Multiresolution analysis by infinitely differentiable compactly supported functions

N. Dyn

School of of Mathematical Sciences
Tel-Aviv University
Tel-Aviv, Israel

A. Ron

Computer Sciences Department
University of Wisconsin-Madison
Madison, Wisconsin 53706, USA

September 1992

ABSTRACT

The paper is concerned with the introduction and study of multiresolution analysis based on the up function, which is an infinitely differentiable function supported on $[0, 2]$. Such analysis is, necessarily, nonstationary. It is shown that the approximation orders associated with the corresponding spaces are spectral, thus making the spaces attractive for the approximation of very smooth functions.

AMS (MOS) Subject Classifications: 41A25 41A30 46C99 39B99

Key Words: up function, shift-invariant spaces, wavelets, multiresolution analysis, spectral approximation orders, stability.

Supported in part by the Israel-U.S. Binational Science Foundation (grant 9000220), by the U.S. Army (Contract DAAL03-G-90-0090), and by the National Science Foundation (grants DMS-9000053, DMS-9102857).

Multiresolution analysis by infinitely differentiable compactly supported functions

N. Dyn & A. Ron

1. Introduction

Multiresolution analysis based on a compactly supported refinable function is limited to generators with a finite degree of smoothness. In this paper we discuss multiresolution analysis with C^∞-compactly supported generators. This is possible if the generator of each space is related to the generator of the next finer space by a mask whose support grows linearly with the resolution of the space. We consider here a particular instance of such analysis based on the up function of Rvachev [Rv], defined as follows.

Let

$$\chi_k(x) := \begin{cases} 2^k, & 0 \le x \le 2^{-k}, \\ 0, & \text{otherwise,} \end{cases}$$

and let σ denote the infinite convolution product

$$\sigma := \chi_1 * \chi_2 * \ldots \tag{1.1}$$

Then the up function, denoted here by ϕ_0, is given by

$$\phi_0 = \chi_0 * \sigma.$$

It follows from its definition that the up function is supported on the interval $[0,2]$ and is infinitely smooth. Approximation properties of this function can be found in [Rv]. In [DDL] it is shown that the up function can be obtained as a limit of a non-stationary subdivision scheme, which employs at a level k the mask of the stationary scheme that corresponds to the B-spline of degree k. As observed in [DDL], this is equivalent to the existence of a family of functions (ϕ_k) satisfying an infinite system of functional equations relating each ϕ_k to (appropriately scaled) shifts of ϕ_{k+1} in terms of the above mentioned mask.

In Section 2 we introduce the relevant functions $(\phi_k)_{k=0}^\infty$ (the first of which is the above mentioned up function), and use them to define the ladder of spaces $S_0 \subset S_1 \subset S_2 \subset \ldots$ with each S_k being the "span" of the $2^{-k}\mathbb{Z}$-shifts of the corresponding ϕ_k. The resolution obtained in this way is **nonstationary** in the sense that ϕ_k is not a dilate of its predecessor. We provide the wavelet decomposition, based on the general theory of nonstationary multiresolution analysis of [BDR2]. We discuss the stability issue and show that the generator we choose for each wavelet space is stable (and even linearly independent), but that the L_2-stability bounds blow up with k at the rate (no faster than) $(\pi/2)^{3k}$. On the other hand we show in Section 3, using the general theory in [BDR1], that the least square approximation from (S_k) is spectral, namely that, for any $r \ge 0$, the $L_2(\mathbb{R})$-error of best approximation to $f \in W_2^r$ from S_k is $o(2^{-rk})$. This means that high resolution of a very smooth f can be achieved for a relatively small k, and in such a case the difficulty of the growth of the stability constants may be less of a problem.

In the paper, we use, for a compactly supported function $f : \mathbb{R} \to \mathbb{C}$, and a function g defined (at least) on $2^{-k}\mathbb{Z}$, the notation

$$f *_k' g := \sum_{j \in 2^{-k}\mathbb{Z}} f(\cdot - j) g(j). \tag{1.2}$$

2. Wavelet decompositions

The up function provides an interesting example of wavelet decompositions via multiresolution. A general discussion of these topics can be found in [BDR2], and is certainly beyond the scope of this paper.

