\mu(x)$ be a scalar function

$$s_\mu(x) = \begin{cases} 0 & 0 \le x < \frac{a}{2} \\ \frac{1}{2\mu} & \frac{a}{2} \le x \le a \end{cases}$$

Each $s_\mu(x)$ is discontinuous, but $s_\mu(x) \Longrightarrow 0$ as $\mu \to \infty$.

1.2 A Special Matrix.

We shall encounter matrices of the form

$$B = B(k, q, L) = \begin{bmatrix} 1 + \frac{\ell_0 k^q}{q!} & k + \frac{\ell_1 k^q}{q!} & \frac{k^2}{2!} + \frac{\ell_2 k^q}{q!} & \cdots & \frac{k^{q-1}}{(q-1)!} + \frac{\ell_{q-1} k^q}{q!} \\ \frac{\ell_0 k^{q-1}}{(q-1)!} & 1 + \frac{\ell_1 k^{q-1}}{(q-1)!} & k + \frac{\ell_2 k^{q-1}}{(q-1)!} & \cdots & \frac{k^{q-2}}{(q-2)!} + \frac{\ell_{q-1} k^{q-1}}{(q-1)!} \\ \frac{\ell_0 k^{q-2}}{(q-2)!} & \frac{\ell_1 k^{q-2}}{(q-2)!} & 1 + \frac{\ell_2 k^{q-2}}{(q-2)!} & \cdots & \frac{k^{q-3}}{(q-3)!} + \frac{\ell_{q-1} k^{q-2}}{(q-2)!} \\ \vdots & \vdots & \vdots & & \vdots \\ \frac{\ell_0 k}{1!} & \frac{\ell_1 k}{1!} & \frac{\ell_2 k}{1!} & \cdots & 1 + \frac{\ell_{q-1} k}{1!} \end{bmatrix}$$

where $L = \mathrm{col}\,(\ell_0, \ell_1, \cdots \ell_{q-1})$.

Lemma 1.1 For sufficiently small k, $\|B\| < e^{\hat{\ell} k}$, where $\hat{\ell} = \max(\ell, 1)$ and $\ell = \sum_{i=0}^{q-1} \ell_i$.

Proof. The sum of the jth row of B is

$$r(j) = 1 + k + \frac{k^2}{2!} + \cdots\cdots + \frac{k^{q-j}}{(q-j)!} + \frac{\ell k^{q-j+1}}{(q-j+1)!}, \quad j = 1, 2, \cdots q .$$

Two cases:

1. $\ell \le 1$. Then, $\|B\| = \max_j |r(j)| < e^k$ (1)

2. $\ell > 1$. For $j = 1, 2, \cdots q-1$

$$r(j) - r(j+1) = \frac{\ell k^{q-j+1}}{(q-j+1)!} + \frac{k^{q-j}}{(q-j)!} - \frac{\ell k^{q-j}}{(q-j)!}$$

$$\le \frac{\ell k^{q-j}}{2\cdot(q-j)!}\left[k - 2\left(1 - \frac{1}{\ell}\right)\right]$$

Hence, if $k < 2(1 - \frac{1}{\ell})$, then $r(j) < r(j+1)$, i.e. the last row has a maximum sum. Therefore, the row-norm is

$$\|B\| = r(q) = 1 + \ell k < e^{\ell k}$$

Combining with (1), we have

$$\|B\| < e^{\hat{\ell} k} .$$

1.3 Two Types of Spline Approximation (Method A and Method B)

For simplicity, a one-dimensional spline of degree p is called a <u>(p)-spline.</u> In $[x_m, x_{m+1}]$, the i^{th} derivative of a (p)-spline has a representation of the form (for $i = p$, the end-point x_{m+1} is excluded):

$$s^i(x) = s^i_m(x) = s^i_m + (x - x_m)\, s^{i+1}_m + \cdots + \frac{(x - x_m)^{p-i}}{(p-i)!}\, s^p_m, \quad i \in P \qquad (2)$$

Matching the derivatives up to order $(p-1)$ at the intermediate grid points leads to the following relation of the spline coefficients

<u>sp-0:</u> $$s^i_{m+1} = s^i_m + h\, s^{i+1}_m + \cdots + \frac{h^{p-i}}{(p-i)!}\, s^p_m , \quad i \in P^1, \quad m \in M$$

Next, consider the following approximation problem:

Problem A Given $\varphi(x) \in C^p[0,a]$. Find a (p)-spline $s^0(x)$ which has the form (2) and satisfies

(i) $s_1^i = D_x^i \varphi(0), \quad i \in P^1$.

(ii) $s_m^{p-1} = D_x^{p-1} \varphi(x_m), \quad m \in M^1$.

Notations: $\varphi_m^i = D_x^i \varphi(x_m)$; $e^i(x) = D_x^i \varphi(x) - s_m^i(x)$; $e_m^i = e^i(x_m)$.

The following method solves Problem A .

Method A .

Step 1. Set $s_m^{p-1} = \varphi_m^{p-1}, \quad m \in M^1$

Step 2. Set $s_1^i = \varphi_1^i, \quad i = 0, 1, 2, \cdots p-2$

$$s_1^p = (s_2^{p-1} - s_1^{p-1})/h$$

Step 3. If the spline coefficients $\{s_m^i, i \in P\}$ in $[x_m, x_{m+1}]$ have been obtained, then those in $[x_{m+1}, x_{m+2}]$ can be calculated as follows:

$$s_{m+1}^i = s_m^i + h\, s_m^{i+1} + \frac{h^2}{2} s_m^{i+2} + \cdots + \frac{h^{p-i}}{(p-i)!} s_m^p, \quad i = 0, 1, \cdots p-2$$

$$s_{m+1}^{p-1} = \varphi_{m+1}^{p-1}$$

$$s_{m+1}^p = (s_{m+2}^{p-1} - s_{m+1}^{p-1})/h .$$

We have the following convergence theorem:

Theorem 1.2

If $\varphi(x) \in C^p[0,a]$, then

(i) $e^p(x) = o(1)$

(ii) $e^i(x) = o(h), \quad i \in P^1$.

If, in addition, $D_x^p \varphi(x)$ satisfies Hölder condition of order $\alpha (0 < \alpha \le 1)$

$$|D_x^p \varphi(x) - D_x^p \varphi(x^1)| \le K|x - x^1|^\alpha, \quad x, x^1 \in [0, a],$$

then

(i') $e^p(x) = O(h^\alpha)$

(ii') $e^i(x) = O(h^{1+\alpha}), \quad i \in P^1$.

Proof.

i) Expanding $\varphi_{m+1}^{p-1} = D_x^{p-1} \varphi(x_{m+1})$ and $s_{m+1}^{p-1}(x_{m+1})$ as Taylor's series about x_m,

$$\varphi_{m+1}^{p-1} = \varphi_m^{p-1} + hD_x^p \varphi(\xi_m), \quad x_m < \xi_m < x_{m+1}$$

$$s_{m+1}^{p-1} = s_m^{p-1} + hs_m^p(\eta_m), \quad x_m < \eta_m < x_{m+1}$$

But, by Step 1 of Method A, $s_m^{p-1} = \varphi_m^{p-1}$, $s_{m+1}^{p-1} = \varphi_{m+1}^{p-1}$. Hence,

$$D_x^p \varphi(\xi_m) - s_m^p(\eta_m) = 0 .$$

Let x be arbitrary in $[x_m, x_{m+1})$. Since $D_x^p \varphi$ is continuous, we have $D_x^p \varphi(\xi_m) = D_x^p \varphi(x) + o(1)$. Also $s_m^p(\eta_m) = s_m^p(x)$. Hence

$$e^p(x) = D_x^p \varphi(x) - s_m^p(x) = o(1).$$

ii) Mathematical induction will be used .

First, $e^{p-1}(x) = e_m^{p-1} + (x-x_m)\, e^p(\xi_m) \quad x_m < \xi_m < x$

$$\therefore \; e^{p-1}(x) = 0 + (x-x_m) \cdot o(1) = o(h)$$

Next, suppose that $e^t(x) = o(h)$ for $t = p-1, p-2, \cdots i+2, i+1$.

For $i \le p-2$, expand $e^i(x)$ about x_m,

$$e^i(x) = e^i_m + (x - x_m)\, e^{i+1}_m + \frac{(x - x_m)^2}{2!} e^{i+2}_m \tag{3}$$
$$+ \cdots + \frac{(x - x_m)^{p-1}}{(p - i)!} e^p(\xi_m)$$

where $x_m < \xi_m < x$. Since there is only a finite number of terms on the right side of (3), we have

$$e^i(x) = e^i_m + (x - x_m)\cdot o(h) + (x - x_m)^{p-i} \cdot o(1), \quad i \le p - 2 \tag{4}$$

For $x = x_{m+1}$,

$$e^i_{m+1} = e^i_m + o(h^2), \qquad i \le p - 2$$
$$\therefore\ e^i_{m+1} = e^i_1 + (m - 1)\cdot o(h^2) = 0 + o(h)$$

Hence, from (4)

$$e^i(x) = o(h) + (x - x_m)\cdot o(h) + (x - x_m)^{p-i} \cdot o(1), \quad i \le p - 2$$
$$= o(h)$$

If $D^p_x\ \varphi(x)$ satisfies Hölder condition of order α, $o(1)$ and $o(h)$ may be replaced by $O(h^\alpha)$ and $O(h^{1+\alpha})$ respectively.

The proofs is completed.

For $\varphi(x) \in C^{p+1}[0,a]$, $p \ge 2$, let us consider another type of spline approximation.

Problem B Given $\varphi(x) \in C^{p+1}[0,a]$, $p \geq 2$. Find a (p)-spline $s^0(x)$ which has the form (2) and satisfies

(i) $s_1^i = D_x^i \varphi(0) \qquad i \in P^1$

(ii) $s_m^{p-2} = \varphi_m^{p-2} \qquad m \in M^1$.

The following method solves Problem B .

Method B

Step 1. Set $s_m^{p-2} = \varphi_m^{p-2} \qquad m \in M^1$

Step 2. Set $s_1^{p-1} = \varphi_1^{p-1}$

$$s_1^p = 2(s_2^{p-2} - s_1^{p-2} - h\, s_1^{p-1})/h^2$$

Step 3. If the spline coefficients $\{s_m^i,\ i \in P\}$ in $[x_m, x_{m+1}]$ have been obtained, then those in $[x_{m+1}, x_{m+2}]$ can be calculated as follows:

$$s_{m+1}^i = s_m^i + h\, s_m^{i+1} + \cdots + \frac{h^{p-i}}{(p-i)!} s_m^p, \quad i = 0, 1, \cdots p-3,\ p-1$$

$$s_{m+1}^{p-2} = \varphi_{m+1}^{p-2}$$

$$s_{m+1}^p = 2(s_{m+2}^{p-2} - s_{m+1}^{p-2} - h\, s_{m+1}^{p-1})/h^2$$

We have the following convergence theorem:

Theroem 1.3

If $\varphi(x) \in C^{p+1}[0,a]$, then

(i) $e^{p}(x) = o(1)$

(ii) $e^{p-1}(x) = o(h)$

(iii) $e^{i}(x) = o(h^2)$, $i = 0, 1, \cdots p-2$

Proof.

For $x \in [x_m, x_{m+1}]$ (when $i = p$, the end point x_{m+1} is excluded),

$$e^{i}(x) = e^{i}_{m} + (x - x_m)\, e^{i+1}_{m} + \cdots + \frac{(x - x_m)^{p-i}}{(p-i)!}\, e^{p}_{m} \tag{5}$$

$$+ \frac{(x - x_m)^{p-i+1}}{(p-i+1)!}\, \varphi^{p+1}(\xi^{i}_{m})$$

where $x_m < \xi^{i}_{m} < x$, $i \in P$.

