

The polynomials in the linear span of integer translates of a compactly supported function

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Abstract. Algebraic facts about the space of polynomials contained in the span of integer translates of a compactly supported function are derived and then used in a discussion of the various quasi-interpolants from that span.

0. Introduction

This note was stimulated by the recent papers [CD], [CJW], and [CL] in which the authors take a new look at the space of integer translates of box splines and, in particular, introduce and highlight the *commutator* of a locally supported pp function φ of several variables. The intent of this note is to offer alternative proofs of some of these results, and to point to some connections with earlier work (e.g., [BH], [DM83], [BJ]), but also to focus more attention on the space Π_φ of *all* polynomials contained in the span of the integer translates of the box spline (or other compactly supported) φ .

The first section collects simple algebraic facts about Π_φ and the action of the linear map

$$\varphi^{*'} : f \mapsto \sum_{j \in \mathbb{Z}^d} \varphi(\cdot - j)f(j)$$

on it.

The second section records that Π_φ is invariant under differentiation and translation, and brings yet another characterization of Π_φ , this time in terms of the Fourier transform of φ .

The final section makes use of these facts about Π_φ in a discussion of the various quasi-interpolants available.

Throughout, I will use standard multi-index notation. I find it convenient to use the special symbol $[\![\]\!]^\alpha$ for the **normalized monomial of degree α** , i.e., for the map given by the rule

$$[\![\]\!]^\alpha : \mathbb{R}^d \rightarrow \mathbb{R} : x \mapsto x^\alpha / \alpha!.$$

With this,

$$\Pi_\alpha := \text{span}([\![\]\!]^\beta)_{\beta \leq \alpha}$$

denotes the space of all polynomials of degree $\leq \alpha$, and

$$\Pi_k := \text{span}([\![\]\!]^\beta)_{|\beta| \leq k}, \quad \Pi_{<k} := \text{span}([\![\]\!]^\beta)_{|\beta| < k}, \quad \Pi := \text{span}([\![\]\!]^\beta)$$

have similarly obvious meaning.

1. The polynomials

Consider the span of integer translates of a compactly supported function φ on \mathbb{R}^d , i.e.,

$$(1.1) \quad S := S_\varphi := \{\varphi * c : c \in \mathbb{R}^{\mathbb{Z}^d}\}.$$

Here I use the convolution product notation

$$(1.2) \quad \varphi * c := \sum_{j \in \mathbb{Z}^d} \varphi(\cdot - j)c(j)$$

since there is no danger of confusion with either the continuous or the discrete convolution product. I find it convenient to use the special notation

$$(1.3) \quad \varphi^{*'} f := \varphi * f|_{\mathbb{Z}^d} = \sum_{j \in \mathbb{Z}^d} \varphi(\cdot - j)f(j)$$

in case f is a function on \mathbb{R}^d , in order to stress the semidiscrete character of this product. Further, since the restriction to \mathbb{Z}^d of a function on \mathbb{R}^d occurs often here, I will employ the abbreviation

$$f| := f|_{\mathbb{Z}^d}$$

for it.

The asymmetry in the semidiscrete convolution product (1.3) is not all that strong since, after all,

$$\varphi *' f = f *' \varphi \quad \text{on } \mathbb{Z}^d.$$

This implies, e.g., that, for $f \in \Pi$ (hence $f *' \varphi \in \Pi$),

$$\varphi *' f = f *' \varphi \Leftrightarrow \varphi *' f \in \Pi,$$

hence

$$(1.4) \quad \Pi_\varphi := \{f \in \Pi : \varphi *' f \in \Pi\} = \{f \in \Pi : \varphi *' f = f *' \varphi\}.$$

It also implies that

$$(1.5) \quad \varphi *' f = f *' \varphi \quad \text{for all } f \in S,$$

since, for $f = \varphi * c$,

$$\begin{aligned} \varphi *' f &= \varphi *' (\varphi| * c) \\ &= \varphi *' (c * \varphi|) \\ &= (\varphi * c) * \varphi| = f *' \varphi. \end{aligned}$$

