ITERATIVE SOLUTION OF THE NEUMANN PROBLEM ON A RECTANGLE BY SUCCESSIVE LINE OVER-RELAXATION

by

Fred W. Dorr

Computer Sciences Technical Report #12
January 1968

Iterative Solution of the Neumann Problem on a Rectangle by Successive Line Over-Relaxation

Fred W. Dorr

Introduction.

Approximate solutions to the system of equations corresponding to a finite difference analog of the Neumann problem can often be found iteratively by successive line over-relaxation (SLOR). The convergence rate of this procedure is governed by the eigenvalues of an associated iteration matrix. These eigenvalues can be calculated explicitly for the Dirichlet problem on a rectangle, but for the Neumann problem this has not been accomplished. Rather, asymptotic estimates have been given for the eigenvalue which governs the rate of convergence. Gilchrist [4] has considered the Jacobi point iterative method for the case of a square with uniform mesh. Parter [6] has given a general treatment of the rates of convergence of iterative methods for elliptic equations, which includes the Neumann problem as a special case. This estimate is used here to prove a recent conjecture of Gary concerning a problem related to the solution by SLOR of the Neumann problem on a rectangle.

The Neumann Problem.

Let R be the rectangle $[0, a] \times [0, b]$ in the (x, y)-plane, and let Γ be the boundary of R. If two functions f and g are given, the Neumann problem is to find a solution $u(x, y)$ to

$$(1)\qquad \begin{cases} \Delta u = f & \text{in } R \\ \dfrac{\partial u}{\partial \nu} = g & \text{on } \Gamma^* \end{cases}$$

where Γ^* is Γ with the four corners deleted, and $\dfrac{\partial u}{\partial \nu}$ is the outer normal derivative of u on Γ^*. We introduce a mesh on $\bar{R}$ of width Δx in the x-direction, and Δy in the y-direction, with

$$(2)\qquad \begin{cases} \Delta x = \dfrac{a}{N_x + 1} \\ \Delta y = \dfrac{b}{N_y + 1} \end{cases}$$

where N_x and N_y are integers. Using the usual approximations for Δu and $\dfrac{\partial u}{\partial \nu}$, we let $\alpha = (\Delta y)^2 (\Delta x)^{-2}$ and write the finite difference equations in matrix form. Define the $N_y \times N_y$ tridiagonal matrices

$$L_1 = \begin{bmatrix} (1+\alpha) & -1 & & & 0 \\ -1 & (2+\alpha) & -1 & & \\ & -1 & \ddots & \ddots & \\ & & \ddots & (2+\alpha) & -1 \\ 0 & & & -1 & (1+\alpha) \end{bmatrix}$$

and

$$L = \begin{bmatrix} (1+2\alpha) & -1 & & & 0 \\ -1 & (2+2\alpha) & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & (2+2\alpha) & -1 \\ 0 & & & -1 & (1+2\alpha) \end{bmatrix}.$$

We now define the $N_x \times N_x$ block matrices

$$D = \begin{bmatrix} L_1 & & & & 0 \\ & L & & & \\ & & \ddots & & \\ & & & L & \\ 0 & & & & L_1 \end{bmatrix}$$

and

$$U = \begin{bmatrix} 0 & -\alpha I & & & 0 \\ & 0 & -\alpha I & & \\ & & \ddots & \ddots & \\ & & & 0 & -\alpha I \\ 0 & & & & 0 \end{bmatrix}$$

Using the usual notation $u_{ij} = u(i\triangle x,\ j\triangle y)$, we then define the vectors

$$u_i = \begin{bmatrix} u_{i1} \\ u_{i2} \\ \vdots \\ u_{iN_y} \end{bmatrix}$$

and

$$u = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_{N_x} \end{bmatrix} .$$

Then we let $\mathcal{L} = D + U + U^T$ and the finite difference equations for the Neumann problem become

$$\mathcal{L}u = \rho$$

where ρ is a vector, of the same form as u, that depends on Δx, Δy, f, and g.

The following properties of $\mathcal{L}$ can be verified by direct computation:

Lemma 1. $\mathcal{L}$ is positive semi-definite. The null space of $\mathcal{L}$ is one-dimensional, and is spanned by the vector $\vec{e}$, all of whose components are 1.

This says that solutions to $\mathcal{L}u = \varphi$ differ only by an additive constant. This is to be expected, since it is a property of analytic solutions to the Neumann problem.