In particular, for $i = p-2$, $p-1$

$$e^{p-2}_{m+1} = e^{p-2}_{m} + h\, e^{p-1}_{m} + \frac{h^2}{2}\, e^{p}_{m} + \frac{h^3}{3!}\, \varphi^{p+1}(\xi^{p-2}_{m}) \tag{6}$$

$$e^{p-1}_{m+1} = e^{p-1}_{m} + h\, e^{p}_{m} + \frac{h^2}{2!}\, \varphi^{p+1}(\xi^{p-1}_{m}) \tag{7}$$

Eliminating e^{p}_{m} from (6) and (7), we get

$$e^{p-1}_{m+1} = -e^{p-1}_{m} + \frac{2}{h}\,(e^{p-2}_{m+1} - e^{p-2}_{m}) - h^2\, [\frac{1}{3}\, \varphi^{p+1}(\xi^{p-2}_{m}) - \frac{1}{2}\, \varphi^{p+1}(\xi^{p-1}_{m})\,] .$$

By induction, it is easy to show that

$$e^{p-1}_{m} = (-1)^{m-1}\, e^{p-1}_{1} + \frac{2}{h} \sum_{\ell=1}^{m-1} (-1)^{m+\ell-1}\, (e^{p-2}_{\ell+1} - e^{p-2}_{\ell})$$

$$+ h^2 \{\frac{1}{3} \sum_{\ell=1}^{m-1} (-1)^{m+\ell}\, \varphi^{p+1}(\xi^{p-2}_{\ell}) - \frac{1}{2} \sum_{\ell=1}^{m-1} (-1)^{m+\ell}\, \varphi^{p+1}(\xi^{p-1}_{\ell})\, \}$$

By Method B, $e_1^{p-1} = 0$ and $e_\ell^{p-2} = 0$, $\ell \in M^1$. Hence,

$$e_m^{p-1} = h^2 \left\{ \frac{1}{3} \sum_{\ell=1}^{m-1} (-1)^{m+\ell} \varphi^{p+1} (\xi_\ell^{p-2}) - \frac{1}{2} \sum_{\ell=1}^{m-1} (-1)^{m+\ell} \varphi^{p+1} (\xi_\ell^{p-1}) \right\} \quad (8)$$

The derivatives $\varphi^{p+1}(x)$ in each sum of (8) have alternating signs. Since $\varphi(x) \in C^{p+1}[0, a]$ and $|\xi_1^{p-2} - \xi_2^{p-2}| \le 2h$, we have $\varphi^{p+1}(\xi_2^{p-2}) - \varphi^{p+1}(\xi_1^{p-2}) = o(1)$, etc. Hence, the expression inside { } of (8) has the same order as $(m-1) \cdot o(1)$. Therefore,

$$e_m^{p-1} = h^2 \cdot (m-1) \cdot o(1) = o(h).$$

Then (7) immediately implies that $e_m^p = o(1)$.

In (5), setting $i = p$, we have, for $x_m \le x < x_{m+1}$,

$$e^p(x) = e_m^p + (x - x_m) \varphi^{p+1} (\xi_m^p)$$
$$= o(1).$$

In (5), setting $i = p - 1$, we have

$$e^{p-1}(x) = e_m^{p-1} + (x - x_m) e_m^p + \frac{(x - x_m)^2}{2} \varphi^{p+1} (\xi_m^{p-1})$$
$$= o(h) + (x - x_m) o(1) + \frac{(x - x_m)^2}{2} \varphi^{p+1} (\xi_m^{p-1})$$
$$= o(h).$$

In (5), setting $i = p - 2$ and recalling $e_m^{p-2} = 0$, we have

$$e^{p-2}(x) = 0 + (x - x_m) e_m^{p-1} + \frac{(x - x_m)^2}{2} e_m^p + \frac{(x - x_m)^3}{6} \varphi^{p+1} (\xi_m^{p-2})$$
$$= (x - x_m) o(h) + \frac{(x - x_m)^2}{2} o(1) + \frac{(x - x_m)^3}{6} \varphi^{p+1} (\xi_m^{p-2})$$
$$= o(h^2).$$

Next, suppose $e^i(x) = o(h^2)$ for $i = p-2, p-3, \cdots k+2, k+1$. Set $i = k$ in (5).

$$e^k(x) = e_m^k + (x - x_m)\, e_m^{k+1} + \cdots + \frac{(x - x_m)^{p-k}}{(p - k)!}\, e_m^p + \frac{(x - x_m)^{p-k+1}}{(p - k + 1)!}$$

$$\varphi^{p+1}(\xi_m^k) = e_m^k + (x - x_m)\, o(h^2) \qquad (9)$$

$$\therefore\ e_{m+1}^k = e_m^k + o(h^3)$$

$$\therefore\ e_m^k = e_1^k + (m - 1)\, o(h^3)$$

Since $e_1^k = 0$,

$$e_m^k = o(h^2)\ .$$

From (9), we have

$$e^k(x) = o(h^2)\ .$$

The proof is completed.

Now, suppose we have a convergent sequence of meshes $\{\Delta_\mu\ ,\ \mu = 1, 2, \cdots\}$ and a corresponding sequence of spline approximations $s^i_{\mu, m}(x)$. Theorem 1.2 (or Theorem 1.3) implies

Corollary 1.1 The derivative $s^i_{\mu, m}(x)$ of the spline approximation obtained by Method A (or Method B) converges uniformly to the corresponding derivative $D^i_x\, \varphi(x)$ of the given function, as $\mu \to \infty$.

Corollary 1.2 The spline coefficients $s^i_{\mu, m}$ obtained by Method A (or Method B) are uniformly bounded.

1.4 (p, q) - splines.

A (p, q)-spline is a polynomial in two variables x and y such that it is a (p)-spline with respect to x and a (q)-spline with respect to y. In R^{mn}, the (i, j)-derivative of a (p, q)-spline has the following representation:

$$
\begin{aligned}
s^{ij}(x,y) = s^{ij}_{mn}(x,y) &= [s^{ij}_{mn} + (x-x_m)\, s^{i+1,j}_{mn} + \dots + \frac{(x-x_m)^{p-i}}{(p-i)!}\, s^{pj}_{mn}] \\
&+ (y-y_n)\, [s^{i,j+1}_{mn} + (x-x_m)\, s^{i+1,j+1}_{mn} + \dots + \frac{(x-x_m)^{p-i}}{(p-i)!}\, s^{p,j+1}_{mn}] \\
&+ \dots\dots\dots\dots\dots \\
&+ \frac{(y-y_n)^{q-j}}{(q-j)!}\, [s^{iq}_{mn} + (x-x_m)\, s^{i+1,q}_{mn} + \dots + \frac{(x-x_m)^{p-i}}{(p-i)!}\, s^{pq}_{mn}] \\
&i \in P, \quad j \in Q .
\end{aligned}
\qquad (10)
$$

(when $i = p$, the line $x = x_{m+1}$ is excluded; when $j = q$, the line $y = y_{n+1}$ is excluded)

As in section 1.3, the following relations between the coefficients hold:

$$\underline{\text{sp-1}}: \quad s^{ij}_{m+1,n} = s^{ij}_{mn} + h\, s^{i+1,j}_{mn} + \frac{h^2}{2} s^{i+2,j}_{mn} + \cdots + \frac{h^{p-i}}{(p-i)!} s^{pj}_{mn}$$

$$i \in P^1, \quad j \in Q, \quad m = 1, 2, \cdots M-1, \quad n \in N$$

$$\underline{\text{sp-2}}: \quad s^{ij}_{m,n+1} = s^{ij}_{mn} + k\, s^{i,j+1}_{mn} + \frac{k^2}{2} s^{i,j+2}_{mn} + \cdots + \frac{k^{q-j}}{(q-j)!} s^{iq}_{mn}$$

$$i \in P, \quad j \in Q^1, \quad m \in M, \quad n = 1, 2, \cdots N-1$$

Next, we would separate our discussions on the $(p-i, q-j)$-splines $s^{ij}(x, y)$ from their coefficients s^{ij}_{mn}. The former is defined over the whole R, whereas the latter is defined only at the grid points of a mesh.

Definitions

Consider a sequence of deleted meshes $\{\Delta_\mu, \mu = 1, 2, 3, \cdots\}$:

$$\Delta_\mu = \begin{cases} x_{\mu,m} = (m-1)h_\mu, & m \in M_\mu, \quad M_\mu h_\mu = a . \\ y_{\mu,n} = (n-1)k_\mu, & n \in N_\mu, \quad N_\mu k_\mu = b . \end{cases}$$

1) For each mesh Δ_μ, the set

$$s^{ij}_\mu = \{s^{ij}_{\mu,mn}, \ m \in M_\mu, \ n \in N_\mu\},$$

which contains the spline coefficients $s^{ij}_{\mu,mn}$ at the grid points $(x_{\mu,m}, y_{\mu,n})$ can be regarded as a discrete function defined on the mesh Δ_μ.

2) $\underline{s}_\mu = \text{col}\,(s^{00}_\mu, s^{10}_\mu, \cdots s^{ij}_\mu, \cdots s^{pq}_\mu)$ denotes a vector function whose $(p+1)(q+1)$ components are the discrete functions defined in 1).

3) $S[\Delta] = \{\underline{s}_1, \underline{s}_2, \cdots \underline{s}_\mu \cdots\}$ denotes the sequence of vectors defined in 2), corresponding to $\{\Delta_\mu, \mu = 1, 2, \cdots\}$.

4) $\underline{s}_\mu(x,y) = \text{col}\,(s_\mu^{00}(x,y),\ s_\mu^{10}(x,y), \cdots s_\mu^{ij}(x,y), \cdots\ s_\mu^{pq}(x,y)\ )$, where $s_\mu^{ij}(x,y)$ are the splines on Δ_μ .

5) $S[R] = \{\underline{s}_1(x,y),\ \underline{s}_2(x,y), \cdots\ \underline{s}_\mu(x,y) \cdots\}$ denotes the sequence of vector defined in 4), corresponding to $\{\Delta_\mu, \mu = 1, 2 \cdots\}$.

6) A sequence $\{s_{1,mn}, s_{2,mn} \cdots\}$ [see 1)] of discrete scalar functions is said to be Arzelà quasi-continuous if it has the property:

Given $\epsilon > 0$, there exist μ_ϵ, $\delta_{1,\epsilon}$ and $\delta_{2,\epsilon}$, independent of the mesh such that if $\mu > \mu_\epsilon$, $|x_{\mu,m} - x_{\mu,\bar{m}}| < \delta_{1,\epsilon}$, $|y_{\mu,n} - y_{\mu,\bar{n}}| < \delta_{2,\epsilon}$

then

$$|s_{\mu,mn} - s_{\mu,\bar{m}\bar{n}}| < \epsilon$$

For a sequence of discrete vector functions, we require that each component satisfies the above property.

For clarity, the following table shows the correspondence between spline coefficients and spline functions.

	spline coefficients	spline functions
scalar function	i) s_μ^{ij}	iv) $s_\mu^{ij}(x,y)$
vector function	ii) $\underline{s}_\mu$	iv) $\underline{s}_\mu(x,y)$
seq. of vector func	iii) $S[\Delta]$	v) $S[R]$

For simplifying discussions, we consider a special sequence of meshes hereafter:

$$\Delta_\mu = \begin{cases} x_{\mu,m} = (m-1)h_\mu, & m \in M_\mu, \quad h_\mu = \dfrac{a}{2^\mu} . \\ y_{\mu,n} = (n-1)k_\mu, & n \in N_\mu, \quad k_\mu = \dfrac{b}{2^\mu} . \end{cases}$$

The following relations will be proved between $S[\Delta]$ and $S[R]$:

i) If $S[\Delta]$ is uniformly bounded, $S[R]$ is also uniformly bounded.

ii) If $S[\Delta]$ is uniformly bounded and Arzelà quasi-continuous, then $S[R]$ forms an Arzelà sequence (see section 1.1).

Theorem 1.4

If $S[\Delta]$ is uniformly bounded by d, the $S[R]$ is uniformly bounded by de^{a+b}.

Proof.

By hypothesis, $|s^{ij}_{\mu,mn}| \leq d$ for every μ, i, j, m, n. By (10), we have

$$|s^{ij}_{\mu,mn}(x,y)| = \left| \sum_{\ell=0}^{p-i} \sum_{t=0}^{q-j} \frac{(x-x_{\mu,m})^{\ell}}{\ell !} \cdot \frac{(y-y_{\mu,n})^{t}}{t!} s^{i+\ell,j+t}_{\mu,mn} \right|$$

$$\leq d \left(\sum_{\ell=0}^{p-i} \frac{|x-x_{\mu,m}|^{\ell}}{\ell !} \right) \left(\sum_{t=0}^{q-j} \frac{|y-y_{\mu,n}|^{t}}{t!} \right)$$

$$< de^{a+b}$$

Lemma 1.2

Let $S[\Delta]$ be uniformly bounded by d. Given $\epsilon > 0$, $\delta_1 > 0$, and $\delta_2 > 0$ there exists $\mu_\epsilon > 0$ such that $\mu \geq \mu_\epsilon$ implies

i) $k_\mu < \delta_1$, $k_\mu < \delta_2$; and

ii) $|s^{ij}_{\mu}(x,y) - s^{ij}_{\mu,mn}| < \epsilon$, $i \in P$, $j \in Q$, $m \in M_\mu$, $n \in N_\mu$.

Proof.

Similar to the proof of theorem 1.4, we have

$$|s_{\mu}^{ij}(x,y) - s_{\mu,mn}^{ij}| < d(e^{|x-x_{\mu,m}| + |y-y_{\mu,n}|} - 1)$$

$$\leq d(e^{h_\mu + k_\mu} - 1)$$

Now, $h_\mu + k_\mu = \frac{a+b}{2^\mu}$. Given $\epsilon > 0$, let μ_0 be the smallest integer such that $\mu_0 \geq \log_2 \left(\frac{a+b}{\log_e(1+\frac{\epsilon}{d})}\right)$, then

$$|s_{\mu}^{ij}(x,y) - s_{\mu,mn}^{ij}| < \epsilon \quad \text{for every } \mu \geq \mu_0 .$$

Next, let μ_1 be the smallest integer such that $h_{\mu_1} = \frac{a}{2^{\mu_1}} < \delta_1$ and $k_{\mu_1} = \frac{b}{2^{\mu_1}} < \delta_2$. The required $\mu_\epsilon = \max(\mu_0, \mu_1)$.