As a consequence, one gets the inclusion

$$(1.6) \quad \Pi \cap S \subseteq \{f \in \Pi : \varphi *' f = f *' \varphi\} = \Pi_\varphi,$$

and the conclusion that

$$\varphi *' : f \mapsto \varphi *' f$$

maps Π_φ into $\Pi \cap S$. This implies that there must be equality throughout (1.6) as soon as the linear map

$$L := \varphi *'|_{\Pi_\varphi}$$

can be shown to be 1–1. But that is easy to do under the assumption that φ is **normalized**, i.e.,

$$\sum_{j \in \mathbb{Z}^d} \varphi(j) = 1.$$

For, under this assumption,

$$(1.7) \quad \begin{aligned} \text{for } f \in \Pi_\varphi, \quad \varphi *' f &= f \sum_{j \in \mathbb{Z}^d} \varphi(j) - \sum_{j \in \mathbb{Z}^d} (f - f(\cdot - j)) \varphi(j) \\ &\in f + \Pi_{< \deg f} \end{aligned}$$

since, for each j , $f - f(\cdot - j) \in \Pi_{< \deg f}$.

The salient facts of this discussion are gathered in the following.

Proposition 1.1. *If φ is normalized, then*

$$(1.8) \quad \begin{aligned} \Pi_\varphi : &= \{f \in \Pi : \varphi *' f \in \Pi\} = \{f \in \Pi : \varphi *' f = f *' \varphi\} \\ &= \Pi \cap S = \{f \in \Pi : \varphi *' f \in f + \Pi_{<\deg f}\}. \end{aligned}$$

Further, $L := \varphi *'|_{\Pi_\varphi}$ is onto, and

$$(1.9) \quad U := 1 - L$$

is degree-reducing. In particular,

$$(1.10) \quad L(\Pi_\varphi \cap \Pi_\alpha) = \Pi_\varphi \cap \Pi_\alpha.$$

As a consequence, $U^k = 0$ on

$$\Pi_{\varphi,k} := \Pi_\varphi \cap \Pi_{<k}.$$

Therefore

$$(1.11) \quad (L|_{\Pi_{\varphi,k}})^{-1} = (1 + U + \dots + U^{k-1})|_{\Pi_{\varphi,k}}.$$

Note that Π_φ is necessarily finite dimensional, since φ is compactly supported. Precisely, for any bounded set G , the set

$$A(G) := \{\alpha \in \mathbb{Z}^d : \varphi(\cdot - \alpha)|_G \neq 0\}$$

is finite, hence if G also has interior, then

$$\dim \Pi_\varphi = \dim \Pi_{\varphi|_G} \leq \#A(G) < \infty.$$

The sharpest bound attainable this way for a piecewise continuous φ would be

$$(1.12) \quad \dim \Pi_\varphi \leq \max_x \#A(\{x\}).$$

In any case, this implies that

$$L^{-1} = 1 + U + U^2 + \dots,$$

with the Neumann series actually finite.

The assumption that φ be normalized is no real restriction except when

$$\sum_{j \in \mathbb{Z}^d} \varphi(j) = 0.$$

In this case, (1.7) shows L to be degree-reducing, hence in particular, not invertible. Consequently, $\Pi \cap S$ may be strictly smaller than Π_φ . For example, with $\varphi = 1$ on $[-1, 0[$, $= -1$ on $[0, 1[$, and $= 0$ otherwise, $\Pi_\varphi = \Pi_1 \neq \Pi_0 = \Pi \cap S$.

2. Invariance

Denote by E the **multivariate shift**, i.e.,

$$E^\alpha f := f(\cdot + \alpha), \quad \alpha \in \mathbb{Z}^d.$$

While it is obvious that $\varphi_{*'}'$ commutes with E , hence Π_φ is invariant under E , some of the other properties of Π_φ derivable from this fact may not be as immediate.

