Multivariate polynomial interpolation: Conjectures concerning GC-sets

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Abstract. GC-sets are subsets T of $\mathbb{R}^d$ of cardinality $\dim \Pi_n$ for which, for each $\tau \in$ T, there are n hyperplanes whose union contains all of T except for τ, thus making interpolation to arbitrary data on T by polynomials of degree $\le n$ uniquely possible. The existing bivariate theory of such sets is extended to the general multivariate case and the concept of a maximal hyperplane for T is highlighted, in hopes of getting more insight into existing conjectures for the bivariate case.

Keywords: Gasca-Maetzu conjecture, geometric characterization, completely factorizable, Lagrange form, maximal polynomial.

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Introduction

As already simple bivariate examples involving two, three, or four points make clear, those interested in *multivariate* polynomial interpolation must contend with the following

Basic Problem. *Given a finite set* T *of data* **sites** τ *in* $\mathbb{R}^d$*, how to choose a space* F *of polynomials so that, for every choice of data* **values**

$$a = (a(\tau) : \tau \in \mathrm{T}) \quad \in \ \mathbb{F}^{\mathrm{T}}$$

(with $\mathbb{F}$ *either* $\mathbb{R}$ *or* $\mathbb{C}$*), there is exactly one element* $f \in F$ *that matches this information, i.e., satisfies*

$$f(\tau) = a(\tau), \qquad \tau \in \mathrm{T}.$$

I will call such F **correct for** T. Others have used “unisolvent” or “poised” or “regular” instead of “correct”.

While there is some work on this problem (see, e.g., [BR1], [BR2], [B]), most of the work to date on multivariate polynomial interpolation has gone into the somewhat different problem of finding sets that are **correct of degree** n, or n**-correct**, for short, i.e., sets $\mathrm{T} \subset \mathbb{R}^d$ correct for

$$F \ := \ \{\sum_{|\alpha| \le n} ()^\alpha c(\alpha) : c(\alpha) \in \mathbb{F}\},$$

with

$$()^\alpha : \mathbb{R}^d \to \mathbb{F} : x \mapsto x^\alpha := x(1)^{\alpha(1)} \cdots x(d)^{\alpha(d)}$$

a nonstandard but handy notation for the monomial with exponent $\alpha \in \mathbb{Z}_+^d$ and

$$|\alpha| := \alpha(1) + \cdots + \alpha(d)$$

its **(total) degree**.

In effect, most of the work has focused, not on interpolation *per se* but, rather, on interpolation as a means for constructing approximations from $\Pi_{\le n}(\mathbb{R}^d)$ under the assumption that there is some leeway in the choice of interpolation sites. At its most intricate, this line of research looks for good interpolation sites, i.e., for sites for which the resulting linear projector has small norm, with [BoCMVX] a particularly striking example.

The aim of this note is much less ambitious. It is to generalize (in a reasonable way, I hope) what is known in the bivariate case about a particular kind of n-correct T to the general multivariate case, in hopes of shedding some light on a conjecture made by Gasca and Maeztu nearly 25 years ago and still essentially unsettled.

Basic linear algebra

I find it most convenient to start a discussion of interpolation with its *inverse*, i.e., with the linear map

$$\delta_{\mathrm{T}} : f \mapsto f|_{\mathrm{T}} := (f(\tau) : \tau \in \mathrm{T}) \in \mathbb{F}^{\mathrm{T}}$$

of restriction to, or evaluation at, the data sites. In terms of this map, (T, F) is correct iff δ_{T} maps F 1-1 onto $\mathbb{F}^{\mathrm{T}}$. Hence, if (T, F) is correct, then the inverse of $\delta_{\mathrm{T}}|_F$ exists and is necessarily a **column map**, i.e., of the form

$$(\delta_{\mathrm{T}}|_F)^{-1} a =: \sum_{\tau \in \mathrm{T}} \ell_\tau a(\tau) =: [\ell_\tau : \tau \in \mathrm{T}] a, \quad a \in \mathbb{F}^{\mathrm{T}},$$

and with $\ell_\tau(\sigma) = \delta_{\tau\sigma}$ for all $\tau, \sigma \in \mathrm{T}$, i.e., $[\ell_\tau : \tau \in \mathrm{T}]$ is the **Lagrange basis** for F with respect to T.