Theorem 1.5

If $S[\Delta]$ is uniformly bounded and Arzelà quasi-continuous, then $S[R]$ is an arzelà sequence.

Proof.

Consider two points (x_t, y_t), $t = 1, 2$, lying in two subregions $R_{\mu, m_t n_t}$, $t = 1, 2$.

Since $S[\Delta]$ is Arzelà quasi-continuous, given $\epsilon > 0$, there exist μ_0, δ_1, δ_2 such that the following property is satisfied:

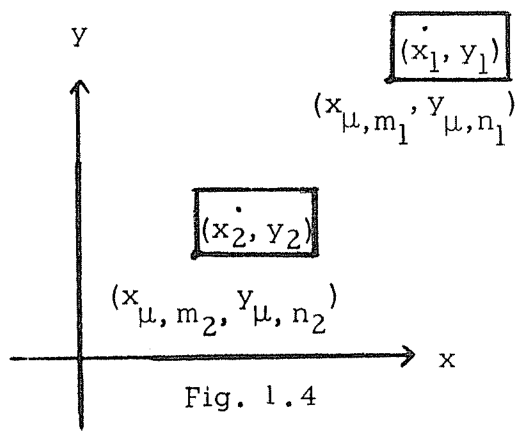

Fig. 1.4

i) If $\mu \geq \mu_0$, $|x_{\mu,m_1} - x_{\mu,m_2}| < \delta_1$, $|y_{\mu,n_1} - y_{\mu,n_2}| < \delta_2$,

then $$|s_{\mu,m_1 n_1}^{ij} - s_{\mu,m_2 n_2}^{ij}| < \frac{\epsilon}{3} .$$

By lemma 1.2, there exists μ_1 such that the following property is satisfied:

ii) $\mu \geq \mu_1$ implies $|x_t - x_{\mu, m_t}| \leq h_\mu < \frac{\delta_1}{3}$,

$$|y_t - y_{\mu, n_t}| \leq k_\mu < \frac{\delta_2}{3}, \quad t = 1, 2, \text{ and}$$

$$|s_\mu^{ij}(x_t, y_t) - s^{ij}_{\mu, m_t n_t}| < \frac{\epsilon}{3}, \quad \text{where } (x_t, y_t) \in$$

$$R_{\mu, m_t n_t}, \quad t = 1, 2 .$$

Now, set $\mu_\epsilon = \max(\mu_0, \mu_1)$, $\delta_{1,\epsilon} = \frac{1}{3}\delta_1$, $\delta_{2,\epsilon} = \frac{1}{3}\delta_2$. If $\mu \geq \mu_\epsilon$, $|x_1 - x_2| < \delta_{1,\epsilon}$ and $|y_1 - y_2| < \delta_{2,\epsilon}$, then both properties i) and ii) are satisfied. It is obvious for ii). i) also follows because $|x_{\mu, m_1} - x_{\mu, m_2}| \leq |x_{\mu, m_1} - x_1| + |x_1 - x_2| + |x_2 - x_{\mu, m_2}|$ $< \frac{1}{3}\delta_1 + \delta_{1,\epsilon} + \frac{1}{3}\delta_1 = \delta_1$ and, similarly, $|y_{\mu, n_1} - y_{\mu, n_2}| < \delta_2$. Hence, we have

$$|s_\mu^{ij}(x_1, y_1) - s_\mu^{ij}(x_2, y_2)| \leq |s_\mu^{ij}(x_1, y_1) - s^{ij}_{\mu, m_1, n_1}| +$$

$$|s^{ij}_{\mu, m_1 n_1} - s^{ij}_{\mu, m_2, n_2}| + |s^{ij}_{\mu, m_2 n_2} - s_\mu^{ij}(x_2, y_2)| < \epsilon,$$

$$i \in P, \quad j \in Q .$$

i.e. $S[R]$ is an Argelà sequence.

SECTION 2. NUMERICAL METHOD OF SOLUTION

2.1 Cauchy Problem (Problem C)

Let R be the region: $0 \le x \le a, \quad 0 \le y \le b,$

$$D_1 \subset E^{(p+1)(q+1)-1} \text{ be bounded, open}$$

$$D = R \times D_1 \subset E^{(p+1)(q+1)+1} .$$

Definition

$\hat{f}[x, y, \underline{u}]$ is an abbreviation for a function of $(p+1)(q+1)+1$ variables, defined over D. These variables are x, y and u^{ij}, $i \in P$, $j \in Q$, $i + j < p + q$.

Problem C

Given two sets of real functions $\{\varphi^j(x),\ j \in Q^1,\ x \in [0, a]\}$ and $\{\psi^i(y),\ i \in P^1,\ y \in [0, b]\}$ and a partial differential equation

PDE $$u^{pq}(x, y) = \hat{f}[x, y, \underline{u}(x, y)] \qquad (1)$$

which satisfy the following assumptions:

1) In D, $\hat{f}[x, y, \underline{u}(x, y)]$ is real, continuous, and bounded, i.e. there exists a constant d such that

$$|\hat{f}[x, y, \underline{u}(x, y)]| \le d .$$

2) In D, $\hat{f}[x, y, \underline{u}(x, y)]$ satisfies a uniform Lipschitz condition

$$|\hat{f}[x, y, \underline{u}(x, y)] - \hat{f}[x, y, \underline{v}(x, y)]| <$$

$$\sum_{\substack{i=0\cdots p \\ j=0\cdots q \\ i+j<p+q}} \ell_{ij} |u^{ij}(x, y) - v^{ij}(x, y)| \qquad \underline{\underline{\text{denoted}}}\ (L, |\underline{u} - \underline{v}|)$$

where $L = \mathrm{col}(\ell_{00}, \ell_{10}, \cdots \ell_{ij}, \cdots \ell_{p,q-1}, \ell_{pq})$, $\ell_{pq} = 0$.

3) $\varphi^j(x) \in C^p[0,a]$, $j \in Q^1$.

4) $\psi^i(y) \in C^q[0,b]$, $i \in P^1$.

5) At the origin $(0, 0)$, the initial functions satisfy the consistency condition:

$$D_x^i \varphi^j(0) = D_y^j \psi^i(0), \quad i \in P^1, \quad j \in Q^1 .$$

By a solution of Problem C, we mean a vector function $\underline{u}(x,y) = \mathrm{col}\,(u^{00}(x,y), u^{10}(x,y), \cdots u^{ij}(x,y), \cdots u^{pq}(x,y)) \in C[R]$, which satisfies the PDE (1) and such that

i) $u^{0j}(x,0) = \varphi^j(x)$ for $x \in [0,a]$, $j \in Q^1$,

ii) $u^{i0}(0,y) = \psi^i(y)$ for $y \in [0,b]$, $i \in P^1$.

Remark: Recall definitions of $\Phi^i(x)$, $\Psi^j(y)$, Φ, Ψ, G^{ij}, I^{ij} from sections 0 and 1. Assumption 5) above is equivalent to $\Phi = \Psi$.

2.2 An Example

Example 2.1

Find a solution of the initial value problem

$$u^{32} = f(x, y, u^{00}, u^{10}, u^{20}, u^{30}, u^{01}, \cdots u^{22}), \quad (x,y) \in R$$

such that i) $u^{00}(x,0) = \varphi^0(x)$, $u^{01}(x,0) = \varphi^1(x)$, $x \in [0,a]$; and

ii) $u^{00}(0,y) = \psi^0(y)$, $u^{10}(0,y) = \psi^1(y)$,

$u^{20}(0,y) = \psi^2(y)$ $\quad y \in [0,b]$,

where f, φ^0, φ^1, ψ^0, ψ^1, ψ^2 satisfy assumptions of Problem C.

The exact solution is to be approximated by a (3, 2)-spline, which has 12 coefficients in each R_{mn}. The following figures show the relationship between the spline coefficients in different subregions:

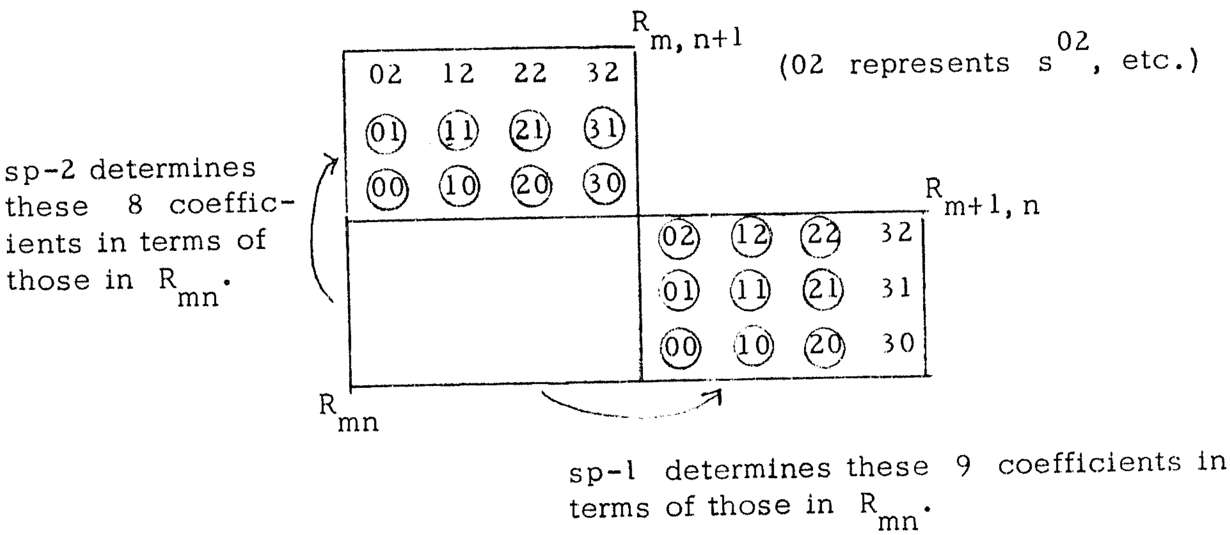

Figure 2.1

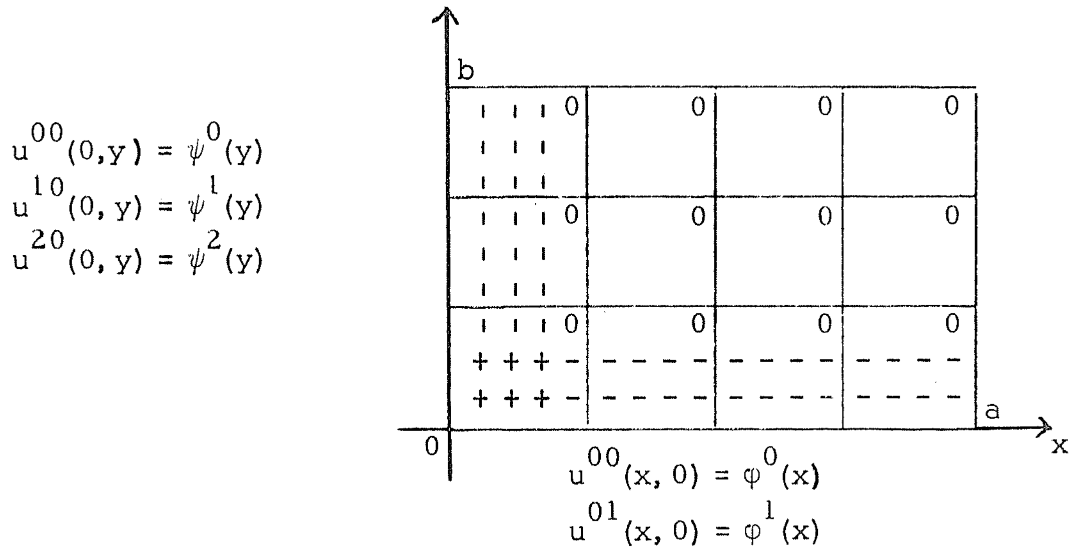

Figure 2.2

The numerical methods to be described in detail in section 2.3 can be outlined roughly here (c.f. Figure 2.2):

Step 1. Determine those coefficients marked with '-' by approximating the initial functions $\varphi^0(x)$, $\varphi^1(x)$ by one-dimensional cubic splines. This will give s^{i0}_{m1} and s^{i1}_{m1}, $i = 0, 1, 2, 3$, $m \in M^1$. (Method A or Method B of section 1.3 may be used.)

Step 2. Determine those coefficients marked with '|' by approximating the initial functions $\psi^0(y)$, $\psi^1(y)$, $\psi^2(y)$ by one-dimensional quadratic splines. This will give s^{0j}_{1n}, s^{1j}_{1n}, and s^{2j}_{1n}, $j = 0, 1, 2$, $n \in N^1$. (Method A or Method B of section 1.3 may be used.)

