

A bound on the L_∞ -Norm of L_2 -Approximation by Splines in Terms of a Global Mesh Ratio

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Abstract. Let $L_k f$ denote the least squares approximation to $f \in \mathbf{L}_1$ by splines of order k with knot sequence $\mathbf{t} = (t_i)_{i=1}^{n+k}$. In connection with their work on Galerkin's method for solving differential equations, Douglas, Dupont and Wahlbin have shown that the norm $\|L_k\|_\infty$ of L_k as a map on \mathbf{L}_∞ can be bounded as follows,

$$\|L_k\|_\infty \leq \text{const}_k M_{\mathbf{t}},$$

with $M_{\mathbf{t}}$ a global mesh ratio, given by

$$M_{\mathbf{t}} := \max_i \Delta t_i / \min\{\Delta t_i : \Delta t_i > 0\}.$$

Using their very nice idea together with some facts about B-splines, it is shown here that even

$$\|L_k\|_\infty \leq \text{const}_k (M_{\mathbf{t}}^{(k)})^{1/2}$$

with the smaller global mesh ratio $M_{\mathbf{t}}^{(k)}$ given by

$$M_{\mathbf{t}}^{(k)} := \max_{i,j} (t_{i+k} - t_i) / (t_{j+k} - t_j).$$

A mesh independent bound for L_2 -approximation by continuous piecewise polynomials is also given.

1. Introduction. This note is an addendum to the clever paper by Douglas, Dupont and Wahlbin [2] in which these authors bound the linear map of least-squares approximation by splines of order k with knot sequence $\mathbf{t} := (t_i)$, as a map on \mathbf{L}_∞ , in terms of the particular global mesh ratio

$$M_{\mathbf{t}} := \max_i \Delta t_i / \min\{\Delta t_i : \Delta t_i > 0\}.$$

Their argument is very elegant. But their result is puzzling in one aspect: The ratio $M_{\mathbf{t}}$ is not a continuous function of \mathbf{t} . If, e.g., \mathbf{t} is uniform, hence $M_{\mathbf{t}} = 1$, and we now let $\mathbf{t} \rightarrow \mathbf{t}^*$ by letting just one knot approach its neighbor, leaving all other knots fixed, then

$$\lim_{\mathbf{t} \rightarrow \mathbf{t}^*} M_{\mathbf{t}} = \infty, \quad \text{while } M_{\mathbf{t}^*} = 2.$$

Correspondingly, their bound goes to infinity as $\mathbf{t} \rightarrow \mathbf{t}^*$, yet is again finite for the particular knot sequence \mathbf{t}^* .

This puzzling aspect is removed below. It is shown that (as asserted in a footnote to [1]) their very nice argument can be used to give a bound in terms of the smaller global mesh ratio

$$(1) \quad M_{\mathbf{t}}^{(k)} := \max_i (t_{i+k} - t_i) / \min_i (t_{i+k} - t_i)$$

which does depend continuously on \mathbf{t} in $\{\mathbf{t} \in \mathbb{R}^{n+k} : t_i \leq t_{i+1}, t_i < t_{i+k}, \text{ all } i\}$.

2. Least-squares approximation by splines of order k . Let $\mathbf{t} := (t_i)_{i=1}^{n+k}$ be a nondecreasing sequence, with $t_i < t_{i+k}$, all i . A spline of order k with knot sequence \mathbf{t} is, by definition, any function of the form

$$\sum_{i=1}^n \alpha_i N_i$$

with $\alpha \in \mathbb{R}^n$ and N_i the normalized B-spline of order k with knots t_i, \dots, t_{i+k} , i.e.,

$$N_i(t) := N_{i,k,\mathbf{t}}(t) := (t_{i+k} - t_i)[t_i, \dots, t_{i+k}](\cdot - t)_+^{k-1}.$$

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In words, for each t , $N_i(t)$ is $(t_{i+k} - t_i)$ times the k th divided difference at t_i, \dots, t_{i+k} of $(s - t)_+^{k-1}$ as a function of s .