For a real parameter $\frac{1}{2} < \gamma < 1$, define

$$\begin{cases} N = \gamma D + U^T \\ P = (\gamma - 1)D - U \end{cases} \tag{3}$$

Now D is positive definite [7, p. 23] so N is non-singular. Thus we can define

$$M = N^{-1} P \quad . \tag{4}$$

The SLOR iteration for the Neumann problem is then

$$\begin{cases} u^{(0)} \quad \text{arbitrary} \\ u^{(n+1)} = M u^{(n)} + N^{-1}\rho \end{cases} \quad . \tag{5}$$

If one writes (5) in terms of the smaller blocks in the matrices involved, it is clear that at each step one solves N_x tridiagonal systems. These can be solved very efficiently, so that the computational effort involved in each iteration is not too great.

We can now state the following convergence result:

<u>Lemma 2.</u> Assume that $\rho^T \vec{e} = 0$, so that $\mathcal{L}u = \rho$ has a solution. Then for any choice of $u^{(0)}$, the iterates $u^{(n)}$ converge to a solution of $\mathcal{L}u = \rho$.

<u>Proof.</u> This follows from a Theorem of Keller [5, p. 285], since the matrix

$$N + N^* - \mathcal{L} = (2\gamma - 1)D$$

is positive definite.

If we define the errors $\epsilon^{(n)} = u - u^{(n)}$, then it is easy to see that $\epsilon^{(n)} = M^n \epsilon^{(0)}$. Thus to examine the rate of convergence of the iterative scheme, we should examine the form of $M^n x$ for an arbitrary vector x. To do this, introduce the following notation:

$$\{\lambda_i \mid 1 \leq i \leq N_x N_y\} = \text{eigenvalues of } M$$

$$\lambda_0 = \max\{|\lambda_i| \;\big|\; |\lambda_i| < 1\}$$

$p(\lambda_i)$ = maximum degree of any Jordan block corresponding to an eigenvalue λ_i of M

$$p_0 = \max\{p(\lambda_i) \mid |\lambda_i| = \lambda_0\}$$

$$J_0 = \{j \mid |\lambda_j| = \lambda_0,\ p(\lambda_j) = p_0\}$$

n_0 = number of elements in J_0 .

Then we have the following:

Theorem 1. Let S be a non-singular matrix such that $S^{-1}MS$ is in Jordan normal form, with the columns $s_1, s_2, \ldots, s_{N_xN_y}$ of S normalized and ordered so that

(i) $s_1 = \vec{e}$

(ii) $(M-\lambda_j I)s_j = 0$ for $2 \le j \le n_0 + 1$

(iii) $(M-\lambda_j I)^{p_0} s_{n_0+1+j} = 0$ but

$$(M-\lambda_j I)^{p_0-1} s_{n_0+1+j} \neq 0 \quad \text{for} \quad 1 \le j \le n_0 \quad .$$

For an arbitrary vector x, let $x = \sum_{j=1}^{N_xN_y} \alpha_j s_j$. Then as $n \to \infty$,

$$\| M^n x - \alpha_1 \vec{e} \|_\infty \sim \binom{n}{p_0-1} \lambda_0^{n+1-p_0} \left\| \sum_{j=2}^{n_0+1} \alpha_{n_0+j} s_j \right\|_\infty$$

Proof. The existence of the s_j in this form follows from the definition of p_0, n_0, and the Jordan normal form (and also because $M\vec{e} = \vec{e}$, and Lemmas 1 and 2 ensure that 1 is a simple eigenvalue of M). We can then write

$$M^n x = \alpha_1 \vec{e} + \sum_{\nu=2}^{N_xN_y} \alpha_\nu M^n s_\nu \quad .$$

Let s_j be the eigenvector associated with an eigenvalue λ_j of M, and let s_{j_i} be the i^{th} generalized eigenvector associated with s_j

and λ_j. Then it is easy to show by induction that

$$M^n s_{j_i} = \sum_{\nu=0}^{j_0-i} \binom{n}{\nu} \lambda_j^{n-\nu} s_{j_{\nu+i}}$$

where j_0 is the degree of this particular Jordan block, and the s_{j_ν} are ordered so that

$$(M-\lambda_j I)\, s_{j_\nu} = s_{j_{\nu+1}} \qquad .$$

Thus we have

$$M^n s_{j_i} \sim \binom{n}{j_0-i} \lambda_j^{n+i-j_0} s_j$$

so that for the asymptotic estimate we neglect all terms except those for which $|\lambda_j| = \lambda_0$, $j_0 = p_0$, and $i = 1$. This completes the proof of the Theorem.