A **multiresolution** begins with a sequence $(\phi_k)_{k\in\mathbb{Z}_+} \subset L_2(\mathbb{R})$. For each k, one denotes by S_k the smallest closed L_2-subspace that contains all the functions

$$\phi_k(\cdot - j), \quad j \in 2^{-k}\mathbb{Z}. \tag{2.1}$$

The nestedness assumption

$$S_k \subset S_{k+1}, \quad k \in \mathbb{Z}_+ \tag{2.2}$$

is in the essence of the process. With (2.2) in hand, one defines the wavelet space W_k to be the orthogonal complement of S_k in S_{k+1},

$$W_k := S_{k+1} \ominus S_k. \tag{2.3}$$

Under mild conditions on the sequence (ϕ_k) (cf. section 4 of [BDR2]; these conditions are always satisfied for compactly supported (ϕ_k), which will be the case here) the wavelet spaces provide an orthogonal decomposition of $L_2(\mathbb{R}) \ominus S_0$, and the subsequent task is then to find efficient (more precisely stable) methods for computing the orthogonal projection $P_k f$ of a given $f \in L_2(\mathbb{R})$ on each of the wavelet spaces. The attraction in these decompositions is that (in many examples) the information on f recorded by $P_k f$ is considered to be "finer" as k increases.

In the original formulation of multiresolution, [Ma], [Me], it was assumed that the ladder (S_k) is **stationary**, namely, that each S_{k+1} is the 2-dilate of S_k. Analysis of nonstationary decompositions can be found in [BDR2], with the guiding example there being exponential B-splines. The wavelet decompositions that correspond to the up function are nonstationary as well, still they form a different variant in this class. We will elaborate on that point in the sequel.

Our sequence $(\phi_k)_k$ can be defined as follows. First, we recall the definition of σ given in (1.1). With B_k the cardinal B-spline of degree k (i.e, with integer breakpoints and with support $[0, k+1]$), we choose

$$\phi_k := (B_k * \sigma)(2^k \cdot). \tag{2.4}$$

Note that $\operatorname{supp}\phi_k = [0,(k+2)/2^k]$. The spaces (S_k) are defined as in the beginning of this section, with respect to the present choice of (ϕ_k). Our discussion here is developed in two steps. First, we will observe below that the spaces (S_k) satisfy the nestedness assumption (2.2). Knowing therefore that the corresponding wavelets spaces $(W_k)_{k\in\mathbb{Z}}$ are well-defined, we will then consider the problem of finding stable generators for the associated wavelet spaces.

Since

$$(\chi_k * f)(2\cdot) = \chi_{k+1} * (f(2\cdot)),$$

and

$$B_k = \underbrace{\chi_0 * ... * \chi_0}_{(k+1)-\text{times}},$$

we find that

$$\phi_k := \underbrace{\chi_k * ... * \chi_k}_{(k+1)-\text{times}} * \chi_{k+1} * ...$$

Substituting $k-1$ for k we get

$$\phi_{k-1} = \underbrace{\chi_{k-1} * ... * \chi_{k-1}}_{k-\text{times}} * \chi_k * \chi_{k+1} * ...$$

Therefore, **a refinement equation** that expresses ϕ_{k-1} as a linear combination of the translates of ϕ_k will be the same as the one that connects the splines

$$\underbrace{\chi_{k-1} * ... * \chi_{k-1}}_{k-\text{times}}, \quad \text{and} \quad \underbrace{\chi_k * ... * \chi_k}_{k-\text{times}}.$$

These splines are (up to the factors 2^{k-1} and 2^k respectively) scales of the B-spline B_{k-1}, and the refinement equation becomes identical to the well-known one for B-splines. Indeed, the solution A_k for the convolution equation

$$\phi_{k-1} = \phi_k *'_k A_k$$

is the k-fold convolution product of the sequence

$$A_k^0(j) := \begin{cases} 1/2, & j = 0, 2^{-k}, \\ 0, & j \in 2^{-k}(\mathbb{Z}\backslash\{0,1\}), \end{cases}$$

that solves the equation

$$\chi_{k-1} = \chi_k *'_k A_k^0.$$

Its Fourier transform has the form

$$\widehat{A}_k(w) = (\frac{1+e^{-iw/2^k}}{2})^k.$$

To conclude, in terms of Fourier transforms we obtained the following refinement equation:

Corollary 2.5.

$$\widehat{\phi}_{k-1} = \widehat{A}_k \widehat{\phi}_k.$$

Since A_k is finitely supported, the corollary shows that ϕ_{k-1} can be expressed as a finite linear combination of $2^{-k}\mathbb{Z}$-shifts of ϕ_k, thereby proving the required nestedness property. Consequently, the corresponding wavelets spaces are well-defined. Note that, while $(\phi_k)_k$ satisfies the same refinement equations that are being satisfied by B-splines, the degree of the associated B-spline changes with k. In particular, the size of the support of the mask sequence A_k grows linearly with k. In contrast, the nonstationary decompositions associated with exponential B-splines (cf. [BDR2: §6],[DL]) employ masks with uniformly bounded support.

Corollary 2.5 allows us to apply standard wavelet techniques (cf. [CW], [JM], [BDR2]). In particular, it is known [BDR2] that the function ψ_k defined by

$$\widehat{\psi}_{k-1}(w) := e^{-iw/2^k}\overline{\widehat{A}_k(w+2^k\pi)}\tau_k(w+2^k\pi)\widehat{\phi}_k(w), \tag{2.6}$$

with

$$\tau_k(w) := \sum_{j\in 2\pi 2^k \mathbb{Z}} |\widehat{\phi}_k(w+j)|^2 \tag{2.7}$$

generates W_k in the sense that the $2^{-k}\mathbb{Z}$-shifts of ψ_k are fundamental in W_k. A standard application of Poisson's summation formula yields that τ_k is a trigonometric polynomial with frequencies in $(2^{-k}\mathbb{Z}) \cap [-(k+1)/2^k, (k+1)/2^k]$. Some straightforward computation then implies that ψ_{k-1} is supported in an interval of length $(k+1)/2^{k-2}$ which is exactly twice the size of the support of ϕ_{k-1}. Since ψ_{k-1} is expressed as a finite linear combination of the shifts of the infinitely differentiable ϕ_k, we conclude that ψ_{k-1} is infinitely differentiable, too.

Now, we turn our attention to the *stability* question. The generator ψ_k is called **stable** if the restriction R_k of $\psi_k *'_k$ to $\ell_2(\mathbb{Z})$ is well-defined, bounded and boundedly invertible. Since the decomposition here is nonstationary, it is also important to make sure that the norms $\|R_k\|$ and $\|R_k^{-1}\|$ are bounded independently of k (cf. [JM] and [BDR3] for detailed discussion of the stability problem).

It is known (cf. section 5 of [BDR2] and especially Remark 5.8 there) that ψ_k is a stable generator of W_k if each $\phi_{k'}$ is a stable generator of $S_{k'}$, $k' \in \mathbb{Z}_+$, and further, $\|R_k\|$ and $\|R_k^{-1}\|$ are bounded by rational expressions in $\|T_{k'}\|$, $\|T_{k'}^{-1}\|$, $k' = k, k+1$, with T_k being the restriction to $\ell_2(\mathbb{Z})$ of $\phi_k *'_k$, and with the rational expressions being independent of k:

Proposition 2.8. *ψ_k in (2.6) is a stable generator of W_k if $\phi_{k'}$, $k' = k, k+1$, is a stable generator of $S_{k'}$. Further, the stability constants associated with $(\psi_k)_k$ are uniformly bounded if the same holds for the stability constants of $(\phi_k)_k$, since*

$$\|R_k\|\|R_k^{-1}\| \leq \text{const}\|T_k\|\|T_k^{-1}\|(\|T_{k+1}\|\|T_{k+1}^{-1}\|)^2.$$

We prove that each ϕ_k is a stable generator of S_k with the aid of the following well-known result (cf. [SF], [DM], [JM] and [BDR3]).