We denote the totality of all splines of order k with knot sequence \mathbf{t} by $\mathbf{S}_{k,\mathbf{t}}$. More detail about $\mathbf{S}_{k,\mathbf{t}}$ is provided in [1] and its references.

Next, let L_k denote the linear projector on \mathbf{L}_1 defined by the condition that $L_k f \in \mathbf{S}_{k,\mathbf{t}}$, and, for all $g \in \mathbf{S}_{k,\mathbf{t}}$, $\int (f - L_k f)g = 0$, i.e., $L_k f$ is the \mathbf{L}_2 -approximation to f in $\mathbf{S}_{k,\mathbf{t}}$. We are interested in estimating the norm $\|L_k\|_p$ of L_k as a map on \mathbf{L}_p . Since

$$\|L_k\|_p = \|L_k\|_q \quad \text{for } 1/p + 1/q = 1,$$

and $\|L_k\|_2 = 1$, interpolation will give a bound on $\|L_k\|_p$ in terms of $\|L_k\|_\infty = \|L_k\|_1$, as is pointed out in [2]. It therefore suffices to consider $\|L_k\|_\infty$.

Let $L_k f = \sum \alpha_j N_j$. Then $\|L_k f\|_\infty \leq \|\boldsymbol{\alpha}\|_\infty$ since $N_i \geq 0$, all i , and $\sum_j N_j \leq 1$, while

$$\sum_j \int N_i N_j \alpha_j = \int N_i f \leq [(t_{i+k} - t_i)/k] \|f\|_\infty, \quad \text{all } i,$$

since $N_i \geq 0$ and $\int N_i = (t_{i+k} - t_i)/k$. Therefore,

$$(2) \quad \|L_k\|_\infty \leq \|G^{-1}\|_\infty$$

with

$$(3) \quad G := G_\infty = E^{1/2} G_2 E^{-1/2},$$

where E is a diagonal matrix,

$$(4) \quad E := [k/(t_{k+1} - t_1), \dots, k/(t_{k+n} - t_n)],$$

and G_2 is the Gramian matrix for the basis $(\overset{2}{N}_i)$ of $\mathbf{S}_{k,\mathbf{t}}$, i.e.,

$$(5) \quad G_2 := \left(\int \overset{2}{N}_i \overset{2}{N}_j \right)_{i,j=1}^n$$

and

$$(6) \quad \overset{p}{N}_i := [k/(t_{i+k} - t_i)]^{1/p} N_i.$$

With this normalization, we are assured of the existence of a positive constant D_k depending only on k and not at all on \mathbf{t} or n so that

$$(7) \quad D_k^{-1} \|\boldsymbol{\alpha}\|_p \leq \left\| \sum_j \alpha_j \overset{p}{N}_j \right\|_p \leq \|\boldsymbol{\alpha}\|_p, \quad \text{all } \boldsymbol{\alpha} \in \mathbb{R}^n$$

(see the theorem on p.539 of [1]). This inequality implies that

$$(8) \quad \|G_2^{-1}\|_\infty \leq \text{const}_k$$

for some const_k depending only on k as we will show below; and, on combining this with (2)-(4), we obtain the desired conclusion

$$(9) \quad \|L_k\|_\infty \leq \text{const}_k (M_{\mathbf{t}}^{(k)})^{1/2}.$$

3. A bound for $\|G_2^{-1}\|_\infty$. With $(\alpha_{ij})_{i,j=1}^n := G_2^{-1}$, let $f_i := \sum_j \alpha_{ij} \overset{2}{N}_j$. Then

$$\int f_i \overset{2}{N}_j = \delta_{ij}, \quad \text{all } j;$$

hence

$$\int \alpha_{ii} \overset{2}{N}_i f_i + \sum_{j \neq i} \alpha_{ij} \overset{2}{N}_j f_i = \alpha_{ii},$$