Step 3. All those coefficients not marked in Figure 2.2 are determined by sp-1 and sp-2 .

Step 4. Substitution of the coefficients obtained from the above steps in the right side of PDE(1) will give those coefficients marked with '0' .

2.3 Method of Spline Approximation for Problem C (Method C)

Our scheme is to approximate the exact solution $\underline{u}(x, y) = \text{col}$ $(u^{00}(x, y), \cdots u^{ij}(x, y), \cdots u^{pq}(x, y))$ of Problem C by a spline $\underline{s}(x, y) =$ $\text{col}\ (s^{00}(x, y),\ s^{ij}(x, y), \cdots\ s^{pq}(x, y))$, where

$$s^{ij}(x, y) = s^{ij}_{mn}(x, y) = \sum_{\ell=0}^{p-i} \sum_{t=0}^{q-j} \frac{(x - x_m)^{\ell}}{\ell\,!} \cdot \frac{(y - y_n)^t}{t!}\ s^{i+\ell\,,\,j+t}_{mn} ,$$

$$(x, y) \in R_{mn} \quad i \in P, \quad j \in Q, \quad m \in M, \quad n \in N .$$

The following method is for finding the $(p + 1)(q + 1)$ coefficients $\{s^{ij}_{mn},\ i \in P,\ j \in Q\}$ in each R_{mn}, $m \in M$, $n \in N$:

Method C

Step 1. By method A (or method B if $\varphi^j(x) \in C^{p+1}$), approximate each $\varphi^j(x)$, $j \in Q^1$, by $s^{0j}(x, 0) = s^{0j}_{m1}(x, 0)$

$$= s^{0j}_{m1} + (x - x_m)\, s^{1j}_{m1} + \frac{(x - x_m)^2}{2!}\ s^{2j}_{m1} + \cdots + \frac{(x - x_m)^p}{p!}\ s^{pj}_{m1} \quad j \in Q^1 .$$

This will give $\{s^{ij}_{m1},\ i \in P,\ j \in Q^1,\ m \in M\}$.

(Compare with step 1 of Example 2.1)

Step 2. By method A (or method B if $\psi^i(y) \in C^{q+1}$), approximate each $\psi^i(y)$, $i \in P^1$, by $s^{i0}(0, y) = s^{i0}_{1n}(0, y)$

$$= s^{i0}_{1n} + (y - y_n)\, s^{i1}_{1n} + \frac{(y - y_n)^2}{2!}\, s^{i2}_{1n} + \dots + \frac{(y - y_n)^q}{q!}\, s^{iq}_{1n},\ i \in P^1$$

This will give $\{s^{ij}_{1n},\ i \in P^1,\ j \in Q,\ n \in N\}$.

(Compare with step 2 of Example 2.1).

<u>Remark</u>: Step 1 determines a matrix of coefficients $\Phi_s = (s^{ij}_{11})$, $i \in P^1$, $j \in Q^1$ in R_{11} for approximating Φ. Step 2 determines another matrix of coefficients $\Psi_s = (s^{ij}_{11})$, $i \in P^1$, $j \in Q^1$ for approximating Ψ. For two reasons, these two matrices are identical:

1) By assumption 5) of Problem C, $\Phi = \Psi$.

2) Method A (or method B) approximates Φ exactly (i.e., $\Phi_s = \Phi$) in step 1, and approximates Ψ exactly (i.e. $\Psi_s = \Psi$) in step 2.

Step 3. In R_{11}, all coefficients except s^{pq}_{11} have been obtained in step 1 and step 2. s^{pq}_{11} can then be calculated by substituting the known coefficients in the right hand side of PDE (1), i.e.

$$s^{pq}_{11} = \hat{f}\,[0, 0, \underline{s}_{11}] .$$

Step 4. In each R_{m1}, $m \in M_2$, $\{s^{ij}_{m1},\ i \in P,\ j \in Q^1,\ m \in M_2\}$ have been obtained in step 1. $\{s^{iq}_{m1},\ i \in P^1,\ m \in M_2\}$ can be calculated by sp-1. Furthermore,

$$s^{pq}_{m1} = \hat{f}\,[x_m, 0, \underline{s}_{m1}] , \quad m \in M_2 .$$

Step 5. In each R_{1n}, $n \in N_2$, $\{s_{1n}^{ij}, \; i \in P^1, \; j \in Q, \; n \in N_2\}$ have been obtained in step 2. $\{s_{1n}^{pj}, \; j \in Q^1, \; n \in N_2\}$ can be calculated by sp-2. Furthermore,

$$s_{1n}^{pq} = \hat{f}\,[0, y_n, \underline{s}_{1n}], \quad n \in N_2 \;.$$

Step 6. For each R_{mn}, $m \in M_2$, $n \in N_2$, suppose the coefficients in $R_{m-1,n}$ and in $R_{m,n-1}$ have been obtained. Then, all coefficients in R_{mn}, except s_{mn}^{pq}, can be calculated by sp-1 and sp-2. Furthermore,

$$s_{mn}^{pq} = \hat{f}\,[x_m, y_n, \underline{s}_{mn}], \quad m \in M_2, \; n \in N_2 \;.$$

SECTION 3. CONVERGENCE OF METHOD C

3.1 $S[\Delta]$ is Uniformly Bounded

For a sequence of meshes $\{\Delta_\mu, \mu = 1, 2, 3 \cdots\}$, Method C provides a sequence of numerical solutions to Problem C, each in the form of a set of spline coefficients $\{s^{ij}_{\mu,mn}, \ i \in P, \ j \in Q, \ m \in M, \ n \in N\}$. As in section 1.4, we can define s^{ij}_μ, $\underline{s}_\mu$, $\underline{s}_{\mu,mn}$, $s[\Delta]$, $\underline{s}_\mu(x, y)$, and $S[R]$.

In theorems 3.1 and 3.2, we shall show that $S[\Delta]$ is uniformly bounded and Arzelà quasi-continuous.

<u>Lemma 3.1</u>

Let $\{s^{ij}_{\mu,m1}, \ i \in P, \ j \in Q^1, \ m \in M\}$ and $\{s^{ij}_{\mu,1n}, \ i \in P^1, \ j \in Q, \ n \in N\}$ be the spline coefficients obtained in step 1 and step 2 of Method C respectively. Then there exists a constant d_2, independent of the mesh, such that

$$|s^{ij}_{\mu,m1}| \le d_2 \qquad i \in P, \ j \in Q^1, \ m \in M ,$$

and

$$|s^{ij}_{\mu,1n}| \le d_2 \qquad i \in P^1, \ j \in Q, \ n \in N .$$

Proof.

Since these coefficients are obtained by Method A (or method B) of section 1.3, this lemma follows immediately from corollary 1.2.

Theorem 3.1

$S[\Delta]$, obtained by Method C on a sequence of meshes $\{\Delta_\mu, \mu = 1, 2 \cdots\}$ is uniformly bounded.

Proof.

It is enough to show that each $s^{ij}_{\mu, mn}$ is bounded by some constant which is independent of the mesh. For notational simplicity, we suppress the subscript μ .

i) For $j = q$, we prove the uniform boundedness of s^{iq}_{mn} by induction with respect to $i \in P$.

Since $s^{pq}_{mn} = \hat{f}[x_m, y_n, \underline{s}_{mn}]$ and $|\hat{f}| \le d$, it follows that $|s^{pq}_{mn}| \le d$.

Next, suppose $|s^{\ell q}_{mn}| \le d_1$ for $\ell = p, p-1, \cdots i+2, i+1$, where d_1 is a constant independent of the mesh. By sp-1

$$s^{iq}_{mn} = s^{iq}_{m-1,n} + h\, s^{i+1,q}_{m-1,n} + \frac{h^2}{2}\, s^{i+2,q}_{m-1,n} + \cdots + \frac{h^{p-i}}{(p-i)!}\, s^{pq}_{m-1,n}, \quad i \in P^1$$

$$\begin{aligned} \therefore \quad |s^{iq}_{mn}| &\le |s^{iq}_{m-1,n}| + h\, d_1 e^h \\ &\le |s^{iq}_{1n}| + (m-1) h\, d_1 e^h \\ &< d_2 + a\, d_1 e^a \equiv d_3 \text{ (say)} \qquad \text{(by lemma 3.1)} \end{aligned}$$

ii) For a fixed i, $i \in P$, we prove the uniform boundedness of s^{ij}_{mn} by induction with respect to $j \in Q$.

As a consequence of i), $|s^{iq}_{mn}| \le d_3$ for each $i \in P$.

Next, suppose $|s^{it}_{mn}| \le d_3$ for $t = q, q-1, \cdots j+2, j+1$.

By sp-2,

$$s^{ij}_{mn} = s^{ij}_{m,n-1} + k\, s^{i,j+1}_{m,n-1} + \frac{k^2}{2}\, s^{i,j+2}_{m,n-1} + \cdots + \frac{k^{q-j}}{(q-j)!}\, s^{iq}_{m,n-1}$$

$$i \in P, \quad j \in Q^1 .$$

$$\therefore \quad |s^{ij}_{mn}| \le |s^{ij}_{m,n-1}| + k\, d_3\, e^k$$

$$\le |s^{ij}_{m1}| + (n-1)k\, d_3\, e^k$$

$$< \quad d_2 + b\, d_3\, e^b \qquad \text{(by lemma 3.1)}$$

3.2 $S[\Delta]$ is Arzelà Quasi-continuous.

For notational simplicity, the subscript μ will be suppressed whenever there is no ambiguity.

Definitions

The following notations will be used throughout this section:

1. $\theta^i(\delta)$, $i = 0, 1, 2, \ldots$, denotes a function which converges uniformly to 0 as $\delta \to 0$.
2. $z^{ij}_m = |s^{ij}_{mn} - s^{ij}_{m\bar{n}}|$, n and $\bar{n}$ are suppressed in the symbol z^{ij}_m .
3. $w^{ij}_n = |s^{ij}_{mn} - s^{ij}_{\bar{m}n}|$, m and $\bar{m}$ are suppressed in the symbol w^{ij}_n .
4. $\underline{w}_n = \text{col}\,(w^{p0}_n, w^{p1}_n, \ldots w^{p,q-1}_n)$
5. $\underline{\alpha} = \text{col}\,\left(\frac{k^{q-1}}{q!}, \frac{k^{q-2}}{(q-1)!}, \ldots \frac{k}{2}, 1\right)$
6. $\|\underline{w}_n\| = \max_{j \in Q^1} w^{pj}_n$
7. $\ell_p = \sum_{j=0}^{q-1} \ell_{pj}$, $\ell_q = \sum_{i=0}^{p-1} \ell_{iq}$, $\ell^1 = \max_{i \in P,\, j \in Q} \ell_{ij}$.

Consider two grid points (x_m, y_n) and $(x_{\bar{m}}, y_{\bar{n}})$ of a mesh. Then

$$|s^{ij}_{mn} - s^{ij}_{\bar{m}\bar{n}}| \le z^{ij}_{\bar{m}} + w^{ij}_{n}$$

We are going to show that both $z^{ij}_{\bar{m}}$ and w^{ij}_{n} are arbitrarily small when the mesh is fine enough and the two grid points are close to each other.

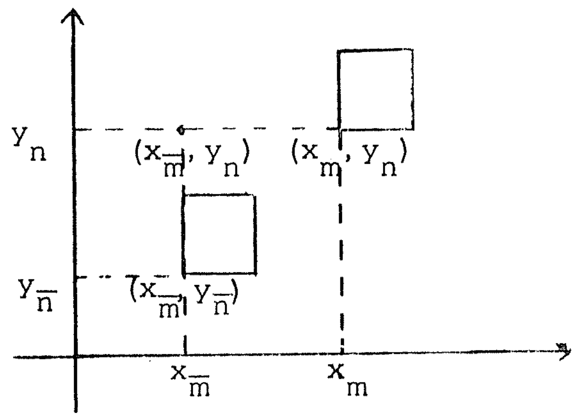

Figure 3.1

Lemma 3.2

Given $\epsilon > 0$, there exist μ_ϵ and δ_ϵ such that, if $\mu \ge \mu_\epsilon$ and $|x_m - x_{\bar{m}}| < \delta_\epsilon$ then $w^{pj}_{l} < \epsilon$, $j \in Q^l$.

Proof.