Result 2.9. *Let ϕ be a compactly supported $L_2(\mathbb{R})$-function. Then the $2^{-k}\mathbb{Z}$-shifts of ϕ are L_2-stable if and only if for every $\theta \in \mathbb{R}$, there exists $j \in 2^{k+1}\pi\mathbb{Z}$ such that $\widehat{\phi}(\theta + j) \neq 0$. In other words, the $2^{-k}\mathbb{Z}$-shifts of ϕ are stable if and only if $\widehat{\phi}$ does not have a $2^{k+1}\pi\mathbb{Z}$-periodic zero.*

We show below that the entire function $\widehat{\phi}_k$ has no $2^{k+1}\pi\mathbb{Z}$-periodic zero in the *complex* domain $\mathbb{C}$. This property is known to be equivalent to $\phi_k *'_k$ being *injective*, a property which is usually referred to as the **linear independence** of the $2^{-k}\mathbb{Z}$-shifts of ϕ_k (cf. [Ro] for details). In view of Result 2.9, this will certainly imply that ϕ_k is a stable generator of S_k.

Corollary 2.10. *The $2^{-k}\mathbb{Z}$-shifts of ϕ_k are linearly independent, hence form a stable basis for the space S_k that they generate. However, the $2^{-k}\mathbb{Z}$-shifts of $\phi_{k'}$ are not L_2-stable, whenever $k' < k$.*

Proof: By Corollary 2.5, we have

$$\widehat{\phi}_{k'} = \widehat{A}_{k'+1}\widehat{\phi}_{k'+1}.$$

Since $\widehat{A}_{k'+1}$ is $2^{k'+2}\pi$-periodic and $k' < k$, it is also $2^{k+1}\pi$-periodic, hence, in view of Result 2.9, the stability of the $2^{-k}\mathbb{Z}$-shifts of $\phi_{k'}$, $k' < k$, forces $A_{k'+1}$ to have no zeros (on $\mathbb{R}$). However, $\widehat{A}_{k'+1}$ vanishes at $2^{k'+1}\pi$, and this proves the second statement of the corollary.

For the linear independence claim, we first remark that basic convergence criteria (cf. e.g. Theorem 15.4 of [Ru]) show that $\widehat{\phi}_k$ vanishes at a point (if and) only if one of its factors $\widehat{\chi}_j$ vanishes there. Secondly, we observe that since $\widehat{\chi}_j(w) = 2^j \int_0^{2^{-j}} e^{-iwt}\,dt$,

$$\widehat{\chi}_j(w) = 0 \quad \iff \quad w \in 2^{j+1}\pi\mathbb{Z}\backslash 0.$$

In particular, for $j < j'$, $\widehat{\chi}_j(w) = 0$ if $\widehat{\chi}_{j'}(w) = 0$. Since $\widehat{\phi}_k$ is the product of factors of the form $\widehat{\chi}_j$, for $j \geq k$, we conclude that the (complex) zeros of $\widehat{\phi}_k$ are identical with these of $\widehat{\chi}_k$. Since $\operatorname{supp}\chi_k = [0, 2^{-k}]$, the $2^{-k}\mathbb{Z}$-shifts of this function are trivially linearly independent, hence $\widehat{\chi}_k$ cannot have a $2^{k+1}\pi\mathbb{Z}$-periodic zero. Therefore, $\widehat{\phi}_k$ does not have such a zero, and consequently its $2^{-k}\mathbb{Z}$-shifts are linearly independent as well. □

While the $2^{-k}\mathbb{Z}$-shifts of ϕ_k are linearly independent hence stable, the stability constants are not uniformly bounded. This assertion is based on the next result, which also provides some estimate on the growth of these constants as $k \to \infty$. A referee's suggestion helped us in improving statement (a) of this result. The referee also made the point that such estimates can be found in the spline theory literature (since they are needed for the estimation of the norm of the odd degree cardinal spline interpolant).

Proposition 2.11. *Let T_k denote the restriction of $\phi_k *'_k$ to $\ell_2(\mathbb{Z})$. Then, for some positive constant c, C*

(a) $\|T_k\|\|T_k^{-1}\| \geq c\left(\frac{\pi}{2}\right)^k$.

(b) $\|T_k\|\|T_k^{-1}\| \leq C\left(\frac{\pi}{2}\right)^k$.