Proposition 2.1. *The linear map $L = \varphi_{*'}'|_{\Pi_\varphi}$ commutes with differentiation, hence with translation, i.e.,*

$$(2.1) \quad LD^\alpha = D^\alpha L, \forall \alpha \in \mathbb{Z}_+^d, \quad E^y L = L E^y, \forall y \in \mathbb{R}^d.$$

Proof: Since Π_φ is a finite-dimensional polynomial subspace, there exists, for each $\alpha \in \mathbb{Z}_+^d$, a weight sequence w of finite support so that

$$(2.2) \quad D^\alpha = \sum_{\beta \in \mathbb{Z}_+^d} w(\beta) E^\beta \quad \text{on } \Pi_\varphi.$$

(For example, with l_i the Lagrange polynomials for the points $0, \dots, k := \max \deg \Pi_\varphi$, we have

$$p = \sum_{0 \leq \beta(j) \leq k} l^\beta E^\beta p(0)$$

for all $p \in \Pi_k(\mathbb{R}) \otimes \dots \otimes \Pi_k(\mathbb{R}) \supseteq \Pi_\varphi$, hence $w(\beta) := D^\alpha l^\beta(0)$, all β , would do.) Thus, $LE = EL$ implies $LD = DL$. But this finishes the proof since

$$(2.3) \quad E^y = \sum_{\alpha} [y]^\alpha D^\alpha.$$

Q.E.D.

Remark. The argument shows that any E -invariant polynomial subspace is D -invariant, hence even translation-invariant, i.e., for any linear subspace P of Π ,

$$(2.4) \quad \begin{aligned} \forall \alpha \in \mathbb{Z}^d, E^\alpha P \subseteq P &\Rightarrow \forall \alpha \in \mathbb{Z}^d, D^\alpha P \subseteq P \\ &\Rightarrow \forall y \in \mathbb{R}^d, E^y P \subseteq P. \end{aligned}$$

Corollary. Π_φ is D -invariant and translation-invariant.

As a simple consequence, consider the polynomials g_α defined in [CJW] by the recurrence

$$(2.5) \quad g_\alpha(x) := x^\alpha - \sum_{j \in \mathbb{Z}^d} \varphi(j) \sum_{\beta \neq \alpha} \binom{\alpha}{\beta} (-j)^{\alpha-\beta} g_\beta(x)$$

and then shown to satisfy

$$(2.6) \quad x^\alpha = \sum_{j \in \mathbb{Z}^d} g_\alpha(j) \varphi(x - j)$$

in case $|\alpha| < m$ and $\Pi_{< m} \subset \Pi_\varphi$. In other words, $g_\alpha = L^{-1}(\cdot)^\alpha$.

Since (2.6) is, offhand, the reason for our interest in the g_α , it would seem more direct to *define* the g_α by (2.6), i.e., to set

$$(2.7) \quad g_\alpha := L^{-1}(\cdot)^\alpha,$$

and then to verify that necessarily (2.5) holds for these g_α , as follows:

$$\begin{aligned} ()^\alpha &= L^{-1}(\varphi^*{}'()^\alpha) = L^{-1}(()^{\alpha*'}\varphi) = \sum_j \varphi(j) \sum_{\beta \leq \alpha} L^{-1}()^\beta \binom{\alpha}{\beta} (-j)^{\alpha-\beta} \\ &= g_\alpha + \sum_j \varphi(j) \sum_{\beta < \alpha} g_\beta \binom{\alpha}{\beta} (-j)^{\alpha-\beta} \end{aligned}$$

using Proposition 1 and the normalization $\sum_j \varphi(j) = 1$. This even shows the validity of (2.6) for any α for which $()^\alpha \in \Pi_\varphi$, since then also $()^\beta \in \Pi_\varphi$ for all $\beta \leq \alpha$, hence the definition $g_\beta := L^{-1}()^\beta$ makes sense for all such β .