For each $j \in Q^l$,

$$s^{0j}(x, 0) = s^{0j}_{ml} + (x - x_m)\, s^{ij}_{ml} + \cdots + \frac{(x-x_m)^p}{p!}\, s^{pj}_{ml}$$

is the spline approximation of $\varphi^j(x)$ obtained by method A (or method B). Hence, by theorem 1.2, $s^{pj}(x, 0)$ converges uniformly to $D^p_x\, \varphi^j(x)$, i.e., there exists μ_1 such that, if $\mu \ge \mu_1$ then

$$|s^{pj}_{ml} - D^p_x\, \varphi^j_m| < \frac{\epsilon}{3} \quad \text{and} \quad |s^{pj}_{\bar{m}l} - D^p_x\, \varphi^j_{\bar{m}}| < \frac{\epsilon}{3}, \qquad m, \bar{m} \in M.$$

Next, since $D^p_x\, \varphi^j(x)$ is uniformly continuous in $[0, a]$, there exist $\delta_\epsilon > 0$ and μ_2 such that, if $\mu \ge \mu_2$ (this guarantees that the mesh is fine enough so that two grid points can be chosen to lie within preassigned distance) and $|x_m - x_{\overline{m}}| < \delta_\epsilon$, then

$$|D_x^p \varphi_m^j - D_x^p \varphi_{\overline{m}}^j| < \frac{\epsilon}{3}, \quad m, \overline{m} \in M .$$

Hence, letting $\mu_\epsilon = \max(\mu_1, \mu_2)$, we have, for $\mu > \mu_\epsilon$,

$$w_1^{pj} = |s_{m1}^{pj} - s_{\overline{m}1}^{pj}| \le |s_{m1}^{pj} - D_x^p \varphi_m^j| + |D_x^p \varphi_m^j - D_x^p \varphi_{\overline{m}}^j| + |D_x^p \varphi_{\overline{m}}^j - s_{\overline{m}1}^{pj}|$$

$$\le \epsilon .$$

This completes the proof.

<u>Lemma 3.3</u>

Given $\epsilon > 0$, there exist μ_ϵ and δ_ϵ such that, if $\mu \ge \mu_\epsilon$ and $|x_m - x_{\overline{m}}| < \delta_\epsilon$ then $w_n^{ij} < \epsilon$, $i \in P$, $j \in Q$, $n \in N$.

Proof.

Assume $m > \overline{m}$. For each $n \in N$, the proof is separated into three parts: i) for $\{w_n^{ij}, \ i \in P^1, \ j \in Q\}$, ii) for $\{w_n^{pj}, \ j \in Q^1\}$ and iii) for w_n^{pq}.

i) By theorem 3.1, there exists a constant d such that

$$|s_{tn}^{ij}| \le d.$$

By sp-1,

$$w_n^{ij} \equiv |s_{mn}^{ij} - s_{\overline{m}n}^{ij}|$$

$$\le h \sum_{t=\overline{m}}^{m-1} |s_{tn}^{i+1,j}| + \frac{h^2}{2} \sum_{t=\overline{m}}^{m-1} |s_{tn}^{i+1,j}| + \cdots + \frac{h^{p-i}}{(p-i)!} \sum_{t=\overline{m}}^{m-1} |s_{tn}^{pj}|$$

$$\leq (m - \overline{m})\, h\, d \left(1 + \frac{h}{2} + \frac{h^2}{3!} + \cdots + \frac{h^{p-i-1}}{(p-i)!}\right)$$

$$\leq (x_m - x_{\overline{m}})\, d\, e^a$$

$$\leq \theta^1 (\delta_\epsilon) \quad , \quad i \in P^1, \; j \in Q$$

ii) We first derive an inequality.

$$w_n^{pq} \equiv | s_{mn}^{pq} - s_{\overline{m}n}^{pq} |$$

$$= | \hat{f}[x_m, y_n, \underline{s}_{mn}] - \hat{f}[x_{\overline{m}}, y_n, \underline{s}_{\overline{m}m}] |$$

$$\leq | \hat{f}[x_m, y_n, \underline{s}_{mn}] - \hat{f}[x_m, y_n, \underline{s}_{\overline{m}n}] | +$$

$$| \hat{f}[x_m, y_n, \underline{s}_{\overline{m}n}] - \hat{f}[x_{\overline{m}}, y_n, \underline{s}_{\overline{m}n}] |$$

$$\leq (L, |\underline{s}_{mn} - \underline{s}_{\overline{m}n}|) + \Omega(\delta_\epsilon, 0)$$

$$\leq \sum_{j=0}^{q-1} \ell_{pj}\, w_n^{pj} + \ell^1 \sum_{i=0}^{p-1} \sum_{j=0}^{q} \theta^1(\delta_\epsilon) + \Omega(\delta_\epsilon, 0) \qquad \text{(by part i))}$$

$$\leq \sum_{j=0}^{q-1} \ell_{pj}\, w_n^{pj} + \theta^2(\delta_\epsilon) \tag{1}$$

where $\ell^1 = \max_{i \in P,\, j \in Q} \ell_{ij}$

$$\Omega(\delta_\epsilon, 0) = \max_{\substack{|x_m - x_{\overline{m}}| < \delta_\epsilon \\ x_m,\, x_{\overline{m}} \in [0, a]}} | \hat{f}[x_m, y_n, \underline{s}_{\overline{m}n}] - \hat{f}[x_{\overline{m}}, y_n, \underline{s}_{\overline{m}n}] |$$

$$\theta^2(\delta_\epsilon) = \ell^1 p(q+1)\, \theta^1(\delta_\epsilon) + \Omega(\delta_\epsilon, 0) \, .$$

Next, for $j \in Q^1$, sp-2 implies

$$w^{pj}_{n+1} \le w^{pj}_n + k\,w^{p,j+1}_n + \cdots + \frac{k^{q-j}}{(q-j)!} w^{pq}_n$$

$$\le \frac{\ell_{p0}k^{q-j}}{(q-j)!} w^{p0}_n + \frac{\ell_{p1}k^{q-j}}{(q-j)!} w^{p1}_n + \cdots + \frac{\ell_{p,j-1}k^{q-j}}{(q-j)!} w^{p,j-1}_n \quad \text{(by (1))}$$

$$+ \left(1 + \frac{\ell_{pj}k^{q-j}}{(q-j)!}\right) w^{pj}_n + \cdots + \left(\frac{k^{q-j-1}}{(q-j-1)!} + \frac{\ell_{p,q-1}k^{q-j}}{(q-j)!}\right) w^{p,q-1}_n$$

$$+ \frac{k^{q-j}}{(q-j)!} \theta^2(\delta_\epsilon) \qquad (2)$$

In vector notations, system (2) can be rewritten as

$$\underline{w}_{n+1} \le B\underline{w}_n + k\,\underline{\alpha}\,\theta^2,$$

where $B = B(k, q, L_p)$ is the matrix defined in section 1.2,

$$L_p = \text{col}(\ell_{p0}, \ell_{p1}, \ldots \ell_{p,q-1}).$$

For $k < 2$, $\|\underline{\alpha}\| = 1$. Hence

$$\|\underline{w}_{n+1}\| \le \|B\| \; \|\underline{w}_n\| + k\,\theta^2 .$$

By lemma 1.1, if $\ell_p = \ell_{p0} + \ell_{p1} + \cdots + \ell_{p,q-1}$ and $\hat{\ell}_p = \max(\ell_p, 1)$, then, for sufficiently small k,

$$\|\underline{w}_{n+1}\| \le e^{\hat{\ell}_p k} \|\underline{w}_n\| + k\,\theta^2 .$$

Therefore,

$$\|\underline{w}_n\| \le e^{\hat{\ell}_p(n-1)k} \|\underline{w}_1\| + \frac{e^{\hat{\ell}_p(n-1)k} - 1}{\hat{\ell}_p k} \cdot k\,\theta^2$$

$$\le e^{\hat{\ell}_p b}\,\epsilon + \frac{e^{\hat{\ell}_p b} - 1}{\hat{\ell}_p}\,\theta^2 \qquad \text{(by lemma 3.2)}$$

$$\equiv \theta^3(\delta_\epsilon)$$

i.e. $w_n^{pq} \leq \theta^3(\delta_\epsilon)$

iii) By (1), $w_n^{pq} \leq \ell^1 q\,\theta^3 + \theta^2$.

Theorem 3.2

$S[\Delta]$, obtained by Method C on a convergent sequence of meshes $\{\Delta_\mu, \mu = 1, 2, \ldots\}$, is Arzelà quasi-continuous.

Proof.

Let $(x_{\mu,m}, y_{\mu,n})$ and $(x_{\mu,\bar{m}}, y_{\mu,\bar{n}})$ be two grid points of Δ_μ. By lemma 3.3, given $\epsilon > 0$, there exist μ_1 and δ_1 such that, if $\mu \geq \mu_1$ and $|x_{\mu,m} - x_{\mu,\bar{m}}| < \delta_1$, then, for sufficiently small k ,

$$|s^{ij}_{\mu,mn} - s^{ij}_{\mu,\bar{m}n}| < \frac{\epsilon}{2}, \quad i \in P, \ j \in Q .$$

Similarly, there exist μ_2 and δ_2 such that, if $\mu \geq \mu_2$ and $|y_{\mu,n} - y_{\mu,\bar{n}}| < \delta_2$, then, for sufficiently small h,

$$|s^{ij}_{\mu,\bar{m}n} - s^{ij}_{\mu,\bar{m}\bar{n}}| < \frac{\epsilon}{2} \quad , \quad i \in P, \quad j \in Q .$$

Hence, for $\mu \geq \max(\mu_1, \mu_2)$, $|x_{\mu,m} - x_{\mu,\bar{m}}| < \delta_1$ and $|y_{\mu,n} - y_{\mu,\bar{n}}| < \delta_2$, we have

$$|s^{ij}_{\mu,mn} - s^{ij}_{\mu,\bar{m}\bar{n}}| < \epsilon \ , \ i \in P, \quad j \in Q .$$

implying

$$\| \underline{s}_{\mu,mn} - \underline{s}_{\mu,\bar{m}\bar{n}} \| < \epsilon ,$$

where $\underline{s}_{\mu,mn} = \mathrm{col}\,(s^{00}_{\mu,mn}, s^{10}_{\mu,mn}, \ldots, s^{pq}_{\mu,mn})$.

3.3 Convergence of Approximate Solutions

In theorem 3.1 and theorem 3.2, we have respectively shown that $S[\Delta]$ is uniformly bounded and Arzelà quasi-continuous. (These are consequences of the Method C and the assumptions on Problem C). By theorem

1.4 and theorem 1.5, S[R] is uniformly bounded and forms an Arzelà sequence. Hence, by theorem 1.1, S[R] contains a subsequence $\{\underline{s}_{\mu^1}(x,y), \underline{s}_{\mu^2}(x,y), \ldots, \}$ which converges uniformly to a vector function in C[R]. More explicitly, this means that there exists a continuous vector function

$$\underline{z}(x,y) = \text{col}(z^{00}(x,y), z^{10}(x,y), \cdots, z^{ij}(x,y), \cdots, z^{pq}(x,y))$$

such that

$$s^{ij}_{\mu^r}(x,y) \Longrightarrow z^{ij}(x,y) \text{ as } \mu^r \to \infty. \quad i \in P, \; j \in Q, \; (x,y) \in R.$$

Moreover, since $s^{ij}_{\mu^r}(x,y)$ is the (i, j)-derivative of $s^{00}_{\mu^r}(x,y)$ and the convergence is uniform, $z^{ij}(x,y)$ is also the (i, j)-derivative of $z^{00}(x,y)$.

Remark The components $s^{pj}_{\mu}(x,y)$, $s^{iq}_{\mu}(x,y)$ $i \in P$, $j \in Q$ of the spline approximations may have jump discontinuities in R ; while the corresponding components $z^{pj}(x,y)$, $z^{iq}(x,y)$, $i \in P$, $j \in Q$ of a limit function are continuous over R .

We summarize these results in a theorem.

Theorem 3.3

The spline approximations, obtained by Method C on a convergent sequence of meshes, contains a subsequence which converges uniformly to a continuous limit function.

In the next section, we shall show that this limit function $\underline{z}(x,y)$ is a solution of our Problem C.

3.4 Convergence of Approximate Solutions to A Solution of Problem C.

In this section, let $\underline{s}_\mu(x, y)$ be the spline approximations and $\underline{z}(x, y)$ be the limit function of a subsequence. We want to show that $\underline{z}(x, y)$ satisfies the PDE and the initial conditions of Problem C.

In the following, we shall suppress the subscript μ wherever there is no ambiguity.

Definitions

1) $\Phi_s^i(x) = \mathrm{col}(s_{ml}^{i0}(x, 0),\ s_{ml}^{i1}(x, 0), \cdots, s_{ml}^{i,q-1}(x, 0)),\ i \in P$.

$\Psi_s^j(y) = \mathrm{col}(s_{ln}^{0j}(0, y),\ s_{ln}^{1j}(0, y), \cdots, s_{ln}^{p-1,j}(0, y)),\ \ j \in Q$.

2) $g_\mu(x, y) = \hat{f}[x_m, y_n, \underline{s}_{mn}],\quad x_m \le x < x_{m+1},\ y_n \le y < y_{n+1}$

i.e. $g_\mu(x, y)$ is a step-function whose value in R_{mn} is equal to the value of $\hat{f}[x, y, \underline{s}]$ evaluated at the left lower corner of R_{mn} (hence also equal to s_{mn}^{pq}).