The rate of convergence.

As is the usual case for SLOR, we reduce the question of finding λ_0 to the problem of estimating the eigenvalues for the corresponding Jacobi iterative procedure. Therefore, we define the following:

$$(6) \qquad \begin{cases} N_1 = D \\ P_1 = -(U+U^T) \\ M_1 = N_1^{-1} P_1 \end{cases}$$

We let $\{\mu_j \mid 1 \le j \le N_x N_y\}$ be the eigenvalues of M_1, and

$$\mu_0 = \max \{ |\mu_j| \;\Big|\; |\mu_j| < 1 \} \quad .$$

The following is then an easy consequence of known facts about $\mathcal{L}$ and the Jacobi method:

Lemma 3. Let μ be an eigenvalue of M_1. Then:

(i) μ is real

(ii) $-\mu$ is an eigenvalue of M_1

(iii) -1 is an eigenvalue of M_1, so the Jacobi method is not always convergent.

We now define the $N_x \times N_x$ block matrix

$$S(\delta) = \begin{bmatrix} \delta I & & & 0 \\ & \delta^2 I & & \\ & & \ddots & \\ 0 & & & \delta^{N_x} I \end{bmatrix}$$

The relationship between the $\{\lambda_j\}$ and $\{\mu_j\}$ is then provided by the following:

Lemma 4. (i) If $Mx = \lambda x$ for $x \ne 0$, then $\lambda \ne 0$. Thus $M_1 y = \mu y$ for $y = S^{-1}(\lambda^{\frac{1}{2}})x$ and $\mu = (\lambda\gamma + 1 - \gamma)(\lambda)^{-\frac{1}{2}}$.

(ii) If $M_1 y = \mu y$, let λ be a root of the equation

$$\gamma\lambda - \mu\lambda^{\frac{1}{2}} + 1 - \gamma = 0 \quad .$$

Then $Mx = \lambda x$ for $x = S(\lambda^{\frac{1}{2}})y$.

Proof. If $Mx = \lambda x$, then $(\lambda N - P)x = 0$. If $\lambda = 0$, then $-Px = 0$, so then

$x = 0$ because $-P$ is non-singular. The rest of the Lemma follows by a standard argument [2, p. 250].

The two SLOR eigenvalues corresponding to $\mu = 1$ and $\mu = -1$ are:

$$\lambda_1 = 1$$

and

$$\lambda_2 = \left(\frac{1-\gamma}{\gamma}\right)^2 .$$

Now $\left(\frac{1-\gamma}{\gamma}\right)^2 < \left(\frac{1-\gamma}{\gamma}\right) < 1$, so that again by a standard argument [2, p. 253], we should choose

$$\gamma_0 = \frac{1}{2}\left(1 + (1-\mu_0^2)^{\frac{1}{2}}\right)$$

since then any other eigenvalues μ_ν less than $|\mu_0|$ in magnitude satisfy

$$\mu_\nu^2 < \mu_0^2 = 4\gamma_0(1-\gamma_0)$$

so that if λ_ν corresponds to μ_ν then

$$|\lambda_\nu| = \left(\frac{1-\gamma_0}{\gamma_0}\right) .$$

Finally, notice that for this choice of γ we have

$$\lambda_C = \left(1 - (1-\mu_0^2)^{\frac{1}{2}}\right)\left(1 + (1-\mu_0^2)^{\frac{1}{2}}\right)^{-1}$$

An estimate for μ_0 is provided by the following Theorem of Parter [6, p. 343]:

Lemma 5. Let Λ be the smallest non-zero eigenvalue λ of the problem

$$\begin{cases} \Delta u + \lambda u = 0 & \text{in } R \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \Gamma^* \\ u \not\equiv 0 & \end{cases}$$

Then $\mu_0 \sim 1 - \frac{\Lambda}{2}(\Delta x)^2 + o(\Delta x \Delta y)$ as Δx and Δy tend to 0.