This leaves unanswered the question of whether the two definitions, (2.5) and (2.7), are equivalent, at least for the range of α for which they both make sense. It also raises the question as to the nature of the polynomials g_α defined by (2.5) when $()^\alpha \notin \Pi_\varphi$.

To answer these, recall that the **Appell sequence** for a continuous linear functional μ on $C(\mathbb{R}^d)$ with $\mu(1) = 1$ is, by definition, the sequence (g_α) determined by the conditions

$$g_\alpha \in \Pi_\alpha, \quad \mu D^\beta g_\alpha = \delta_{\beta\alpha}.$$

There is, in fact, exactly one such sequence for given μ since the linear system

$$\mu D^\beta \left(\sum_{\gamma \leq \alpha} \prod^\gamma a_\gamma \right) = \delta_{\beta\alpha}$$

for the power coefficients (a_γ) for g_α has a unit triangular coefficient matrix. Backsubstitution therefore provides the formula

$$g_\alpha = \prod^\alpha - \sum_{\beta \neq \alpha} \mu \prod^{\alpha-\beta} g_\beta,$$

whose correctness can also be verified directly by induction on α :

$$\begin{aligned} \mu D^\gamma g_\alpha &= \mu D^\gamma \prod^\alpha - \sum_{\beta \neq \alpha} \mu \prod^{\alpha-\beta} \mu D^\gamma g_\beta \\ &= \mu \prod^{\alpha-\gamma} - \mu \prod^{\alpha-\gamma} = 0 \end{aligned}$$

for $\gamma < \alpha$, while $\mu D^\alpha g_\alpha = \mu D^\alpha \prod^\alpha = \mu(1) = 1$. With existence and uniqueness established, facts about the Appell sequence, such as symmetries which reflect those of μ , or that $D^\beta g_\alpha = g_{\alpha-\beta}$, follow immediately.

In our case, $\mu : f \mapsto \varphi^*{}'f(0)$, hence, for $\prod^\alpha \in \Pi_\varphi$,

$$\delta_{\beta\alpha} = \mu D^\beta g_\alpha = \varphi^*{}'(D^\beta g_\alpha)(0) = D^\beta(\varphi^*{}'g_\alpha)(0),$$

which, together with the fact that $\varphi^*{}'g_\alpha \in L\Pi_\alpha = \Pi_\alpha$, shows that

$$(2.6') \quad \varphi^*{}'g_\alpha = \prod^\alpha.$$

The resulting different normalization of g_α as compared with (2.5) or (2.7) avoids all those factorials.

Dahmen and Micchelli [DM83] consider the polynomial space

$$(2.8) \quad \{p \in \Pi : p(D)\widehat{\varphi} = 0 \text{ on } 2\pi\mathbb{Z}^d \setminus \{0\}\},$$

with $\widehat{\varphi}$ the Fourier transform of φ . It seems slightly more convenient to consider instead

$$\widetilde{\Pi}_\varphi := \{p \in \Pi : p(-iD)\widehat{\varphi} = 0 \text{ on } 2\pi\mathbb{Z}^d \setminus \{0\}\}.$$

They prove that any affinely invariant (i.e., translation- and scale-invariant) subspace of (2.8), hence of $\widetilde{\Pi}_\varphi$, is contained in Π_φ . But their proof can be made to show more.

Proposition 2.2. Π_φ is the largest E -invariant subspace of $\tilde{\Pi}_\varphi$.