Basic facts about n-correct sets

From now on, assume that $\mathrm{T} \subset \mathbb{R}^d$ is n-correct, hence

$$p = \sum_{\tau \in \mathrm{T}} \ell_\tau p(\tau)$$

is the **Lagrange form** for $p \in \Pi_{\le n}$.

(1) Fact [ChY: (c)(iii)]. *Each ℓ_τ is of (exact) degree n.*

Proof: If $\deg \ell_\tau < n$, then multiplication of ℓ_τ with some degree 1 polynomial vanishing at τ would give a nontrivial polynomial in $\Pi_{\le n}$ vanishing on T, contradicting the n-correctness of T. □

One approach to the construction of n-correct sets uses a divide-and-conquer strategy. For its description, the following notions are needed.

Definition. *For all $p \in \Pi$,*

$$Z(p) := \{z \in \mathbb{R}^d : p(z) = 0\}, \quad Z_{\mathrm{T}}(p) := \mathrm{T} \cap Z(p).$$

(2) Fact [CaG1]. *For all $p \in \Pi_{\le n} \backslash 0$, $\# Z_{\mathrm{T}}(p) \le \dim \Pi_{\le n} - \dim \Pi_{\le n - \deg p}$.*

Proof: If $\# Z_{\mathrm{T}}(p) > \dim \Pi_{\le n} - \dim \Pi_{\le n-\deg p}$, hence $\#(\mathrm{T} \backslash Z(p)) < \dim \Pi_{\le n - \deg p}$, then there would be $q \in \Pi_{\le n - \deg p} \backslash 0$ vanishing on $\mathrm{T} \backslash Z(p)$, and then pq would be a nontrivial element of $\Pi_{\le n}$ vanishing on all of T, contradicting the n-correctness of T. □

Definition. *Call $p \in \Pi_{\le n} \backslash 0$* **maximal (for T)** *in case*

$$\# Z_{\mathrm{T}}(p) = \dim \Pi_{\le n} - \dim \Pi_{\le n - \deg p}$$

or, equivalently,

$$\#(\mathrm{T} \backslash Z(p)) = \dim \Pi_{\le n - \deg p}.$$

(3) Fact. *If p is maximal (for T), then:*
(i) $\mathrm{T} \backslash Z(p)$ *is $(n - \deg p)$-correct;*
(ii) *p divides any $q \in \Pi_{\le n}$ for which $Z_{\mathrm{T}}(q) \supset Z_{\mathrm{T}}(p)$;*
(iii) $[\ell_\tau / p : \tau \in \mathrm{T} \backslash Z(p)]$ *is an unnormalized Lagrange basis for $\Pi_{\le n - \deg p}$ with respect to $\mathrm{T} \backslash Z(p)$;*
(iv) $Z_{\mathrm{T}}(p)$ *is correct for $\Pi_{\le n}(Z(p)) := \Pi_{\le n}|_{Z(p)}$.*

Proof: Let $r := \deg p$. If $\mathrm{T} \backslash Z(p)$ were not correct for $\Pi_{\le n-r}$, then, since $\#(\mathrm{T} \backslash Z(p)) = \dim \Pi_{\le n-r}$, there would be a nontrivial $q \in \Pi_{\le n-r}$ vanishing on $\mathrm{T} \backslash Z(p)$, hence pq would be a nontrivial polynomial in $\Pi_{\le n}$ vanishing on all of T, in contradiction to the n-correctness of T. This proves (i). It follows that, for any $q \in \Pi$, there is some $f \in \Pi_{\le n-r}$ that matches q/p on $\mathrm{T} \backslash Z(p)$. If now also $Z_{\mathrm{T}}(q) \supseteq Z_{\mathrm{T}}(p)$, then fp is in $\Pi_{\le n}$ and matches q on all of T, hence must equal q in case $q \in \Pi_{\le n}$. This proves (ii). In particular, for every $\tau \in \mathrm{T} \backslash Z(p)$, p divides ℓ_τ, hence ℓ_τ / p is in $\Pi_{\le n-r}$ and vanishes on every $\sigma \in \mathrm{T} \backslash Z(h)$ other than τ, hence is, up to scaling, the Lagrange polynomial from $\Pi_{\le n-r}$ for this $\tau \in \mathrm{T} \backslash Z(p)$. Finally, as to (iv), the linear map $f \mapsto f|_{Z(p)}$ carries $\Pi_{\le n}$ onto $\Pi_{\le n}(Z(p))$ and has $p \Pi_{\le n-r} \subset \Pi_{\le n}$ in its kernel, hence $\dim \Pi_{\le n}(Z(p)) \le \dim \Pi_{\le n} - \dim \Pi_{\le n-r} = \# Z_{\mathrm{T}}(p)$, while $[\ell_\tau|_{Z(p)} : \tau \in Z_{\mathrm{T}}(p)]$ is a 1-1 map into $\Pi_{\le n}(Z(p))$, hence necessarily is a basis for it, and is a right inverse for $\delta_{Z_{\mathrm{T}}(p)}$. □

