Linear Complementarity as Absolute Value Equation Solution

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Abstract

We consider the linear complementarity problem (LCP): $Mz + q \geq 0, z \geq 0, z'(Mz + q) = 0$ as an absolute value equation (AVE): $(M + I)z + q = |(M - I)z + q|$, where M is an $n \times n$ square matrix and I is the identity matrix. We propose a concave minimization algorithm for solving (AVE) that consists of solving a few linear programs, typically two. The algorithm was tested on 500 consecutively generated random solvable instances of the LCP with n =10, 50, 100, 500 and 1,000. The algorithm solved 100% of the test problems to an accuracy of 10^{-8} by solving 2 or less linear programs per LCP problem.

Keywords: linear complementarity, absolute value equation, linear programming

1 Introduction

We consider the linear complementarity problem (LCP) [2, 7, 3]:

$$Mz + q \geq 0, z \geq 0, z'(Mz + q) = 0, \tag{1.1}$$

where $M \in \mathbb{R}^{n\times n}$ and $q \in \mathbb{R}^n$ are given, and no assumptions are made on the $n \times n$ matrix M. This NP-complete problem [1], which may not have a solution, is easily shown to be equivalent (see Lemma 2.1 below) to the following absolute value equation:

$$(M + I)z + q = |(M - I)z + q|, \tag{1.2}$$

where $|\cdot|$ denotes absolute value. We shall construct a concave minimization problem, (2.3) below, that can efficiently solve the LCP by linearizing a concave objective function. Another concave minimization approach to the LCP was given in [5] using a different and more complex concave objective function, but without any computational results. In Section 2 we outline the theory behind our approach and in Section 3 we state our iterative algorithm that consists of solving a succession of linear programs with modified objective functions. In Section 4 we give computational results that show the effectiveness of our approach by solving 100% of a sequence of 500 randomly generated consecutive LCPs in $\mathbb{R}^{10}$ to $\mathbb{R}^{1,000}$ to an accuracy of 10^{-8} . Section 5 concludes the paper.

We describe now our notation and some background material. The feasible region of the LCP (1.1) is the set $Z = \{z|Mz + q \geq 0,\ z \geq 0\}$. The scalar product of two vectors x and y in a n-dimensional real space will be denoted by $x'y$. For a linear program $\min_{z\in Z} c'z$ with a vertex solution, the notation:

$$\text{arg vertex } \min_{z\in Z} c'z,$$

will denote the set of vertex solutions of the linear program. For $z \in \mathbb{R}^n$, the norm $\|z\|_\infty$ will denote the infinity norm $\max_{i=1,\ldots,n} |z_i|$. For an $m \times n$ matrix A, A_i will denote the ith row of A. The identity matrix

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in a real space of arbitrary dimension will be denoted by I, while a column vector of ones of arbitrary dimension will be denoted by e and a column of zeros by 0. For a concave function $f: \mathbb{R}^n \to \mathbb{R}$, a supergradient $\partial f(z)$ of f at z is a vector in $\mathbb{R}^n$ satisfying:

$$f(y) - f(z) \leq \partial f(z)(y - z),$$

for any $y \in \mathbb{R}^n$. The set of supergradients of f at the point z is nonempty, convex, compact and reduces to the ordinary gradient $\nabla f(z)$, when f is differentiable at z [8, 9]. For a vector $z \in \mathbb{R}^n$, $\text{diag}(z)$ consists of an $n \times n$ diagonal matrix with entries of $z_i,\ i = 1, \ldots, n$. The abbreviation "s.t." stands for "subject to".

2 LCP Solution as an AVE

We begin with the following simple lemma.

LEMMA 2.1. *The LCP (1.1) is equivalent to the AVE (1.2).*

Proof Let AVE (1.2) hold. If $(M - I)_i z + q_i \geq 0$ then it follows from (1.2) that $z_i = 0,\ M_i z + q_i \geq 0$, and hence $z_i(M_i z + q_i) = 0$. If on the other hand, $(M - I)_i z + q_i \leq 0$ then it follows from (1.2) that $M_i z + q_i = 0,\ z_i \geq 0$, and hence again $z_i(M_i z + q_i) = 0$. Consequently, $Mz + q \geq 0, z \geq 0, z'(Mz + q) = 0$ and we have a solution to the LCP (1.1).

Conversely, let the LCP (1.1) hold. If $z_i = 0,\ M_i z + q_i \geq 0$, then:

$$(M + I)_i z + q_i = (M - I)_i z + q_i = |(M - I)_i z + q_i|.$$

On the other hand if $z_i \geq 0,\ M_i z + q_i = 0$ then:

$$(M + I)_i z + q_i = -(M - I)_i z - q_i = |(M - I)_i z + q_i|.$$

Hence AVE (1.2) holds. $\square$

Our approach consists of representing AVE (1.2) as minimizing the difference between the two terms of the inequality:

$$(M + I)z + q \geq |(M - I)z + q|,$$

which is equivalent to

$$(M + I)z + q \geq (M - I)z + q \geq -(M + I)z - q,$$

which in turn is equivalent to:

$$z \geq 0, Mz \geq -q.$$

We then minimize the difference $(M + I)z + q - |(M - I)z + q|$ as the following concave minimization problem:

$$\min_z e'((M + I)z + q - |(M - I)z + q|) \text{ s.t. } Mz \geq -q,\ z \geq 0. \tag{2.3}$$

We immediately note for this concave minimization problem:

$$\min_{z \in Z} f(z), \tag{2.4}$$

where:

$$f(z) = e'((M + I)z + q - |(M - I)z + q|), \text{ and } Z = \{z \mid Mz \geq -q,\ z \geq 0\}, \tag{2.5}$$

that $f: \mathbb{R}^n \to \mathbb{R}$ is a concave function on $\mathbb{R}^n$ and Z is a polyhedral set that does not contain lines going to infinity in both directions. For such a problem, since f is bounded below on Z, by zero here, it follows that it has a vertex solution [9, Corollary 32.3.4]. We now state and establish finite termination of a stepless successive linearization algorithm for solving the concave minimization problem (2.5).