i.e.,

$$(10) \quad \|f_i\|_2^2 = \alpha_{ii}.$$

Therefore, by (7),

$$D_k^{-2} \alpha_{ii}^2 \leq D_k^{-2} \sum_j |\alpha_{ij}|^2 \leq \|f_i\|^2 = \alpha_{ii},$$

hence, as $\alpha_{ii} = \|f_i\|_2^2 \neq 0$ (G_2^{-1} is invertible!), we have $\alpha_{ii} \leq D_k^2$; and so, $\|f_i\|_2 \leq D_k$ and

$$(11) \quad \left(\sum_j |\alpha_{ij}|^2 \right)^{1/2} \leq D_k \|f_i\|_2 = D_k (\alpha_{ii})^{1/2} \leq D_k^2.$$

This shows that

$$\|G_2^{-1}\|_\infty = \max_i \sum_j |\alpha_{ij}| \leq n^{1/2} \max_i \left(\sum_j |\alpha_{ij}|^2 \right)^{1/2} \leq n^{1/2} D_k^2$$

and so bounds $\|G_2^{-1}\|_\infty$ in terms of only k and n . From this, one obtains

$$\|G^{-1}\|_\infty \leq (n M_{\mathbf{t}}^{(k)})^{1/2} D_k^2,$$

a bound in terms of the desired global mesh ratio, except that the bound goes to infinity with the number of mesh points. Note that we can express $M_{\mathbf{t}}^{(k)}$ in terms of n and the local mesh ratio

$$m_{\mathbf{t}}^{(k)} := \max_{|i-j|=1} (t_{i+k} - t_i) / (t_{j+k} - t_j);$$

hence, we even have a bound on $\|G^{-1}\|_\infty$ in terms of that *local* mesh ratio but, alas, involving also n .

In order to remove this dependence on n , we use the ideas of Douglas, Dupont and Wahlbin [2] to prove the following lemma.

Lemma 1. *There exist const_k and $\lambda_k \in (0, 1)$ independent of n or \mathbf{t} so that, for all i and j ,*

$$|\alpha_{ij}| \leq \text{const}_k (\lambda_k)^{|i-j|}.$$

Proof: We observed earlier that the function $f_i = \sum_j \alpha_{ij} \overset{2}{N}_j$ is orthogonal to $\text{span}(\overset{2}{N}_j)_{j \neq i}$. Hence, for any $m > i$,

$$f_{i,m} := \sum_{m \leq j} \alpha_{ij} \overset{2}{N}_j$$

is orthogonal to f_i and, therefore, also orthogonal to $f_{i,m-k+1}$ since the latter function agrees with f_i on the support of $f_{i,m}$. This proves that

$$(12) \quad \|f_{i,m-k+1}\|_2^2 + \| - f_{i,m} \|_2^2 = \|f_{i,m-k+1} - f_{i,m}\|_2^2$$

from which we conclude that

$$\left\| \sum_{m-k < j} \alpha_{ij} \overset{2}{N}_j \right\|_2^2 \leq \left\| \sum_{m-k < j < m} \alpha_{ij} \overset{2}{N}_j \right\|_2^2$$

or, with the inequality (7),

$$(13) \quad \sum_{m-k < j < m} |\alpha_{ij}|^2 \geq D_k^{-2} \sum_{m-k < j} |\alpha_{ij}|^2, \quad m = i+1, i+2, \dots$$

Faced with a similar inequality, Douglas, Dupont and Wahlbin [2] make use of what amounts to the following discrete Gronwall inequality:

Lemma 2. *If the sequence a_0, a_1, \dots satisfies*

$$(14) \quad |a_m| \geq c \sum_{m \leq j} |a_j|, \quad m = 0, 1, 2, \dots,$$

for some $c \in (0, 1)$, then $\lambda := 1 - c \in (0, 1)$ and

$$(15) \quad |a_m| \leq |a_0| \lambda^m / c, \quad m = 0, 1, 2, \dots$$

Proof: Let $A_m := \sum_{m \leq j} |a_j|$. Then (14) reads

$$A_m - A_{m+1} \geq c A_m, \quad \text{all } m,$$

or, $A_{m+1} \leq (1 - c)A_m$, all m , therefore, with $\lambda := 1 - c$,

$$A_{m+j} \leq \lambda^j A_m, \quad \text{all } m, j,$$

and so,

$$|a_m| = A_m - A_{m+1} \leq A_m \leq \lambda^m A_0 \leq |a_0| \lambda^m / c. \quad \text{Q.E.D.}$$

In order to apply this lemma to (12), we pick $m_0 > i$ and let

$$J_m := \{j \in \mathbb{Z} : m_0 + (k-1)(m-1) \leq j < m_0 + (k-1)m\}, \quad m = 0, 1, \dots$$

Then, with

$$a_m := \sum_{j \in J_m} |\alpha_{ij}|^2, \quad \text{all } m,$$

we obtain from (12) that

$$a_m \geq D_k^{-2} \sum_{m \leq j} a_j, \quad m = 0, 1, 2, \dots;$$

hence, from the lemma,

$$\max_{j \in J_m} |\alpha_{ij}| \leq a_m^{1/2} \leq D_k (1 - D_k^{-2})^{m/2} a_0^{1/2}$$

while, by (11),

$$a_0^{1/2} \leq \left(\sum_j |\alpha_{ij}|^2 \right)^{1/2} \leq D_k^2.$$

This proves the asserted exponential decay of $|\alpha_{ij}|$ for $j > i$; but G_2 is symmetric. Q.E.D.

It follows at once that

$$(16) \quad \|G_2^{-1}\|_\infty \leq \text{const}_k 2 / (1 - \lambda_k).$$

In view of the discussion at the end of Section 2, we have therefore proved the following theorem.

Theorem 1. *There exists a constant c depending only on k so that the norm $\|L_k\|_\infty$ of \mathbf{L}_2 -approximation by splines of order k with knot sequence \mathbf{t} , as a map on \mathbf{L}_∞ , satisfies*

$$\|L_k\|_\infty \leq c (M_{\mathbf{t}}^{(k)})^{1/2}$$

with the global mesh ratio $M_{\mathbf{t}}^{(k)}$ given by

$$M_{\mathbf{t}}^{(k)} := \max_{i,j} (t_{i+k} - t_i) / (t_{j+k} - t_j).$$

and extend \mathbf{x} to a $(k-1)$ -periodic function $\mathbf{y} = (y_i)_1^n$ on all of $(1, \dots, n)$. This is possible since $x_k = x_1$ by symmetry. Then, for $i = m(k-1) + I$, we have from (17) and (19) that

$$\sum_j G_{ij}(-1)^{i+j} y_j = \sum_{j=1}^k \widehat{G}_{Ij}(-)^{I+j} x_j = 1, \quad I = 2, \dots, k-1; \quad m = 0, \dots, r,$$

and also

$$\begin{aligned} \sum_j G_{ij}(-)^{i+j} y_j &= (\Delta \xi_{m-1} / (\xi_{m+1} - \xi_{m-1})) \sum_j \widehat{G}_{kj}(-)^{k+j} x_j \\ &\quad + (\Delta \xi_m / (\xi_{m+1} - \xi_{m-1})) \sum_j \widehat{G}_{1j}(-)^{1+j} x_j = 1 \end{aligned}$$

for $I = 1; \quad m = 0, \dots, r+1$.

This proves with (18) that

$$\|G^{-1}\|_\infty = \|\mathbf{y}\|_\infty = \|\mathbf{x}\|_\infty = \|\widehat{G}^{-1}\|_\infty. \quad \text{Q.E.D.}$$

References

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