Proof: The proof in [DM83] is based on the observation that, by Poisson's summation formula [would need some assumptions, else mollify],

$$\varphi *' p(x) = \sum_{\alpha} \varphi(x - \alpha) p(\alpha) =: \sum_{\alpha} \psi(\alpha) = \sum_{\alpha} \widehat{\psi}(2\pi\alpha),$$

while, for any $p \in \Pi$, the function $\psi : y \mapsto \varphi(x - y)p(y)$ has the Fourier transform

$$\widehat{\psi}(\eta) = e^{-ix\eta} (p(x - iD)\widehat{\varphi})(-\eta).$$

If now $p \in P$, with P an E -invariant (hence D -invariant) subspace of $\tilde{\Pi}_\varphi$, then

$$\widehat{\psi}(2\pi\alpha) = (p(x - iD)\widehat{\varphi})(2\pi\alpha) = \sum_{\beta} \llbracket x \rrbracket^{\beta} (D^{\beta} p(-iD)\widehat{\varphi})(2\pi\alpha) = 0$$

for $\alpha \neq 0$, hence

$$\begin{aligned} \varphi *' p(x) &= (p(x - iD)\widehat{\varphi})(0) \\ &= \sum_{\alpha} D^{\alpha} p(x) \llbracket -iD \rrbracket^{\alpha} \widehat{\varphi}(0) \\ (2.9) \quad &= p(x)\widehat{\varphi}(0) + \sum_{|\alpha| > 0} D^{\alpha} p(x) \llbracket -iD \rrbracket^{\alpha} \widehat{\varphi}(0), \end{aligned}$$

showing that $\varphi *' p \in \Pi$, i.e., $p \in \Pi_\varphi$.

On the other hand, if $p \in \Pi_\varphi$, then

$$\varphi *' p = \sum_{\alpha} e^{-2\pi i \alpha \langle \cdot \rangle} (p(\cdot - iD)\widehat{\varphi})(-2\pi\alpha)$$

is a polynomial, and this is possible only if

$$p(\cdot - iD)\widehat{\varphi}(2\pi\alpha) = 0, \quad \forall \alpha \neq 0,$$

showing that $p \in \tilde{\Pi}_\varphi$. Q.E.D

While Π_φ has been shown in [BH] to be dilation-invariant in case φ is a box spline, it is not clear that Π_φ is necessarily dilation-invariant for arbitrary φ . For this, I note that a polynomial subspace P is dilation-invariant if and only if P **stratifies**, i.e., $P = \sum_k P \cap \Pi_k^0$, with

$$\Pi_k^0 := \text{span}(\llbracket \cdot \rrbracket^{\alpha})_{|\alpha|=k}.$$

Hence, $\text{span}\{\llbracket \cdot \rrbracket^{2,0} + \llbracket \cdot \rrbracket^{0,1}, \llbracket \cdot \rrbracket^{1,0}, 1\}$ provides a simple example of an E -invariant polynomial subspace which is not dilation-invariant.

3. Quasi-interpolants

The space Π_φ is of interest because it characterizes the local approximation order obtainable from S , or, more precisely, from the **ladder** (S_h) associated with S . To recall,

$$S_h := \sigma_h(S),$$

with

$$\sigma_h f : x \mapsto f(x/h).$$

Further, the **local approximation order** of S is the largest k for which

$$\text{dist}(f, S_h) = O(h^k)$$

for all smooth f , with the distance measured in some norm, e.g., the max–norm on some bounded domain, and the support of the approximation to f within h of the support of f .

In [FS], Fix and Strang give a characterization of the local approximation order from the ladder (S_h) which, in the terms of Section 1, can be phrased thus: it is the largest k for which

$$(3.1) \quad U := 1 - \varphi*' \text{ is degree-reducing on } \Pi_{<k}.$$

Proposition 1.1 shows that we can state this condition more simply as

$$(3.2) \quad \Pi_{<k} \subseteq \Pi_\varphi.$$

To be precise, [FS] consider the “controlled” approximation order, which turns out to be the same as the local approximation order; cf. [BJ].

Fix and Strang use in their proof a quasi–interpolant whose construction relies on Fourier transform arguments which, in a univariate context, can already be found in Schoenberg’s basic spline paper [S] and which appear in the proof of Proposition 2.2. This makes it easy to recall their construction here.

