

# On Local Spline Approximation by Moments

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1. This note is intended to generalize the statements of [1]. Incidentally it should justify some of the steps taken in [1].

2. Let  $m$  be a positive integer,  $\pi: 0 = x_0 < x_1 < \dots < x_n = 1$  a partition of the unit interval, and denote by  $S = S_\pi$  the set of spline functions on  $[0, 1]$  of degree  $2m - 1$  with (interior) joints  $x_1, \dots, x_{n-1}$ . We wish to investigate the behavior of

$$(1) \quad \text{dist}(f, S) = \min_{s \in S} \|f - s\|_\infty,$$

for  $f \in C[0, 1]$ , as the mesh of  $\pi$ ,  $|\pi| = \max_i |x_{i+1} - x_i|$ , goes to zero. As is pointed out in [1],

$$(2) \quad \text{dist}(f, S_\pi) = O(|\pi|^k)$$

will not hold for  $k > 2m$ , except for the trivial case that  $f$  is a polynomial of degree  $\leq 2m - 1$ . It is further stated there that if  $f \in C^{2m}[0, 1]$  and if the numbers

$$(3) \quad M_\pi = \max_{|i-j|=1} (x_{i+1} - x_i)/(x_{j+1} - x_j)$$

stay bounded, then there exists  $K$  independent of  $f$  or  $\pi$  and  $s_\pi \in S_\pi$  s.t.

$$\|f(x) - s_\pi(x)\| \leq K |\pi|^{2m} \|f^{(2m)}\|_\infty, \quad \text{all } x \in [x_m, x_{n-m}].$$

It is one result of this note that in fact

$$(4) \quad \text{dist}(f, S_\pi) = O(|\pi|^{2m}),$$

for  $f \in C^{2m}[0, 1]$ , and that (4) holds even without the assumption of bounded mesh ratios  $M_\pi$ .

The argument in [1] relies on a linear approximation scheme, called local spline approximation by moments, which realizes the convergence rate  $O(|\pi|^{2m})$ . Briefly, the approximation  $P_\pi f$  to  $f$  is defined by

$$(5) \quad (P_\pi f)(x) = p(x) + \sum_{\tau} G(x, x_\tau) \int_0^1 W_\tau(t) f^{(2m)}(t) dt.$$

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