Proof: We recall ([Me]) that

$$C_k := \|T_k\|^2\|T_k^{-1}\|^2 = \frac{\sup_{w\in\mathbb{R}} K_k(w)}{\inf_{w\in\mathbb{R}} K_k(w)}, \tag{2.12}$$

where

$$K_k(w) := \sum_{\alpha \in 2^{k+1}\pi \mathbb{Z}} |\widehat{\phi}_k(w+\alpha)|^2 = \sum_{\alpha \in 2\pi \mathbb{Z}} |(\widehat{B}_k \widehat{\sigma})(w+\alpha)|^2.$$

In order to estimate C_k from below (as required for the proof of (a)), we note first that, since, for each k, $\widehat{B}_k(0) = 1$, and also $\widehat{\sigma}(0) = 1$, we have that $\|K_k\|_\infty \geq 1$. This takes care of the numerator in (2.12). As for the denominator, we observe that

$$|\widehat{B}_k(w)| = \Big(\frac{|e^{-iw} - 1|}{|w|}\Big)^k,$$

and that

$$\frac{|e^{-iw} - 1|}{|w|} < 1 \quad \text{on } \mathbb{R}\backslash 0. \tag{2.13}$$

This implies, in particular, that $\|\widehat{\sigma}\|_\infty \leq 1$, hence that

$$K_k(w) \leq \sum_{\alpha \in 2\pi \mathbb{Z}} |\widehat{B_k}(w+\alpha)|^2 =: \widetilde{K}_k(w).$$

Thus, the value $\widetilde{K}_k(\pi)$ bounds $\inf_w K_k(w)$ from above. The number $\widetilde{K}_k(\pi)$ can be estimated as follows:

$$\widetilde{K}_k(\pi) = \sum_{\alpha \in 2\pi \mathbb{Z}} \frac{2^{2k}}{(\pi + 2\pi\alpha)^{2k}} = 2\Big(\frac{2}{\pi}\Big)^{2k} \sum_{j=0}^{\infty} (1+2j)^{-2k}.$$

The sum $\sum_{j=0}^{\infty}(1+2j)^{-2k}$ clearly decreases monotonely to 1, hence, in particular, is uniformly bounded in k. In summary, we have obtained the estimate

$$C_k \geq \frac{1}{\widetilde{K}_k(\pi)} \geq c_k \Big(\frac{2}{\pi}\Big)^{2k},$$

with $(c_k)_k \to 2$ as $k \to \infty$. This proves (a).

To prove (b), we first conclude from (2.13) that $(K_k)_k$ is a non-increasing sequence. This implies the estimate

$$\|K_k\|_\infty \leq \|K_0\|_\infty = \text{const}.$$

In addition, for $w \in [-\pi, \pi]$, we estimate the sum that defines K_k below by its 0-term:

$$K_k(w) \geq |\widehat{\sigma}(w)|^2 \left(\frac{|e^{-iw} - 1|}{|w|}\right)^{2k} \geq \text{const} \left(\frac{2}{\pi}\right)^{2k},$$

where, in the last inequality, we have used the facts that (i): $\widehat{\sigma}$ vanishes nowhere on $[-\pi, \pi]$, (ii): the minimal value of $\frac{|e^{-iw}-1|}{|w|}$ (assumed at $w = \pi$) is $2/\pi$.

Claim (b) is then obtained by combining the estimates in the two last displays. □

From Proposition 2.8, we conclude that the stability constants associated with the wavelets $(\psi_k)_k$ grow no faster than $O((\pi/2)^{3k})$.

3. Approximation Orders

As a natural continuation of the previous discussion, we consider L_2-approximation orders of the spaces (S_k) generated by (ϕ_k) of (2.4). Given $r > 0$, let W_2^r be the usual potential space

$$W_2^r := \{f \in L_2(\mathbb{R}) : \ \|f\|_{W_2^r} := (2\pi)^{-1/2}\|(1+|\cdot|)^r \widehat{f}\|_{L_2(\mathbb{R})} < \infty\}. \tag{3.1}$$

We say that $(S_k)_k$ (or, $(\phi_k)_k$) has approximation order $r > 0$ (in the L_2-norm) if

$$E_k(f) := \mathrm{dist}_{L_2(\mathbb{R})}(f, S_k) \tag{3.2}$$

satisfies

$$E_k(f) \le \mathrm{const}_r \, \|f\|_{W_2^r} 2^{-kr}, \quad \forall f \in W_2^r, \ \forall \text{ sufficiently large } k. \tag{3.3}$$