Define the quasi–interpolant Q on Π by the rule

$$Qf := \varphi*' Ff$$

with

$$Ff := \sum_{\alpha} a_{\alpha} (-iD)^{\alpha} f$$

and $a_{\alpha} := \llbracket D \rrbracket^{\alpha} (1/\widehat{\varphi})(0)$ the Taylor coefficients for $1/\widehat{\varphi}$. Dahmen and Micchelli [DM83] prove that Q reproduces any affinely invariant subspace of (2.8), but, again, their argument supports a stronger claim, namely that

$$(3.3) \quad Q_{\Pi_\varphi} = 1.$$

For, if $p \in \Pi_\varphi$, then also $Fp \in \Pi_\varphi$ since Π_φ is D –invariant, hence, by (2.9),

$$\begin{aligned} Qp &= \sum_{\alpha} (D^{\alpha} Fp) \llbracket -iD \rrbracket^{\alpha} \widehat{\varphi}(0) \\ &= \sum_{\alpha} \sum_{\beta} a_{\beta} (-iD)^{\alpha+\beta} p \llbracket D \rrbracket^{\alpha} \widehat{\varphi}(0) \\ &= \sum_{\gamma} (-iD)^{\gamma} p \sum_{\alpha+\beta=\gamma} \llbracket D \rrbracket^{\beta} (1/\widehat{\varphi})(0) \llbracket D \rrbracket^{\alpha} \widehat{\varphi}(0) \\ &= \sum_{\gamma} (-iD)^{\gamma} p \delta_{0\gamma} = p. \end{aligned}$$

The construction is finished by noting that (3.3) only depends on the action of F on Π_φ , hence a local quasi–interpolant on smooth functions which reproduces Π_φ can be obtained in the form

$$(3.4) \quad Qf := \varphi*'(\lambda*f),$$

with

$$(3.5) \quad (\lambda*f)(x) := \lambda f(\cdot + x),$$

and λ any locally supported linear functional which agrees on Π_φ with $p \mapsto Fp(0)$.

The construction idea in [BH] seems more direct. There the locally supported bounded linear functional (on whatever normed linear space X you may wish to carry out approximation from $S \cap X$) is constructed as an extension of the linear functional

$$(3.6) \quad p \mapsto (L^{-1}p)(0).$$

Since $L = \varphi^{*\prime}|_{\Pi_\varphi}$ commutes with E , so does L^{-1} . Thus, for $p \in \Pi_\varphi$,

$$(\lambda * p)(j) = (L^{-1}p)(j) = (L^{-1}p(\cdot + j))(0),$$

hence

$$Qp = \varphi^{*\prime}(L^{-1}p) = p.$$

In order to obtain a quasi-interpolant of the optimal order k , the extension λ only needs to match (3.6) on $\Pi_{<k}$. For example, one obtains the Strang-Fix quasi-interpolant by expressing the extension as a linear combination of the linear functionals

$$(3.7) \quad f \mapsto (-iD)^\alpha f(0), \quad |\alpha| < k,$$

i.e., in the form

$$\lambda f = \sum_{|\alpha| < k} a_\alpha (-iD)^\alpha f(0).$$

The weights a_α are uniquely determined by the requirement that this linear functional match (3.6) on $\Pi_{<k}$ since (3.7) is maximally linearly independent over $\Pi_{<k}$. In particular,

$$a_\alpha = L^{-1}[[i \cdot]]^\alpha(0) = i^\alpha g_\alpha(0),$$

by (2.6'). This shows, incidentally, that

$$[[D]]^\alpha(1/\widehat{\varphi})(0) = i^\alpha g_\alpha(0).$$

If point evaluation is continuous on X , then the linear functional λ can be written as a linear combination of evaluations at the integer points near 0. For, by (1.11),

$$(L^{-1})|_{\Pi_{<k}} = (1 + U + \cdots + U^{k-1})|_{\Pi_{<k}},$$

while, from (1.9),

$$(Uf)(j) = (c * f|_1)(j),$$

with

$$c := \delta - \varphi|_1$$

and δ the unit sequence, i.e., $\delta(j) = \delta_{0j}$. Hence

$$(L^{-1}p)(0) = p^{[k]}(0)$$

with $p^{[k]}$ obtained inductively in the following computation:

$$(3.8) \quad p^{[r]} := \begin{cases} 0 & \text{if } r = 0; \\ p|_1 + c * p^{[r-1]} & \text{if } r > 0. \end{cases}$$