ALGORITHM 2.2. **Successive Linearization Algorithm (SLA)** *Start with* $z^0 = 0 \in \mathbb{R}^n$. *Having* z^i *determine* z^{i+1} *as follows:*

$$z^{i+1} \in \arg \text{vertex} \min_{z \in Z} \partial f(z^i)(z - z^i). \tag{2.6}$$

Stop if $z^i \in Z$ *and* $\partial f(z^i)(z^{i+1} - z^i) = 0$.

By invoking [5, Theorem 3] we have the following finite termination result for the SLA Algorithm (2.2).

PROPOSITION 2.3. **Finite Termination of SLA 2.2** *The SLA 2.2 generates a finite sequence of feasible iterates* $\{z^1, z^2, \ldots, z^{\bar{i}}\}$ *of strictly decreasing objective function values;* $f(z^1) > f(z^2) > \ldots > f(z^{\bar{i}})$, *such that* $z^{\bar{i}}$ *satisfies the minimum principle necessary optimality condition:*

$$\partial f(z^{\bar{i}})(z - z^{\bar{i}}) \geq 0, \ \ \forall z \in Z. \tag{2.7}$$

We turn now to the specific application of the SLA 2.2 to the LCP (1.1).

3 LCP Solution via Linear Programming

We begin by stating our successive linear programming algorithm as follows.

ALGORITHM 3.1. **LCP SLA** *Choose a termination tolerance (typically* $tol = 10^{-8}$*) and a maximum number of iterations itmax (typically* $itmax = 10$*).*

(I) Initialize the algorithm by by choosing $z^0 = 0$ *and set iteration number* $i = 0$.

(II) While $\|diag(z^i)(Mz^i + q)\|_\infty > tol$ *and* $i < itmax$ *perform the following two steps.*

(III) Solve the following linear programming problem which is a linearization of(2.3) around z^i:

$$\min_z e'((M + I)z + q - diag(sign((M - I))z^i + q))(M - I)(z - z^i)) \text{ s.t. } Mz \geq -q, \ z \geq 0. \tag{3.8}$$

(IV) Set $i = i + 1$ *and go to Step (II).*

It follows immediately by Proposition 2.3 that the LCP SLA Algorithm 3.1 terminates at in a finite number of steps at a point satisfying the minimum principle necessary optimality condition for our concave minimization problem (2.3) which is not guaranteed to be an LCP (1.1) solution. However, computationally this appears to be the case as demonstrated in the next section where exact solutions are obtained for 100% of 500 consecutively generated random solvable LCP problems to an accuracy of 10^{-8}.

4 Computational Results

We implemented our algorithm by solving 500 solvable random instances of the linear complementarity problem (1.1) consecutively generated. Elements of the matrix M were random numbers picked from a uniform distribution in the interval $[-5, 5]$. A solution z with random components chosen from the interval [0,5] was generated with approximately half of its components being zero. Finally the vector q was generated such that the generated z solves the LCP (1.1). All computations were performed on 4 Gigabyte machine with a 3159MHz CPU running amd64-rhe16 Linux. We utilized the CPLEX linear programming code [4] within MATLAB [6] to solve our linear programs.

Of the 500 test problems, 100% were solved exactly to an ∞-norm tolerance of 10^{-8}. The maximum number of iterations was set at 10 for $n = 10, 50, 100, 500, 1000$. The computational results are summarized in Table 1. It is interesting to note that 100% solution rate was achieved among the

500 problems described in Table 1 even though Proposition 2.3 does not guarantee a solution of LCP (1.1) by Algorithm 2.2. One possible justification for our solution rate is that all our test problems have a solution and it is not a simple matter to generate an LCP (1.1) that has *no* solution, to test our algorithm on, nor can our algorithm determine with any certainty that no solution exists for such a problem.

Problem Size n	Number of LCPs out of 100 Solved with ∞-norm Error $\leq tol = 10^{-8}$	Average # of Iterations per Problem	Time in Seconds for Solving 100 LCPs
10	100	1.98	0.1112
50	100	2.00	0.4449
100	100	2.00	2.5863
500	100	2.00	352.11
1,000	100	2.00	3991.1

Table 1: Computational Results for 500 Randomly Generated Consecutive LCPs

5 Conclusion and Outlook

We have proposed a finite concave-minimization-based linear programming formulation for solving the NP-hard linear complementarity problem. The method consists of solving a succession of linear programs. In 100% of 500 consecutive instances of solvable random test problems, the proposed algorithm solved the problem to an accuracy of 10^{-8}. Possible future work may consist of precise sufficient conditions under which the proposed formulation and solution method is guaranteed to terminate in a finite number of steps.

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