Here, "sufficiently large" may depend on r, but not on f. If, in addition to the above,

$$2^{kr} E_k(f) \to 0, \quad \text{as } k \to \infty, \ \forall f \in W_2^r, \tag{3.4}$$

we say that (S_k) has **density order** r**.** The notion of density orders extends to $r = 0$; in such a case the definition is reduced to the requirement that

$$E_k(f) \to 0, \quad \forall f \in L_2(\mathbb{R}).$$

Although it is not obvious from the definition, $(\phi_k)_k$ has all approximation orders $\le j$ whenever it has the approximation order j. We will show that the approximation by $(S_k)_k$ is **spectral** which means, by definition, that $(\phi_k)_k$ has all positive approximation orders.

Our precise result is as follows:

Theorem 3.5. *The sequence of spaces (S_k) has density order r for every $r \ge 0$.*

The fact that all *approximation* orders are obtained is plausible, since each ϕ_k is obtained by smoothing $B_k(2^k\cdot)$, and for a fixed j, the functions $(B_j(2^k\cdot))_k$ are well-known to have approximation order $j+1$. However, the fact that we obtain even all *density* orders seems to be less expected. Furthermore, in what follows we show that for a very smooth function f (whose Fourier transform decays exponentially), $E_k(f)$ decays exponentially, as well.

A general comprehensive discussion of L_2-approximation orders and density orders for shift invariant spaces is given in [BDR1]. Instead of deriving Theorem 3.5 from those results, we will apply the approach taken there to our special (and much simpler) case. This will result at tighter estimates for $E_k(f)$. Similar results in other p-norms are available as well. For example, spectral approximation in the uniform norm can be proved by employing the results of [BR].

Our analysis of the approximation orders goes as follows: let $I = [-t, t] \subset [-\pi, \pi]$. In order to estimate $E_k(f)$, we localize f on the Fourier domain. Precisely, we multiply $\widehat{f}$ by the characteristic function η_k of the interval $2^k I$, to obtain the function

$$g_k := (\eta_k \widehat{f})^\vee, \tag{3.6}$$

(with $f^\vee$ the inverse Fourier transform of f) and use the straightforward bound

$$E_k(f) \leq E_k(g_k) + \|((1-\eta_k)\widehat{f})^\vee\|. \tag{3.7}$$

We refer to the first term in the above sum as the **approximation error**, or the **projection error**, and to the second term as the **truncation error**.

The decay rate of the truncation error is clearly independent of (ϕ_k), and depends on the smoothness class of f. In particular, the following can be easily proved (cf. [BDR1]):

Lemma 3.8. *Let $I := [-t, t]$ be some neighborhood of the origin, and let η_k be the characteristic function of $2^k I$. Let $f \in W_2^r$, $r \geq 0$. Then*

$$\|((1-\eta_k)\widehat{f})^\vee\|_{L_2(\mathbb{R})} \leq t^{-r} 2^{-kr} \|f\|_{W_2^r} \varepsilon_k(f, t),$$

where $0 \leq \varepsilon_k(f,t) \leq 1$ and converges to 0 as k tends to ∞.

In view of Lemma 3.8, a proof of Theorem 3.5 requires the study of the behaviour of the projection error. To estimate this, we employ the following result from [BDR1]. In what follows we use, for $f \in L_2(\mathbb{R})$, the notation $S(f)$ to denote the smallest closed subspace of $L_2(\mathbb{R})$ that contains all the shifts of f.