3) $$I_{mn}^{ij} = I^{ij}[x_m, y_n, g_\mu(x, y)] = \int_0^{y_n}\int_0^{\eta_{q-j-1}}\cdots\int_0^{\eta_1}\int_0^{x_m}\int_0^{\xi_{p-i-1}}\cdots\int_0^{\xi_1} g_\mu(\alpha, \beta)\, d\xi\, d\eta$$

$$G_{mn}^{ij} = G^{ij}[x_m, y_n, \Phi_s^i(x_m),\ \Psi_s^j(y_n),\ \Phi_s]$$
$$= (X^i(x_m),\ \Psi_s^j(y_n)) + (\Phi_s^i(x_m),\ Y^j(y_n)) - X^i(x_m)\,\Phi_s\, Y^j(y_n)$$

With these definitions, it is obvious that the spline approximations obtained by Method C have representations of the form

$$s^{ij}(x, y) = I^{ij}[x, y, g_\mu(x, y)] + G^{ij}[x, y, \Phi_s^i(x), \Psi_s^j(y),\ \Phi_s] .$$

Lemma 3.4

Let $s_{mn}^{ij}(x,y)$ be the spline approximations obtained by Method C, and $\Phi_s^i(x), \Psi_s^j(y)$ be defined as in 1), then as $\mu \to \infty$,

$$\Phi_s^i(x) \Longrightarrow \Phi^i(x) = \mathrm{col}(D_x^i \varphi^0(x), D_x^i \varphi^1(x), \cdots, D_x^i \varphi^{q-1}(x)), \quad i \in P,$$

$$\Psi_s^j(y) \Longrightarrow \Psi^j(y) = \mathrm{col}(D_y^j \psi^0(y), D_y^j \psi^1(y), \cdots, D_y^j \psi^{p-1}(y)), \quad j \in Q.$$

Proof.

Since $\Phi_s^i(x)$ and $\Psi_s^j(y)$ are obtained by Method A (or Method B), lemma 3.4 follows immediately from corollary 1.1.

Lemma 3.5

Let $g_\mu(x,y)$ be defined in 2) and $\underline{z}(x,y)$ be a limit function of the spline approximations obtained by Method C. Then, $g_\mu(x,y)$ converges uniformly to $\hat{f}[x,y,\underline{z}(x,y)]$ as $\mu \to \infty$.

Proof.

Consider an arbitrary point $(x,y) \in R_{mn}$,

$$|g_\mu(x,y) - f[x,y,\underline{z}(x,y)]|$$

$$= |\hat{f}[x_m, y_n, \underline{s}_{mn}] - \hat{f}[x_m, y_n, \underline{s}(x,y)]| + |\hat{f}[x_m, y_n, \underline{s}(x,y)]| - \hat{f}[x_m, y_n, \underline{z}(x,y)]|$$

$$+ |\hat{f}[x_m, y_n, \underline{z}(x,y)] - \hat{f}[x,y,\underline{z}(x,y)]|$$

$$\leq (L, |\underline{s}_{mn} - \underline{s}(x,y)|) + (L, |\underline{s}(x,y) - \underline{z}(x,y)|) + \Omega(\delta_{1,\epsilon}, \delta_{2,\epsilon})$$

By lemma 1.2, the first term is less than $\ell'\epsilon$. Since $\underline{s}(x,y) \Longrightarrow \underline{z}(x,y)$ (by theorem 3.3), the second term is less than $\ell'\epsilon$. Here, $\ell' = \sum_{i+j<p+q} \ell_{ij}$ and Ω is the modulus of continuity. Hence,

$$|g_\mu(x,y) - f[x,y,\underline{z}(x,y)]| < 2\,\ell'\epsilon + \Omega(\delta_{1,\epsilon}, \delta_{2,\epsilon})$$

The right hand side can be made aribitrarily small.

Theorem 3.4

Let $\underline{z}(x, y)$ be a limit function of the spline approximation obtained by method C . Then, $\underline{z}(x, y)$ is a solution of Problem C.

Proof.

The spline approximation can be represented as

$$s^{ij}(x, y) = G^{ij}[x, y, \Phi^i_s(x), \Psi^j_s(y), \Phi_s] + I^{ij}[x, y, g_\mu(x, y)] \qquad (3)$$

Now, $\Phi_s = \Phi$, because Method A (or Method B) approximates the initial function values at the origin exactly. Also, as $\mu \to \infty$,

$$s^{ij}(x, y) \Longrightarrow z^{ij}(x, y) \qquad \text{(by theorem 3.3)}$$

$$\Phi^i_s(x) \Longrightarrow \Phi^i(x) \qquad \text{(by lemma 3.4)}$$

$$\Psi^j_s(y) \Longrightarrow \Phi^j(y) \qquad \text{(by lemma 3.4)}$$

$$g_\mu(x, y) \Longrightarrow \hat{f}[x, y, \underline{z}(x, y)] \qquad \text{(by lemma 3.5)}$$

Since convergence is uniform, taking limit under integral sign is justified. Hence, from (3), we have

$$z^{ij}(x, y) = G^{ij}[x, y, \Phi^i(x), \Psi^j(y), \Phi] + I^{ij}[x, y, \hat{f}[s, y, \underline{z}(x, y)]] \quad i \in P, \ j \in Q .$$

This is just the integral form of Problem C .

Corollary 3.1

Let $u^{00}(x, y) \in C^{p,q}[R]$. Then, there exists a sequence of (p, q)-splines $\{s^{00}_\mu(x, y)\}$ such that $s^{ij}_\mu(x, y) \Longrightarrow u^{ij}(x, y)$ as $\mu \to \infty$. $i \in P$, $j \in Q$

SECTION 4. ERROR ESTIMATION

4.1 An a posterior Error Bound

We consider only truncation errors,

Let $\underline{u}(x, y) = \text{col}(u^{00}(x, y),\ u^{10}(x, y), \cdots u^{pq}(x, y))$ be an exact solution,

$\underline{s}(x, y) = \text{col}(s^{00}(x, y),\ s^{10}(x, y), \cdots s^{pq}(x, y))$ be any approximation,

$\underline{e}(x, y) = \text{col}(e^{00}(x, y),\ e^{10}(x, y), \cdots e^{pq}(x, y))$ be any function which

satisfies the assumptions of theorem 4.1.

Definitions

1) For the function $\underline{u}(x, y)$:

$$\Phi^i(x) = \text{row}\,(u^{i0}(x, 0),\ u^{i1}(x, 0), \cdots u^{i, q-1}(x, 0)) \ , \quad i \in P$$

$$= \text{row}\,(D_x^i \varphi^0(x),\ D_x^i \varphi^1(x), \cdots D_x^i \varphi^{q-1}(x))$$

$$\Psi^j(y) = \text{col}\,(u^{0j}(0, y),\ u^{1j}(0, y), \cdots u^{p-1, j}(0, y)) \ , \quad j \in Q$$

$$= \text{col}\,(D_y^j \psi^0(y),\ D_y^j \psi^1(y), \cdots D_y^j \psi^{p-1}(y))$$

$$\Phi = \begin{bmatrix} \varphi^0(0) & \varphi^1(0) & \cdots & \varphi^{q-1}(0) \\ D_x^1 \varphi^0(0) & D_x^1 \varphi^1(0) & \cdots & D_x^1 \varphi^{q-1}(0) \\ \vdots & \vdots & & \vdots \\ D_x^{p-1} \varphi^0(0) & D_x^{p-1} \varphi^1(0) & \cdots & D_x^{p-1} \varphi^{q-1}(0) \end{bmatrix}$$

$$\hat{g}[\underline{u}(x, y)] = G^{ij}[x, y, \Phi^i(x), \Psi^j(y), \Phi]$$

$$= (X^i(x), \Psi^j(y)) + (\Phi^i(x), Y^j(y)) - X^i(x)\Phi Y^j(y)$$

$$= (X^i(x), \Psi^j(y) - \Phi Y^j(y)) + (\Phi^i(x) - X^i(x)\Phi,\ Y^j(y))$$

$$+ \ X^i(x)\,\Phi\, Y^j(y) \qquad (1)$$

2) For the function $\underline{s}(x, y)$:

$\Phi^i_s(x)$, $\Psi^j_s(y)$, Φ_s, $\hat{g}[\underline{s}(x, y)]$ are similar quantities corresponding to those defined in 1).

3) For the function $\underline{e}(x, y)$:

$E^i(x)$, $\hat{E}^j(y)$, E, $\hat{g}[\underline{e}(x, y)]$ are similar quantities corresponding to those defined in 1).

The following theorem gives an a posterior bound on the error:

<u>Theorem 4.1</u>

If i) $|\Phi - \Phi_s| \leq E$

ii) $|u^{pj}(x, 0) - s^{pj}(x, 0)| \leq e^{pj}(x, 0) \qquad j \in Q^1$

$|u^{iq}(0, y) - s^{iq}(0, y)| \leq e^{iq}(0, y) \qquad i \in P^1$

iii) $u^{pq}(x, y) = \hat{f}[x, y, \underline{u}(x, y)] \qquad (x, y) \in R$

$|s^{pq}(x, y) - \hat{f}[x, y, \underline{s}(x, y)]| \leq d(x, y) \qquad (x, y) \in R$

iv) $|\hat{f}[x, y, \underline{u}(x, y)] - \hat{f}[x, y, \underline{s}(x, y)]| \leq (L, \underline{e}(x, y))$ in D

v) $e^{pq}(x, y) \geq (L, \underline{e}(x, y)) + d(x, y) \qquad (x, y) \in R$

then $|u^{ij}(x, y) - s^{ij}(x, y)| \leq e^{ij}(x, y) \qquad (x, y) \in R$

$i \in P, \; j \in Q, \; i + j < p + q$.

This theorem is a generalization of some simpler cases [6].

<u>Discussions</u> In particular, if $\underline{s}(x, y)$ is the spline approximation obtained by Method C, then we have

1) $\Phi_s = \Phi$. Hence, E may be taken as a zero matrix.

2) In step 1 of Method C, $u^{pj}(x, 0) = D^p_x \varphi^j(x)$, $j \in Q^1$, and is,

therefore, known. $s^{pj}(x, 0)$ is the spline approximation obtained by Method A (or Method B). Thus, $e^{pj}(x, 0)$, $j \in Q^1$ can be easily approximated. Similarly, $e^{iq}(0, y)$, $i \in P^1$, can be easily approximated.

3) $d(x, y)$ can be approximated after getting the spline $\underline{s}(x, y)$.

4) In practice, we solve the differential equation

$$e^{pq}(x, y) = (L, \underline{e}(x, y)) + d(x, y)$$

instead of the differential inequality v). This needs a more accurate approximation for $d(x, y)$. Hence, an error bound is obtained by solving an initial value problem similar to Problem C.

Before proving theorem 4.1, we develop a few lemmas:

Lemma 4.1

Under condition ii) of theorem 4.1,

$$|[\Phi^i(x) - X^i(x)\Phi] - [\Phi^i_s(x) - X^i(x)\Phi_s]| \leq E^i(x) - X^i(x)\, E\,, \quad x \geq 0, \quad i \in P \tag{2}$$

Proof.

Define $r^{ij}(x) = u^{ij}(x, 0) - s^{ij}(x, 0)$ and $\alpha_\ell = \dfrac{x^{\ell - i}}{(\ell - i)!}$. The $(j+1)$th component of the vector inequality (2) takes the form

$$\left| r^{ij}(x) - \sum_{\ell = i}^{p-1} \alpha_\ell \; r^{\ell j}(0) \right| \leq e^{ij}(x, 0) - \sum_{\ell = i}^{p-1} \alpha_\ell \; e^{\ell j}(0, 0), \quad x \geq 0,\ i \in P,\ j \in Q^1 \tag{3}$$

Hence it is sufficient to show that (3) holds. But, we have

$$e^{ij}(x,0) - \sum_{\ell=i}^{p-1} \alpha_\ell\, e^{\ell j}(0,0) - [r^{ij}(x) - \sum_{\ell=i}^{p-1} \alpha_\ell\, r^{\ell j}(0)]$$

$$= [e^{ij}(x,0) - r^{ij}(x)] - \sum_{\ell=i}^{p-1} \alpha_\ell\, [e^{\ell j}(0,0) - r^{\ell j}(0)]$$

$$= \frac{x^{p-i}}{(p-i)!}\, [e^{pj}(\xi,0) - r^{pj}(\xi)]\ , \quad \text{where } 0 < \xi < x$$

$$\geq 0$$

The last inequality follows from condition ii) of theorem 4.1.

Hence,

$$r^{ij}(x) - \sum_{\ell=i}^{p-1} \alpha_\ell\, r^{\ell j}(0) \leq e^{ij}(x,0) - \sum_{\ell=i}^{p-1} \alpha_\ell\, e^{\ell j}(0,0) \tag{4}$$

Similarly, we have

$$-[r^{ij}(x) - \sum_{\ell=i}^{p-1} \alpha_\ell\, r^{\ell j}(0)] \leq e^{ij}(x,0) - \sum_{\ell=i}^{p-1} \alpha_\ell\, e^{\ell j}(0,0) \tag{5}$$

(4) and (5) together imply (3).