For the Neumann problem on a rectangle, these eigenvalues are [1, p. 429]

$$\mu_{mn} = \pi^2 (m^2 a^{-2} + n^2 b^{-2})$$

where $m, n \geq 0$. Thus

$$\Lambda = \pi^2 c^{-2}$$

for $c = \max(a, b)$. We can summarize this in the following:

Theorem 2. If SLOR is used to solve the Neumann problem on a rectangle with relaxation parameter $\gamma_0 \sim \frac{1}{2}(1 + \pi c^{-1}(\Delta x))$ then the rate of convergence is governed by the quantity

$$\lambda_0 \sim 1 - 2\pi c^{-1}(\Delta x)$$

Application to the results of Gary.

In [3], Gary considered the following class of problems: Let $R(\alpha) = [0, a] \times [0, \alpha^{\frac{1}{2}} a]$ for $0 < \alpha \leq 1$. Solve the Neumann and Dirichlet problems on this region by SLOR with Nx and Ny fixed, and let the rates of convergence be $\lambda_N(\alpha)$ and $\lambda_D(\alpha)$ respectively.

From computational results, Gary conjectured that $\lambda_D(\alpha)$ decreases as α decreases, but that $\lambda_N(\alpha)$ is independent of α.

The Jacobi eigenvalues for the Dirichlet problem are

$$\mu_{rs}(D) = (\alpha \cos (r\pi a^{-1}\Delta x))(\alpha+1 - \cos(s\pi b^{-1}\Delta y))^{-1}$$

so that

$$\mu_D(\alpha) \sim 1 - \frac{\pi^2}{2}(a^{-2}+b^{-2})(\Delta x)^2$$
$$= 1 - \frac{\pi^2}{2} a^{-2}(1+\alpha^{-1})(\Delta x)^2$$

and so

$$\lambda_D(\alpha) \sim 1 - 2\pi a^{-1}(1+\alpha^{-1})^{\frac{1}{2}}(\Delta x)$$

Thus as α decreases, so does $\lambda_D(\alpha)$.

For the Neumann problem, since $0 < \alpha \leq 1$ we have $c = a$, so that

$$\lambda_N(\alpha) \sim 1 - 2\pi a^{-1}(\Delta x)$$

and $\lambda_N(\alpha)$ is independent of α.

<u>Remarks</u>.

(1) These results are all for vertical SLOR, and this is the reason for the dependence of λ_0 only on Δx in Theorem 2. Corresponding results also hold for horizontal SLOR.

(2) Computations performed on the CDC 3600 at the University of Wisconsin have yielded rates of convergence in good agreement with those predicted by Theorem 1. These results suggest

that $p_0 = 1$ for the Neumann problem, and this agrees with the usual assumption that this is the case for the Dirichlet problem.

(3) Gilchrist has shown [4] that, in the special case where $a = b$ and $\Delta x = \Delta y$, for the <u>point</u> Jacobi scheme for the Neumann problem we have

$$\mu_0 \sim 1 - \frac{1}{4} \pi^2 a^{-2} (\Delta x)^2$$

Thus the convergence rate for the Jacobi line scheme for the Neumann problem is twice that of the Jacobi point scheme. This agrees with the results for the Dirichlet problem [2, p. 270].

(4) In [3, p. 221], Gary notes that the convergence rate depends on the function u_{ij} he chooses. This is true only to the extent that the choice of function determines the coefficients α_ν in the expansion of the vector u in terms of generalized eigenvectors. Thus the magnitude of the error may be changed (at a given number of iterations), but the rate of convergence will not be altered by the choice of function.

REFERENCES

1. R. Courant and D. Hilbert, Methods of Mathematical Physics, vol. I, Interscience, New York, 1963.

2. G. Forsythe and W. Wasow, Finite-Difference Methods for Partial Differential Equations, Wiley, New York, 1965.

3. J. Gary, "On Convergence Rates for Line Overrelaxation", Math. Comp., 21(1967), pp. 220-223.

4. B. Gilchrist, "The Convergence Rates of Iterative Solutions of Laplace's Equation", unpublished.

5. H. Keller, "On the Solution of Singular and Semidefinite Linear Systems by Iteration", SIAM J. Numer. Anal. Series B, 2(1965), pp. 281-290.

6. S. V. Parter, "On Estimating the "Rates of Convergence" of Iterative Methods for Elliptic Difference Equations", Trans. Amer. Math. Soc., 114(1965), pp. 320-354.

7. R. S. Varga, Matrix Iterative Analysis, Prentice-Hall, Englewood Cliffs, N.J. 1962.