

The limit at the origin of a smooth function space

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Abstract

The map $H \rightarrow H_{\downarrow}$ assigns to each finite-dimensional space of smooth functions a homogeneous polynomial space of the same dimension. We discuss applications of this map in the areas of multivariate polynomial interpolation, box spline theory and polynomial ideals.

§1. Introduction

Let A_0 be the space of all s -dimensional complex-valued functions analytic at the origin. For $f \in A_0$, we write its power series expansion at the origin as

$$f = f_0 + f_1 + f_2 + \dots, \quad (1)$$

where, for each j , f_j is a homogeneous polynomial of degree j . We denote by f_{\downarrow} the *least term of f* , i.e., the homogeneous polynomial f_k with $k := \max\{j : f_j = 0, \forall i < j\}$. For a finite-dimensional subspace H of A_0 , we define

$$H_{\downarrow} := \text{span}\{f_{\downarrow} : f \in H\}. \quad (2)$$

Note that $\deg f_{\downarrow} = k$ if and only if $T_k(f) = 0 \neq T_{k+1}(f)$, where $T_k : H \rightarrow \pi_{<k} : f \mapsto f_0 + f_1 + \dots + f_{k-1}$. This means that

$$\dim\{f_{\downarrow} : f \in H, \deg f_{\downarrow} = k \text{ or } f = 0\} = \dim \ker T_k - \dim \ker T_{k+1}. \quad (3)$$

Summing this equation over all k , we obtain

Proposition 1. *The space H_{\downarrow} is a homogeneous polynomial space of the same dimension as H .*

In [2], we provide the following simple algorithm for the computation of a basis for H_{\downarrow} from a given basis for H . Its description uses the inner product

$$\langle p, q \rangle := p(D)q(0) = q(D)p(0) = \sum_{\alpha \in \mathbb{Z}_+^s} \frac{D^\alpha p(0)D^\alpha q(0)}{\alpha!}, \quad (4)$$

with $p(D) := \sum_{\alpha} (D^\alpha p)(0)/\alpha!$ D^α the differential operator induced by the polynomial p .

Approximation Theory VI

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Algorithm 2 [2]. Given the basis (p_j) of the finite-dimensional subspace H of A_0 .

For $k = 1, 2, \dots$, carry out the following three steps:

$$\text{Step 1.} \quad q_k \leftarrow p_k - \sum_{j < k} q_j \frac{\langle r_j, p_k \rangle}{\langle r_j, q_j \rangle}$$

$$\text{Step 2.} \quad r_k \leftarrow q_{k \downarrow}$$

$$\text{Step 3.} \quad q_j \leftarrow q_j - q_k \frac{\langle r_k, q_j \rangle}{\langle r_k, q_k \rangle} \quad \text{if } \deg r_k > \deg r_j.$$

Then (r_j) is bi-orthogonal to (q_j) and provides a homogeneous orthogonal basis for H_{\downarrow} .

§2. H_{\downarrow} and multivariate polynomial interpolation

Let Θ be a finite subset of \mathbb{R}^s (\mathbb{C}^s will do as well). For each $\theta \in \Theta$, let P_{θ} be a finite-dimensional polynomial space. In the interpolation problem $IP(\Theta; P)$, we seek a polynomial space Q such that, for every smooth function f , there exists a unique $q_f \in Q$ satisfying

$$p(D)f(\theta) = p(D)q_f(\theta), \quad \forall \theta \in \Theta, \quad p \in P_{\theta}. \quad (5)$$

We have

Theorem 3 [2]. For given $IP(\Theta; P)$, define $H := \text{span}\{e_{\theta}p : \theta \in \Theta, p \in P_{\theta}\}$. Then the space H_{\downarrow} solves $IP(\Theta; P)$, and is of least degree among all the solutions Q of that interpolation problem in the sense that

$$\dim(\pi_j \cap H_{\downarrow}) \geq \dim(\pi_j \cap Q), \quad \forall j, Q.$$

§3. The polynomials in a box spline space

The polynomial space associated with a given (polynomial) box spline B_X is defined as follows: Let $X \subset \mathbb{R}^s \setminus 0$ be a spanning multiset for \mathbb{R}^s and

$$\mathbb{K}(X) := \{K \subset X : \text{span}(X \setminus K) \neq \mathbb{R}^s\}.$$

Also, for $Z \subset X$, define the homogeneous polynomial $p_Z := \prod_{x \in Z} \langle x, \cdot \rangle$, with $\langle x, y \rangle$ the scalar product of $x, y \in \mathbb{R}^s$. The polynomial space $\mathcal{H}(X)$, defined by

$$\mathcal{H}(X) = \{f \in \pi : p_K(D)f = 0, \quad \forall K \in \mathbb{K}(X)\}, \quad (6)$$

is of importance in box spline theory since the box spline B_X is a piecewise- $\mathcal{H}(X)$ function.