This gives

$$(L^{-1}p)(0) = \sum_{j \in \mathbb{Z}^d} C(j)p(j) \quad \text{all } p \in \Pi_{<k},$$

with the weight sequence C of finite support since c has finite support.

This construction was arrived at by different means by Chui and Diamond [CD], who added the following very useful observation. *If φ is symmetric, then U reduces the degree by at least 2*, since (1.7) can be written in the form

$$(1.7') \quad \text{for } f \in \Pi_\varphi, \quad \varphi *' f = f \sum_{j \in \mathbb{Z}^d} \varphi(j) + \sum_{j \in \mathbb{Z}^d} (f(\cdot + j) - 2f + f(\cdot - j))\varphi(j)/2.$$

This implies that, on $\Pi_\varphi \cap \Pi_k$, $U^{\lfloor k/2 \rfloor}$ already vanishes, hence only half the iteration (3.8) is necessary in this case.

Even for a symmetric φ , the support of the resulting λ may be far from minimal. Since we are only interested in extending a linear functional from Π_φ , a support consisting of $(\dim \Pi_\varphi)$ points is sufficient. These points can be chosen from \mathbb{Z}^d since \mathbb{Z}^d is total for Π . It would be interesting to find out whether they could be chosen as neighbors.

Such questions of minimal support for λ have been answered quite elegantly by Dahmen and Micchelli in case φ is a box spline. They find in [DM85] that the $(\dim \Pi_\varphi)$ integer points in the (right-continuous) support of φ are linearly independent over Π_φ , and so conclude the existence of an extension from Π_φ involving just these $(\dim \Pi_\varphi)$ point evaluations.

I note that the quasi-interpolant construction in [BJ] takes the opposite tack. Instead of constructing an appropriate λ as a linear combination of certain point evaluations, a compactly supported function $\psi \in S$ is constructed there so that $\psi *'$ already reproduces Π_φ .

References

- [BH] C. de Boor and K. Höllig, B-splines from parallelepipeds, *J. Analyse Math.***42**(1982/83), 99–115.
- [BJ] C. de Boor and R. Q. Jia, Controlled approximation and a characterization of the local approximation order, *Proc. Amer. Math. Soc.***95**(1985), 547–553.
- [CD] C. K. Chui and H. Diamond, A natural formulation of quasi-interpolation by multivariate splines, *Proc. Amer. Math. Soc.***99**(1987), 643–646.
- [CJW] Charles K. Chui, K. Jetter, and J. D. Ward, Cardinal interpolation by multivariate splines, *Math. Comp.***48**(178)(1987), 711–724.
- [CL] C. K. Chui and M.-J. Lai, A multivariate analog of Marsden’s identity and a quasi-interpolation scheme, *Constr. Approx.***3**(1987), 111–122.
- [DM83] W. Dahmen and C. A. Micchelli, Translates of multivariate splines, *Linear Algebra Appl.***52**(1983), 217–234.
- [DM85] W. Dahmen and C. A. Micchelli, On the solution of certain systems of partial difference equations and linear independence of translates of box splines, *Trans. Amer. Math. Soc.***292**(1985), 305–320.
- [FS] G. Fix and G. Strang, Fourier analysis of the finite element method in Ritz-Galerkin theory, *Studies in Appl. Math.***48**(1969), 265–273.
- [S] I. J. Schoenberg, Contributions to the problem of approximation of equidistant data by analytic functions, Parts A and B, *Quart. Appl. Math.***4**(1946), 45–99, 112–141.