Result 3.9. *Let ξ be a function in L_2 and let $g \in L_2$ be a function whose Fourier transform $\widehat{g}$ is supported in $I \subset [-\pi, \pi]$. Let Pg be the orthogonal projection of g on $S(\xi)$. Then*

$$\|g - Pg\|_{L_2(\mathbb{R})} = (2\pi)^{-1/2} \|\widehat{g} \Lambda_\xi\|_{L_2(I)},$$

where

$${\Lambda_\xi}^2 := 1 - \frac{|\widehat{\xi}|^2}{\sum_{j \in 2\pi\mathbb{Z}} |\widehat{\xi}(\cdot + j)|^2}.$$

We intend to apply the last result to the function $g := g_k(2^{-k}\cdot)$, with g_k as in (3.6). We first note that, for any $f \in L_2(\mathbb{R})$, by dilating,

$$E_k(f) = 2^{-k/2} \operatorname{dist}(f(2^{-k}\cdot), S(\phi_k(2^{-k}\cdot))).$$

Thus, we can use Result 3.9 with respect to $\xi := \phi_k(2^{-k}\cdot) = B_k * \sigma$. We estimate Λ_ξ as follows. We first denote

$$\mathrm{M}_\xi^2 := \sum_{j \in 2\pi\mathbb{Z}\backslash 0} |\widehat{\xi}(\cdot + j)|^2.$$

Then

$$\Lambda_\xi^2 = \frac{\mathrm{M}_\xi^2}{|\widehat{\xi}|^2 + \mathrm{M}_\xi^2} \leq \frac{\mathrm{M}_\xi^2}{|\widehat{\xi}|^2}.$$

Further,

$$|\widehat{\xi}(w)| = |\widehat{B}_k(w)||\widehat{\sigma}(w)| = |\tau_k(w)||w|^{-k}|\widehat{\sigma}(w)|,$$

with τ_k a 2π-periodic trigonometric polynomial. We conclude that, for $w \in I$,

$$\frac{\mathrm{M}_\xi(w)^2}{|\widehat{\xi}(w)|^2} = \sum_{j\in 2\pi\mathbb{Z}\backslash 0} \frac{|\widehat{\xi}(w+j)|^2}{|\widehat{\xi}(w)|^2} = \sum_{j\in 2\pi\mathbb{Z}\backslash 0} |w/(w+j)|^{2k}|\widehat{\sigma}(w+j)/\widehat{\sigma}(w)|^2 \le$$

$$\frac{|w|^{2k}}{|\widehat{\sigma}(w)|^2} \sup_{j\in 2\pi\mathbb{Z}\backslash 0} \|(\cdot+j)^{-2k}\|_{L_\infty(I)} \sum_{j\in 2\pi\mathbb{Z}\backslash 0} |\widehat{\sigma}(w+j)|^2.$$

Since $\sigma \in C^\infty(\mathbb{R})$, $\widehat{\sigma}$ is rapidly decaying, and therefore

$$\sum_{j\in 2\pi\mathbb{Z}\backslash 0} |\widehat{\sigma}(w+j)|^2 =: q(w)^2$$

converges to a smooth bounded function (the boundedness can be proved as follows: if we add to the sum the summand for $j = 0$, we get a periodic function which is bounded due to its continuity. Since each summand, including the $j = 0$ one, is clearly bounded, it follows that the above sum is, too).

Since $w \in I = [-t, t]$, $\inf_{j\in 2\pi\mathbb{Z}\backslash 0} |w + j| = 2\pi - t$, and we obtain the following estimate

$$\Lambda_\xi(w) \le (2\pi - t)^{-k}|w|^k \,\|q/\widehat{\sigma}\|_{L_\infty(I)} =: \mathrm{const}\,(2\pi - t)^{-k}|w|^k. \tag{3.10}$$

Consequently, since the Fourier transforms of $f(2^{-k}\cdot)$ and $g_k(2^{-k}\cdot)$ coincide on I, we have for $k \ge r$

$$\begin{aligned}(2\pi)^{1/2}E_k(g_k) =& 2^{-k/2}\|\widehat{f(2^{-k}\cdot)}\Lambda_\xi\|_{L_2(I)}\\ \le& \|\,|\cdot|^{-r}\Lambda_\xi\|_{L_\infty(I)}\, 2^{k/2}\|\,|\cdot|^r\widehat{f}(2^k\cdot)\|_{L_2(I)}\\ \le& \mathrm{const}\,(2\pi-t)^{-k}t^{k-r}\|\,|2^{-k}\cdot|^r\widehat{f}(\cdot)\|_{L_2(\mathbb{R})}\\ \le& \mathrm{const}\;(2\pi-t)^{-k}t^{k-r}2^{-kr}\|f\|_{W_2^r}.\end{aligned}$$

Theorem 3.5 now follows when combining the last estimate (say, with $t := 1$) for the projection error, with the estimate for the truncation error from Lemma 3.8.