In the context of *exponential* box splines, one deals with the following generalization of the above space: we associate with each direction $x \in X$

an (arbitrary) constant λ_x and, correspondingly, we define the possibly non-homogeneous polynomials $p_{Z,\lambda} := \prod_{x \in Z} (\langle x, \cdot \rangle - \lambda_x)$. The exponential space $\mathcal{H}(X, \lambda)$ is then defined analogously as

$$\mathcal{H}(X, \lambda) := \{f \text{ is entire} : p_{K,\lambda}f = 0, \forall K \in \mathbb{K}(X)\}. \quad (7)$$

With the aid of $\mathcal{H}(X, \lambda)$, we identify elements in $\mathcal{H}(X)$. For $f \in \mathcal{H}(X, \lambda)$ and $K \in K(X)$,

$$0 = p_{K,\lambda}(D)f = p_K(D)f_{\downarrow} + \text{higher order terms}, \quad (8)$$

which implies that $p_K(D)f_{\downarrow} = 0$ and hence $f_{\downarrow} \in \mathcal{H}(X)$. Consequently,

$$\mathcal{H}(X, \lambda)_{\downarrow} \subset \mathcal{H}(X),$$

hence, by Proposition 1,

$$\dim \mathcal{H}(X, \lambda) \leq \dim \mathcal{H}(X),$$

regardless of the choice of λ . Since, for a generic $\lambda \in \mathbb{C}^X$, $\mathcal{H}(X, \lambda)$ is spanned by $\#\mathbb{B}(X)$ different exponentials, [1], with $\mathbb{B}(X)$ the multiset of all bases for \mathbb{R}^s from X , one concludes that

$$\#\mathbb{B}(X) \leq \dim \mathcal{H}(X),$$

a result which is due to Dahmen and Micchelli, [4].

§4. A basis for $\mathcal{H}(X)$

We know from [3] that

$$\mathcal{H}(X, \lambda)_{\downarrow} = \mathcal{H}(X). \quad (9)$$

For a generic λ , the exponentials e_{θ} in $\mathcal{H}(X, \lambda)$ form a basis for it. The relevant set Θ of frequencies θ is easily determined: Each $B \in \mathbb{B}(X)$ provides a $\theta = \theta_B$ as the unique solution of the linear system $\langle x, \theta \rangle = \lambda_x, \forall x \in B$. Thus, a basis for the polynomial space $\mathcal{H}(X)$ can be obtained as follows:

Step 1. *Compute the exponential basis for a suitable $\mathcal{H}(X, \lambda)$.*

Step 2. *Apply to this basis the Algorithm 2 for the construction of a basis for H_{\downarrow} from a basis for H .*

Note that the algorithm requires the determination of the least term of functions. This presents no numerical problem in the present situation in case $X \subset \mathbb{Z}^s$. For, then λ can be chosen so that each θ is rational, and the algorithm’s calculations can be carried out in exact (i.e., integer) arithmetic.

§5. Subspaces of $\mathcal{H}(X)$

The observations based on (8) made about the action of differential operators on $\mathcal{H}(X)$ and $\mathcal{H}(X, \lambda)$ can be formulated in terms of polynomial ideals and extended to more general settings, [3]. We omit here these details, yet describe the application of these extensions to subspaces of $\mathcal{H}(X)$.

Note that the elements of $\mathbb{K}(X)$ are exactly all subsets of X which intersect every element of $\mathbb{B}(X)$. Suppose now that \mathbb{B}_1 is a subset of $\mathbb{B}(X)$. Let $\mathbb{K}_1 := \{K \in X : K \cap B \neq \emptyset, \forall B \in \mathbb{B}_1\}$. Define

$$\mathcal{H}_1 := \{f : p_K(D)f = 0, \forall K \in \mathbb{K}_1\}.$$

One checks that $\mathcal{H}_1 \subset \mathcal{H}(X)$.

Theorem 4 [3].

$$\#\mathbb{B}_1 \leq \dim \mathcal{H}_1. \tag{10}$$

The inequality in the theorem is sometimes strict. To guarantee equality, one may choose \mathbb{B}_1 to be **order-closed**: suppose that X is ordered, $X = \{x_1, \dots, x_{\#X}\}$ say. This order induces a partial ordering on $\mathbb{B}(X)$:

$$B_1 = \{y_1, \dots, y_s\} \leq B_2 = \{z_1, \dots, z_s\} \iff y_j \leq z_j, \forall j.$$

We call $\mathbb{B}_1 \subset \mathbb{B}(X)$ *order-closed* if the condition

$$B_1 \leq B_2, B_2 \in \mathbb{B}_1 \implies B_1 \in \mathbb{B}_1$$

holds for all $B_1, B_2 \in \mathbb{B}(X)$.

Theorem 5 [3]. *If \mathbb{B}_1 is an order-closed subset of $\mathbb{B}(X)$, then*

$$\#\mathbb{B}_1 = \dim \mathcal{H}_1.$$

Similar results hold for subspaces of the more general space $\mathcal{H}(X, \lambda)$. These results allow us to identify the local approximation order of subspaces of $\mathcal{H}(X, \lambda)$, [3].

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