<u>Lemma 4.2</u>

Under condition ii) of theorem 4.1,

$$\left|[\Psi^j(y) - \Phi\, Y^j(y)] - [\Psi^j_s - \Phi_s\, Y^j(y)]\right| \leq \hat{E}^j(y) - EY^j(y), \quad y \geq 0, \quad j \in Q$$

Proof.

Similar to lemma 4.1.

<u>Lemma 4.3</u>

Under conditions i) and ii) of theorem 4.1,

$$\left|\hat{g}[\underline{u}(x,y)]\right| - \hat{g}[\underline{s}(x,y)]\big| \leq \hat{g}[\underline{e}(x,y)], \quad x \geq 0, \quad y \geq 0$$

Proof.

By (1), $\hat{g}[\underline{u}(x,y)] = (X^i(x), \Psi^j(y) - \Phi Y^j(y)) + (\Phi^i(x) - X^i(x)\Phi,\ Y^j(y)) + X^i(x)\Phi Y^j(y)$

$$\hat{g}[\underline{s}(x,y)] = (X^i(x), \Psi_s^j(y) - \Phi_s Y^j(y)) + (\Phi_s^i(x) - X^i(x)\Phi_s, Y^j(y)) + X^i(x)\Phi_s Y^j(y)$$

By lemma 4.1, lemma 4.2 and condition i) of theorem 4.1,

$$|\hat{g}[\underline{u}(x,y)] \le \hat{g}[\underline{s}(x,y)]| \le (X^i(x), \hat{E}^j(y) - EY^j(y)) + (E^i(x) - X^i(x)E, Y^j(y))$$
$$+ X^i(x)\, EY^j(y)$$
$$= \hat{g}[\underline{e}(x,y)]$$

<u>Proof of theorem 4.1</u>

The (i, j)-derivative of a function can be expressed as

$$u^{ij}(x,y) = \hat{g}[\underline{u}(x,y)] + I^{ij}[x, y, u^{pq}(x,y)], \quad i \in P, \quad j \in Q$$

Similar expressions hold for $s^{ij}(x,y)$ and $e^{ij}(x,y)$. Hence,

$$|u^{ij}(x,y) - s^{ij}(x,y)| \le |\hat{g}[\underline{u}(x,y)] - \hat{g}[\underline{s}(x,y)]| + |I^{ij}[x, y, u^{pq}(x,y)] - I^{ij}[x, y, s^{pq}(x,y)]|$$
$$\le \hat{g}[\underline{e}(x,y)] + |I^{ij}[x, y, \hat{f}[x, y, \underline{u}(x,y)]] - I^{ij}[x, y, \hat{f}[x, y, \underline{s}(x,y)]]|$$
$$+ |I^{ij}[x, y, \hat{f}[x, y, \underline{s}(x,y)]] - I^{ij}[x, y, s^{pq}(x,y)]|.$$
$$\le \hat{g}[\underline{e}(x,y)] + I^{ij}[x, y, (L, \underline{e}(x,y)) + d(x,y)]$$
$$\le \hat{g}[\underline{e}(x,y)] + I^{ij}[x, y, e^{pq}(x,y)]$$
$$= e^{ij}(x,y).$$

4.2 An a posterior Error Estimate

<u>Definitions</u>

1) $\underline{u}(x,y)$ = exact solution of Problem C.

2) $\underline{s}(x,y)$ = any approximation obtained by Method C.

3) $\underline{e}(x,y) = \underline{u}(x,y) - \underline{s}(x,y)$

4) $r(x,y) = s^{pq}(x,y) - \hat{f}[x, y, \underline{s}(x,y)], \quad (x,y) \in R$

5) $\partial \hat{f}\,[x, y, \underline{u}(x, y)] = \text{col}(\hat{f}_{u^{00}}[x, y, \underline{u}(x, y)], \hat{f}_{u^{10}}[x, y, \underline{u}(x, y)], \ldots$
$\hat{f}_{u^{p,q-1}}[x, y, \underline{u}(x, y)],\ 0)$

is a vector whose components are the partial derivatives of $\hat{f}$ with respect to the variables u^{ij}, $i \in P$, $j \in Q$.

I. Linear Case

Let $\hat{f}\,[x, y, \underline{u}(x, y)]$ be linear with respect to the variables u^{ij}. Then,

$$e^{pq}(x, y) = \hat{f}\,[x, y, \underline{e}(x, y)] - r(x, y) .$$

This is a partial differential equation similar to that of Problem C, with an additional term $-\, r(x, y)$ on the right hand side. Hence we have

Theorem 4.2

The error $\underline{e}(x, y)$ is a solution of the following

Problem E

Find a solution of

$$e^{pq}(x, y) = \hat{f}\,[x, y, \underline{e}(x, y)] - r(x, y), \qquad (x, y) \in R \tag{6}$$

which satisfies

i) $e^{0j}(x, 0) = \varphi^{j}(x) - s^{0j}(x, 0), \quad j \in Q^{1}, \quad x \in [0, a]$

and ii) $e^{i0}(0, y) = \psi^{i}(y) - s^{i0}(0, y), \quad i \in P^{1}, \quad y \in [0, b]$

where $\hat{f}[x, y, \underline{u}(x, y)]$, $\varphi^{j}(x)$, $\psi^{i}(y)$ are defined as in Problem C.

Remark: All assumptions on Problem C are also valid on Problem E. Hence, Method C can be applied to obtain an error estimate for Problem C.

II. Non-linear Case

When $\hat{f}[x, y, \underline{u}(x, y)]$ is non-linear with respect to u^{ij}, we can linearize it about $\underline{s}(x, y)$, i.e.

$$\hat{f}[x, y, \underline{u}(x, y)] \doteqdot \hat{f}[x, y, \underline{s}(x, y)] + (\partial\hat{f}[x, y, \underline{s}(x, y)], \underline{u}(x, y) - \underline{s}(x, y)).$$

Thus, $$e^{pq}(x, y) \doteqdot (\partial\hat{f}[x, y, \underline{s}(x, y)], \underline{e}(x, y)) - r(x, y) \tag{7}$$

Hence, we can obtain an error estimate by solving an initial value problem similar to Problem E, with the right hand side of (6) replaced by that of (7).

4.3 Growth of Error and Instability of Method C.

Definition

A numerical method for solving Problem C is said to be stable if, for any $\hat{f}[x, y, \underline{u}(x, y)]$ and any initial functions satisfying the assumptions of Problem C, the errors $e^{ij}(x, y)$ remain bounded as x or y increases.

In Section 3, we have shown that, as $h \to 0$ and $k \to 0$, $\underline{e}(x, y)$ converges to $\underline{0}$ uniformly. In this section, we want to investigate, for fixed h and k, how $\underline{e}(x, y)$ varies as x or y increases. Since

$$u^{ij}(x, y) = I^{ij}[x, y, u^{pq}(x, y)] + (X^i(x), \Psi^j(y)) + (\Phi^i(x), Y^j(y)) - X^i(x)\Phi Y^j(y),$$
$$i \in P, \quad j \in Q,$$

$$s^{ij}(x, y) = I^{ij}[x, y, s^{pq}(x, y)] + (X^i(x), \Psi^j_s(y)) + (\Phi^i_s(x), Y^j(y)) - X^i(x)\Phi_s Y^j(y),$$
$$i \in P, \quad j \in Q,$$

after subtraction, we have, for $i \in P$, $j \in Q$,

$$e^{ij}(x, y) = \underbrace{I^{ij}[x, y, u^{pq}(x, y) - s^{pq}(x, y)]}_{\text{part I}} + \underbrace{(X^i(x), \hat{E}^j(y)) + (E^i(x), Y^j(y)) - X^i(x)EY^j(y)}_{\text{part II}}, \tag{8}$$

Equation (8) indicates that the error $e^{ij}(x, y)$ arises from two sources. They are discussed separately as follows:

Part I

Accumulation of the difference $u^{pq}(x, y) - s^{pq}(x, y)$: -

Part I of (8) is $I^{ij}[x, y, u^{pq}(x, y) - s^{pq}(x, y)]$. Suppose that there exist two functions $\epsilon_1(x, y)$ and $\epsilon_2(x, y)$ such that

$$\epsilon_1(x, y) \le |u^{pq}(x, y) - s^{pq}(x, y)| \le \epsilon_2(x, y), \quad 0 \le x \le a, \quad 0 \le y \le b.$$

Then, for $0 \le x \le a$ and $0 \le y \le b$, we have

$$\frac{x^{p-i}}{(p-i)!} \cdot \frac{y^{q-j}}{(q-j)!} \epsilon_1(x, y) \le |I^{ij}[x, y, u^{pq}(x, y) - s^{pq}(x, y)]| \le \frac{x^{p-i}}{(p-i)!} \cdot \frac{y^{q-j}}{(q-j)!} \epsilon_2(x, y).$$

Since $|u^{pq}(x, y) - s^{pq}(x, y)| \Longrightarrow 0$ as $h \to 0$ and $k \to 0$, $\epsilon_1(x, y)$ are small if h, k are small.

Part II

Propagation of initial errors: -

Part II of (8) is $(X^i(x), \hat{E}^j(y)) + (E^i(x), Y^j(y)) - X^i(x)\, E\, Y^j(y)$.

i) $$(X^i(x), \hat{E}^j(y)) = e^{ij}(0, y) + x e^{i+1, j}(0, y) + \cdots + \frac{x^{\ell-i}}{(\ell-i)!} e^{\ell j}(0,y) + \cdots + \frac{x^{p-i-1}}{(p-i-1)!} e^{p-i, j}(0, y) \tag{9}$$

where $e^{\ell j}(0, y) = D_y^j \psi^\ell(y) - s^{\ell j}(0, y)$ is the error arising from approximating the initial function $\psi^\ell(y)$ along the y-axis. (9) shows how these initial errors are propagated as x and y increase.

The situation is much clearer if we consider two particular cases:

Case 1

For a fixed $y \in [0,b]$, suppose

$$e^{\ell j}(0,y) = \epsilon(y)$$

and $$e^{tj}(0,y) = 0 \quad , \quad t = i, i+1, \cdots \ell-1, \ell+1, \ldots p-1 .$$

For these particular initial errors, (9) takes the form

$$(X^i(x), \hat{E}^j(y)) = \epsilon(y) \cdot \frac{x^{\ell-i}}{(\ell-i)!}$$

The following table shows how this single error $\epsilon(y)$ is propagated as x increases.

x	0	1	2	3	4	5	...
$(X^i(x), \hat{E}^j(y))$	0	1	4	8	16	25	...

$\ell = 2, \quad i = 0, \quad$ unit is $\frac{\epsilon(y)}{2}$.

Table 4.1

Case 2

For a fixed $y \in [0,b]$, suppose

$$e^{tj}(0,y) = \epsilon(y), \quad t = i, \; i+1, \; \ldots \; p-1 .$$

For these particular initial errors, (9) takes the form

$$(X^i(x), \hat{E}^j(y)) = \epsilon(y) \left(\sum_{t=0}^{p-i-1} \frac{x^t}{t!} \right)$$

The following table shows how these errors are propagated as x increases.

x	0	1	2	3	4	5	...
$(X^i(x), \hat{E}^j(y))$	2	5	10	17	26	37	...

$p = 3, \quad i = 0, \quad \text{unit is } \frac{\epsilon(y)}{2}$

Table 4.2

In general, since the initial errors $e^{tj}(0, y)$ may have different signs, the error $e^{ij}(x, y)$ at (x, y) does not grow so fast as in cases 1 and 2. But, it is very unlikely that they may cancel each other completely.

ii) $(E^i(x), Y^j(y)) = e^{ij}(x, 0) + y\, e^{i, j+1}(x, 0) + \ldots + \frac{y^{q-j-1}}{(q-j-1)!}\, e^{i, q-1}(x, 0)$

This part shows how the initial errors $e^{it}(x, 0)$ made along the x-axis are propagated as x and y increase. Similar phenomena as in i) occur.

iii) $X^i(x)\, E\, Y^j(y) = \sum_{\ell=0}^{p-i-1} \sum_{t=0}^{q-j-1} \frac{x^\ell}{\ell!} \frac{y^t}{t!}\, e^{\ell t}(0, 0)$

This part shows how the initial errors $e^{\ell t}(0, 0)$ made at the origin are propagated as x and y increase. In Method C, $e^{\ell t}(0, 0) = 0$, $\ell \in P^1$, $t \in Q^1$.

The above discussion can be summarized in a theorem.

Theorem 4.3

The numerical Method C for solving Problem C is unstable.

Proof.