Finally, for very smooth functions, better rates can be derived. First, upon substituting $r = k$, $t = \pi$ in the above estimate and in Lemma 3.8, we obtain:

Corollary 3.11. *If $f \in W_2^k$ for some k then*

$$E_k(f) \le \mathrm{const}\pi^{-k}\|f\|_{W_2^k}2^{-k^2}.$$

Concrete improvements of the rates of Theorem 3.5 require knowledge on the rate of growth of $(\|f\|_{W_2^k})_{k\in\mathbb{Z}_+}$. A typical example follows.

Example 3.12. Assume that f is so smooth so that its Fourier transform decays exponentially at ∞. Precisely, assume that $|\widehat{f}(w)| \le \mathrm{const}e^{-\alpha|w|}$, for some positive α. Then $\|f\|_{W_2^k} = O((k+1)!\alpha^{-k})$, and therefore we conclude from the last theorem that

$$E_k(f) \le \mathrm{const}\,(k+1)!(\pi\alpha)^{-k}2^{-k^2},$$

and therefore the error decays in this case exponentially in 2^k.

References

[BDR1] C. de Boor, R.A. DeVore and A. Ron, Approximation from shift-invariant subspaces of $L_2(\mathrm{I\!R}^d)$, CMS TSR #92-2, University of Wisconsin-Madison, July 1991, Trans. Amer. Math. Soc., to appear.

[BDR2] C. de Boor, R.A. DeVore and A. Ron, On the construction of multivariate (pre)wavelets, Constructive Approximation, Special Issue on Wavelets **9** (1993), 123–166.

[BDR3] C. de Boor, R. DeVore and A. Ron, The structure of shift invariant spaces and applications to approximation theory, CMS-TSR 92–08, University of Wisconsin-Madison, February 1992, J. Functional Anal., to appear.

[BR] C. de Boor and A. Ron, Fourier analysis of approximation power of principal shift-invariant spaces, Constr. Approx. **8** (1992), 427–462.

[CW] C.K. Chui and J.Z. Wang, A general framework for compactly supported splines and wavelets, J. Appprox. Theory **71 (3)** (1992), 263–304.

[DDL] N. Dyn, G. Derfel and D. Levin, Generalized functional equations and subdivision processes, preprint, July 1992.

[DL] N. Dyn and D. Levin, Stationary and non-stationary binary subdivision schemes, in *Mathematical Methods in Computer Aided Geometric Design*, T. Lyche and L. Schumaker, eds., Academic Press, (1992), 209-216.

[DM] W. Dahmen and C.A. Micchelli, Recent progress in multivariate splines, *Approximation Theory IV*, eds. C.K. Chui, L.L. Schumaker, J. Ward. Academic Press, 1983, 27-121.

[JM] R-Q. Jia and C.A. Micchelli, Using the refinement equation for the construction of pre-wavelets II: powers of two, *Curves and Surfaces* P.J. Laurent, A. Le Méhaué, and L.L. Schumaker eds., Academic Press, New York, 1991, 209–246.

[Ma] S.G. Mallat, Multiresolution approximations and wavelet orthonormal bases of $L^2(\mathrm{I\!R})$. Trans. Amer. Math. Soc. **315**(1989), 69–87.

[Me] Y. Meyer, "Ondelettes et Opérateurs I: Ondelettes", Hermann Éditeurs, 1990

[Ro] A. Ron, A necessary and sufficient condition for the linear independence of the integer translates of a compactly supported distribution, Constructive Approx. **5** (1989), 297–308.

[Ru] W. Rudin, Real and Complex Analysis, Mc Graw-Hill, 1974.

[Rv] V.A. Rvachev, Compactly supported solutions of functional-differential equations and their applications, Russian Math. Surveys **45**:1 (1990), 87–120.

[SF] G. Strang and G. Fix, A Fourier analysis of the finite element variational method. C.I.M.E. II Ciclo 1971, in *Constructive Aspects of Functional Analysis* ed. G. Geymonat 1973, 793-840.