Note that the degree of q comes into the proof of (ii) in an essential way. There is no way to extend the argument to q whose degree exceeds the degree of correctness of T which, in turn, figures in the maximality of p.

(4) Fact. *If Σ is $(n-r)$-correct, and T is correct for $\Pi_{\le n}(Z(p))$ for some polynomial p of degree r with $\Sigma \cap Z(p) = \emptyset$ for which $Z(p) \subseteq Z(q)$ implies $p|q$, then $\Sigma \cup \mathrm{T}$ is n-correct.*

Proof: If $f \in \Pi_{\le n}$ vanishes on T, then, by the correctness of T for $\Pi_{\le n}(Z(p))$, f vanishes on all of $Z(p)$, hence is of the form $f = qp$ for some $q \in \Pi_{\le n-r}$ by assumption, and if f also vanishes on Σ, then, since p does not vanish on Σ, this q necessarily vanishes on Σ, hence, by the correctness of Σ for $\Pi_{\le n-r}$, is 0, therefore also $f = 0$. This shows that $\delta_{\Sigma\cup\mathrm{T}}$ is 1-1 on $\Pi_{\le n}$ while $\#(\Sigma \cup \mathrm{T}) = \dim \Pi_{\le n-r} + \dim \Pi_{\le n}(Z(p)) = \dim \Pi_{\le n}$, hence $\Sigma \cup \mathrm{T}$ is n-correct. □

For the special case of a bivariate p of degree 1, this "recipe" is used in [Coa] and goes back at least to [R] where n-correct sets in $\mathbb{R}^2$ are constructed inductively, starting with a 1-point set T and, assuming that the current T is $(n-1)$-correct, choosing an arbitrary $(n+1)$-set from a line that does not intersect T and then adjoining that set to T (using the elementary fact, valid in the general multivariate case, that $\deg(p) = 1$ implies that p is a factor of any polynomial that vanishes on $Z(p)$). Such **Radon sets** were already constructed in [Be], and even for $\mathbb{R}^d$, but in a more complicated way.

For interpolation by complex-valued polynomials on $\mathbb{C}^d$, the implication $Z(p) \subseteq Z(q) \Longrightarrow p|q$ needed in (4) is available for any polynomial p without repeated factors (see, e.g., [CoLO: p. 178]), a fact used, e.g., in [LLF] for such recipes involving polynomials of degree greater than 1.

Recipes of Chung and Yao

In [ChY], Chung and Yao provide the following recipe for the construction of an n-correct set.

Let H be a collection of $d+n$ hyperplanes in $\mathbb{R}^d$ **in general position**, meaning that, for $r = 1, 2, \ldots$, the intersection of any r of these is a $(d-r)$-dimensional flat. Then, in particular, for any d-set K in H, there is exactly one point, call it τ_K, common to all $h \in K$, and the resulting map,

$$\binom{H}{d} \to \mathbb{R}^d : K \mapsto \tau_K,$$

from the collection of all d-sets in H to $\mathbb{R}^d$, is 1-1, hence its range, call it T_H, has cardinality $\#\binom{H}{d} = \binom{n+d}{d} = \dim \Pi_{\le n}$ while, for any τ_K, the union of the n hyperplanes in $H\backslash K$ contains $\mathrm{T}_H\backslash\tau_K$ but not τ_K. Therefore, denoting by h also any particular polynomial of degree 1 for which $Z(h)$ is the hyperplane h,

$$[\prod_{h\in H, h(\tau)\ne 0} h/h(\tau) : \tau \in \mathrm{T}_H]$$

is a map into $\Pi_{\le n}$ and a right inverse for δ_{T_H}, hence a basis for $\Pi_{\le n}$, hence T_H is n-correct.