Consider the particular problem in which $f[x, y, \underline{u}(x, y)] \equiv 0$, and $\varphi^j(x) \equiv \psi^i(y) \equiv 0$, $i \in P^1$ and $j \in Q^1$. The solution is obviously the zero function, i.e., $\underline{u}(x, y) \equiv \underline{0}$. Part II of this section indicates clearly how $e^{ij}(x, y)$ propagates as x and y increase.

SECTION 5. EXAMPLES

5.1 The following are some of the examples run on CDC 1604. Each took about one minute. They show clearly the properties of convergence and instability. The error bounds and error estimates are realistic.

Notations

u^{ij} - exact solution; s^{ij} - spline approximation; $e^{ij} = u^{ij} - s^{ij}$

Example 5.1

Consider the equation

$$u^{22} = (u^{00} - x^3 - y)(x^2 y^2 - 2) - 4y\,(u^{01} - 1)$$

with initial conditions

$$\varphi^0(x) = x^3,\ \varphi^1(x) = 1 + x \qquad x \in [0,\ .5]\ ,$$

$$\psi^0(y) = \psi^1(y) = y \qquad y \in [0,\ .5]\ .$$

The exact solution is $u^{00} = x^3 + y + \sin xy$.

The following tables show some approximate values and errors.

(h = .05, k = .1):

Table 5.1 Approximations and Errors

x	y	s^{00}	e^{00}	s^{01}	e^{01}
.05	.0	1.2500000 E-4	.0	1.0500000	.0
.15	.0	3.3750000 E-3	.0	1.1500000	.0
.30	.0	2.7000000 E-2	1.8189894 E-12	1.3000000	2.9103830 E-11
.45	.0	9.1125000 E-2	3.6379788 E-12	1.4500000	.0
.05	.15	1.5762500 E-1	-7.0307578 E-08	1.0500000	-1.4062389 E-06
.15	.15	1.7587477 E-1	-1.6640151 E-06	1.1499906	-2.8592214 E-05
.30	.15	2.2199742 E-1	-1.2607932 E-05	1.2998969	-2.0057778 E-04
.45	.15	3.0861544 E-1	-4.1684507 E-05	1.4496175	-6.4230213 E-04
.05	.30	3.1512500 E-1	-5.6248973 E-07	1.0500000	-5.6248973 E-06
.15	.30	3.4837031 E-1	-1.0498574 E-05	1.1499438	-9.5601339 E-05
.30	.30	4.1694844 E-1	-6.9894922 E-05	1.2993814	-5.9554668 E-04
.45	.30	5.2593381 E-1	-2.1850059 E-04	1.4477061	-1.8004926 E-03
.05	.45	4.7262500 E-1	-1.8983701 E-06	1.0500000	-1.2655713 E-05
.15	.45	5.2085531 E-1	-3.1559888 E-05	1.1498500	-1.9160562 E-04
.30	.45	6.1178352 E-1	-1.9320555 E-04	1.2983511	-1.0806616 E-03
.45	.45	7.4282250 E-1	-5.7862695 E-04	1.4438901	-3.0850587 E-03

Table 5.2 Approximations and Errors

x	y	s^{10}	e^{10}	s^{11}	e^{11}
.1	.0	3.0000000 E-02	9.0949470 E-13	1.0	-1.1641532 E-09
.2	.0	1.2000000 E-01	5.4569682 E-12	1.0	-2.3283064 E-09
.3	.0	2.7000000 E-01	5.8207661 E-11	1.0	-2.3283064 E-09
.4	.0	4.8000000 E-01	6.5483619 E-11	1.0	-1.1641532 E-09
.1	.1	1.3000000 E-01	-5.0000781 E-06	1.0	-1.4999908 E-04
.2	.1	2.2000000 E-01	-1.9999559 E-05	1.0	-5.9996900 E-04
.3	.1	3.7000000 E-01	-4.4996799 E-05	1.0	-1.3498336 E-03
.4	.1	5.8000000 E-01	-7.9989390 E-05	1.0	-2.3994679 E-03
.1	.2	2.2999250 E-01	-3.2498931 E-05	9.9985000 E-01	-4.4996847 E-04
.2	.2	3.1999250 E-01	-1.1498026 E-04	9.9910002 E-01	-1.4994916 E-03
.3	.2	4.6988751 E-01	-2.4739945 E-04	9.9775014 E-01	-3.1474434 E-03
.4	.2	6.7979002 E-01	-4.2968342 E-04	9.9580049 E-01	-5.3919604 E-03
.1	.3	3.2996250 E-01	-9.7490571 E-05	9.9955001 E-01	-8.9983807 E-04
.2	.3	4.1977502 E-01	-3.1485615 E-04	9.9730031 E-01	-2.6976065 E-03
.3	.3	5.6943762 E-01	-6.5180026 E-04	9.9325211 E-01	-5.3884480 E-03
.4	.3	7.7895044 E-01	-1.1078444 E-03	9.8740772 E-01	-8.9645469 E-03

Example 5.2

Consider the equation

$$u^{22} = \sin((u^{11})^2 + (u^{12})^2) - u^{10} - u^{20} + x \sin y - \sin x^2$$

with initial conditions

$$\varphi^0(x) = e^{-x}, \quad \varphi^1(x) = \frac{1}{2}x^2 \qquad x \in [0, .2]$$

$$\psi^0(y) = 1 + y^3, \ \psi^1(y) = -1 \qquad y \in [0, .2]$$

The exact solution is $u^{00} = e^{-x} + y^3 + \frac{1}{2}x^2 \sin y$.

The following table shows the convergence of some approximate values. The values in each column are alterantively the errors at the grid points and intermediate points in the interval [0, .2] of the x-axis.

Table 5.3 Error e^{00} along the line $y = 0.1$

h = k =	.1	.05	.025	.0125
	.0	.0	.0	.0
	1.9658617 E-5	2.5429617 E-6	3.2209209 E-7	4.0512532 E-08
	-1.6458071 E-6	-5.1732059 E-8	-1.6298145 E-9	-5.8207661 E-11
	-2.5356465 E-5	-2.7711067 E-6	-3.3226388 E-7	-4.0978193 E-08
	-7.2594412 E-6	-2.3051689 E-7	-7.2614057 E-9	-2.3283064 E-10
		2.1908490 E-6	3.1095988 E-7	4.0148734 E-08
		-5.3268741 E-7	-1.6865670 E-8	-5.3842086 E-10
		-3.3702963 E-6	-3.5129779 E-7	-4.1589374 E-08
		-9.5478026 E-7	-3.0369847 E-8	-9.6042641 E-10
			2.8408249 E-7	3.9319275 E-08
			-4.7715730 E-8	-1.5133992 E-09
			-3.8587314 E-7	-4.2695319 E-08
			-6.8859663 E-8	-2.1827873 E-09
			2.4194014 E-7	3.7965947 E-08
			-9.3743438 E-8	-2.9685907 E-09
			-4.3548062 E-7	-4.4266926 E-08
			-1.2230885 E-7	-3.8999133 E-09
				3.6132406 E-08
				-4.9330993 E-09
				-4.6347850 E-08
				-6.0681487 E-09
				3.3789547 E-08
				-7.3632691 E-09
				-4.8908987 E-08
				-8.7457011 E-09

Example 5.3

Consider the equation

$$u^{22} = (1 + \cos x)\, u^{20} + \sin x \cdot u^{11} - \frac{3}{2} x \cos y \;(\cos x + 2) + \frac{3}{4}$$
$$x^2 \sin x \sin y$$

with initial conditions

$$\varphi^0(x) = \frac{1}{4}x^3 + \sin x, \quad \varphi^1(x) = \sin x \qquad x \in [0,\ 1.5],$$

$$\psi^0(y) = 0 \qquad , \quad \psi^1(y) = e^y \qquad y \in [0,\ 1.5].$$

The exact solution is $u^{00} = \frac{1}{4} x^3 \cos y + e^y \sin x$.

The following tables show the error estimates and error bounds ($h = k = .5$):

Table 5.4 Error estimation of s^{00}

x	y	approximations	actual errors	error estimates	error bounds
.0	.0	.0	.0	.0	.0
.5	.0	5.0015632 E-2	-5.2122377 E-6	-1.1724445 E-5	1.5647778 E-5
1.0	.0	1.0009387 E-1	-1.0450503 E-5	-3.9134085 E-5	6.2708205 E-5
.0	.5	.0	.0	.0	8.1240516 E-6
.5	.5	5.2578197 E-2	-5.3302638 E-6	5.6274321 E-6	4.3806464 E-5
1.0	.5	1.0521264 E-1	-1.0963982 E-5	-4.6301612 E-6	1.1390555 E-4
.0	1.0	.0	.0	.0	1.7279674 E-5
.5	1.0	5.5272301 E-2	-5.6827885 E-6	2.0772753 E-5	7.6938392 E-5
1.0	1.0	1.1059424 E-1	-1.2497507 E-5	2.6452856 E-5	1.7534358 E-4

Table 5.5 Error estimation of s^{02}

x	y	approximations	actual errors	error estimates	error bounds
.0	.0	.0	.0	.0	.0
.5	.0	5.1302359 E-2	-1.3544401 E-3	-1.3570399 E-3	1.4126718 E-3
1.0	.0	1.0251113 E-1	-2.9277147 E-3	-3.2764423 E-3	3.1122553 E-3
.0	.5	.0	.0	.0	.0
.5	.5	5.3928039 E-2	-1.4175935 E-3	-1.4195270 E-3	1.4808246 E-3
1.0	.5	1.0775334 E-1	-3.0510387 E-3	-3.3971861 E-3	3.2547727 E-3
.0	1.0	.0	.0	.0	.0
.5	1.0	5.6688644 E-2	-1.4842134 E-3	-1.4855863 E-3	1.5526934 E-3
1.0	1.0	1.1326577 E-1	-3.1815314 E-3	-3.5264054 E-3	3.4055042 E-3

Example 5.4

Consider the equation

$$u^{32} = u^{10}\left[6 + \log(1 + u^{00})\cdot(6 + x(.1 - y))\right]$$

with initial conditions

$$\varphi^{0}(x) = e^{.1x} - 1, \quad \varphi^{1}(x) = -x\,e^{.1x} - 1 \qquad x \in [0, 4] \ ,$$

$$\psi^{0}(y) = 0, \quad \psi^{1}(y) = .1 - y, \quad \psi^{2}(y) = (.1 - y)^{2}, \ y \in [0, .2] \ .$$

The exact solution is $u^{00} = e^{x(.1-y)} - 1$.

The following table shows how the error of s^{00} increases as x increases ($h = .04$, $k = .05$):

Table 5.6 Instability of Method C

x \ y	.0	.05
.0	.0	.0
.32	-7.1031536 E-10	-6.8583631 E-7
.64	-2.8085196 E-09	-5.6429535 E-6
.96	-6.3773768 E-09	-1.9591223 E-5
1.28	-1.1408702 E-08	-4.7772748 E-5
1.60	-1.8018909 E-08	-9.5988964 E-5
1.92	-2.6193447 E-08	-1.7064013 E-4
2.24	-3.6088750 E-08	-2.7876687 E-4
2.56	-4.7679350 E-08	-4.2809425 E-4
2.88	-6.1045284 E-08	-6.2707825 E-4
3.20	-7.6201104 E-08	-8.8495482 E-4
3.52	-9.3335984 E-08	-1.2117920 E-3
3.84	-1.1234806 E-07	-1.6185443 E-3

REFERENCES

1. J. T. Day, On the approximate solution of differential equations, Ph.D. thesis, University of Wisconsin, 1963.

2. J. B. Diaz, On existence, uniqueness, and numerical evaluation of solutions of ordinary and hyperbolic differential equations. Tech. Note BN-216, AFOSR-TN-60-1059, Sept., 1960, The Institute for Fluid Dyn. and Appl. Math., University of Maryland.

3. Yu I. Kovach, Proof of an existence and uniqueness theorem for the solution of the Cauchy problem by the method of two-sided approximation for the generalized 2n-wave system. USSR Comp. Math. and Math. Phy. Vol. 5, No. 3, 1965, pp. 208.

4. Beatriz Margolis, Existence, uniqueness and convergence of successive approximations for ordinary and hyperbolic partial differential equations. Ph.D. thesis. University of Maryland, 1966.

5. R. H. Moore, On approximations of the solutions of the Goursat problem for second order quasi-linear equations, Ph.D. thesis. University of Michigan, June, 1959.

6. W. Walter, Differential und Integral Ungleichungen, Springer Verlag, Berlin, 1964.

7. G. Zwirner, Sull'approssimasione degli integrali del sistema differenziale

$$\frac{\partial^2 z}{\partial x \partial y} = f(x, y, z), \quad z(x_0, y) = \psi(y), \quad z(x, y_0) = \varphi(x) ,$$

Atti dell'Istituto Veneto di Scienze, Lettere ed Arti., Cl. Sci. Fis. Mat. Nat. 109, 219-231 (1950/51).

ACKNOWLEDGEMENT

I would like to thank Professor J. B. Rosen for his encouragement and assistance in the writing of this report.