Chung and Yao call any such T_H a **natural lattice**.

Note that, in [Bo], L. Bos extends this elegant construction to suitable collections of higher-degree algebraic varieties.

The above proof of the n-correctness of T_H uses nothing more than the fact that, for each $\tau \in \mathrm{T}_H$, there are $\le n$ hyperplanes whose union contains $\mathrm{T}_H\backslash\tau$ but not τ. Recognizing this, [ChY] introduce the following

Definition [ChY]. T *in $\mathbb{R}^d$ satisfies the* **geometric characterization of degree** *n if $n > 0$ and, for each $\tau \in \mathrm{T}$, there is a set H_τ of $\le n$ hyperplanes whose union contains $\mathrm{T}\backslash\tau$ but not τ.*

Call such a T a **GC$_n$-set** for short, or even just a **GC-set** if the degree of correctness doesn't matter. It is an n-correct set for which all Lagrange polynomials have as their zero set the union of $\le n$ hyperplanes (as [ChY: Theorem 1] makes clear).

To be sure, the restriction to $n > 0$ does not occur in [ChY] but is made here in order to avoid some niggling discussions. Also, [ChY] requires that $\#H_\tau = n$ for all $\tau \in \mathrm{T}$ even though this already follows from the seemingly weaker requirement that $\#H_\tau \le n$; see (6) below.

(5) Fact. *Any* 1*-correct set is a* GC_1*-set.*

Proof: All the Lagrange polynomials for a 1-correct set are affine polynomials, hence have hyperplanes as their zero sets. □

Basic properties of GC-sets

From now on, let T be a GC_n-set, hence

$$\ell_\tau = \prod_{h\in H_\tau} \frac{h}{h(\tau)}, \quad \tau \in \mathrm{T},$$

where, here and below, we continue our convenient practice of denoting, for any hyperplane h in $\mathbb{R}^d$, by the very same letter h also any one polynomial of degree 1 whose zero set is h.

The various properties of GC_n-sets given in this section can be found in the literature (see, especially, [GM], [CaG1], [CaG2], [CaG3]) but usually only for $d = 2$ and, usually, with different proofs.

(6) [ChY: (c)(iv)]. *For all* $\tau \in \mathrm{T}$, $\#H_\tau = n$*. In particular,* ℓ_τ *has no repeated factors.*

Proof: $\#H_\tau = \deg \ell_\tau = n$, the second equality by (1). □

Definition. $H_\mathrm{T} := \cup_{\tau\in\mathrm{T}} H_\tau$.

(7) Fact. *For all* $h \in H_\mathrm{T}$,
(i) h *is the unique hyperplane containing* $Z_\mathrm{T}(h)$*;*
(ii) $d \le \#Z_\mathrm{T}(h) \le \dim \Pi_{\le n} - \dim \Pi_{<n}$.

Proof: If k were a hyperplane different from h with $Z_\mathrm{T}(k) = Z_\mathrm{T}(h)$, then the flat spanned by $Z_\mathrm{T}(h)$ would lie in the $(d-2)$-dimensional flat $h \cap k$ and not contain τ, hence we could find a (proper) hyperplane l containing $Z_\mathrm{T}(h)$ and τ and, in this way, find the nontrivial polynomial $(\ell_\tau/h)l$ in $\Pi_{\le n}$ which vanishes on T, contradicting the n-correctness of T. This proves (i), hence the first inequality in (ii), while the second inequality is a special case of (2). □

Definition. $K_\tau := \{h \in H_\mathrm{T} : h(\tau) = 0\}$.

(8) Fact.
(i) For all $\tau \in \mathrm{T}$, $\cap_{h\in K_\tau} Z(h) = \{\tau\}$.
(ii) For all $\tau \in \mathrm{T}$, $\#H_\mathrm{T} \ge \#H_\tau + \#K_\tau \ge n + d$.
(iii) $\#H_\mathrm{T} \ge n + d$*, with equality iff* T *is natural.*

Proof: If $p \in \Pi_{\le n}$ and $p(\tau) = 0$, then

$$p = \sum_{\sigma\ne\tau} \ell_\sigma p(\sigma),$$

and each of these ℓ_σ has at least one of the $h \in K_\tau$ as a factor. Consequently,

$$Z(p) \supseteq \bigcap_{h\in K_\tau} Z(h) \supseteq \{\tau\}.$$

But since p is otherwise arbitrary here, this implies that

$$\{\tau\} = \bigcap_{p\in\Pi_{\le n}, p(\tau)=0} Z(p) \supseteq \bigcap_{h\in K_\tau} Z(h) \supseteq \{\tau\},$$

proving (i). It follows that $\#K_\tau \ge d$ while $H_\tau \cap K_\tau = \emptyset$ and, by (6), $\#H_\tau = n$, proving (ii), hence the inequality in (iii). Further, $\#H_\mathrm{T} = n + d$ in case T is a natural lattice (from the very definition of a natural lattice) while, conversely, $\#H_\mathrm{T} = n + d$ implies, with (ii), that $\#K_\tau = d$ for all $\tau \in \mathrm{T}$ showing H_T to be in general position, hence T is natural. □

Definition. $\mathrm{T}_h := \{\tau \in \mathrm{T} : h \in H_\tau\}$.

(9) Proposition [S]. $\#\mathrm{T}_h \le \dim \Pi_{<n}$, *with equality iff h is maximal.*

The proof in [S] makes use of elementary ideal theory. But, in trying to understand the result, I came across a simpler proof. For it, note that T_h is necessarily a subset of

$$\mathrm{T}\backslash Z(h) = \{\tau \in \mathrm{T} : h(\tau) \ne 0\}$$

but there is, offhand, no reason to believe that these two sets coincide. In fact, the following is true.

(10) Fact. $\#\mathrm{T}_h \le \dim \Pi_{<n} \le \#(\mathrm{T}\backslash Z(h))$, *with equality in one or the other iff h is maximal iff there is equality in both.*

Proof: The set $\{\ell_\tau/h : \tau \in \mathrm{T}_h\}$ is linearly independent and in $\Pi_{<n}$, hence cannot be larger than $\dim \Pi_{<n}$, proving the first inequality, with equality iff $\{\ell_\tau/h : \tau \in \mathrm{T}_h\}$ is a basis for $\Pi_{<n}$, hence $\{\ell_\tau : \tau \in \mathrm{T}_h\}$ is a basis for $h\Pi_{<n}$, and this basis vanishes on $\mathrm{T}\backslash\mathrm{T}_h$, hence all elements of $h\Pi_{<n}$ vanish on that set, including h; in other words, then $Z_\mathrm{T}(h) \subseteq \mathrm{T}\backslash\mathrm{T}_h \subseteq Z_\mathrm{T}(h)$, hence $Z_\mathrm{T}(h) = \mathrm{T}\backslash\mathrm{T}_h$, therefore $\#Z_\mathrm{T}(h) = \#\mathrm{T} - \#\mathrm{T}_h = \dim \Pi_{\le n} - \dim \Pi_{<n}$, i.e., h is maximal. The second inequality is a special case of (2), with the equality case exactly the definition of h being maximal. Conversely, if h is maximal then, by (3)(ii), $\mathrm{T}_h = \mathrm{T}\backslash Z_\mathrm{T}(h)$, and this implies that both inequalities must be equalities. □

Conjectures

About 25 years ago, Gasca and Maeztu made the following conjecture.

GM Conjecture ([GM]). *For every GC_n-set T in $\mathbb{R}^2$, there is $h \in H_\mathrm{T}$ with $\#Z_\mathrm{T}(h) = n + 1$.*

If true, it would imply, by induction, that every planar GC-set can be obtained by the Radon recipe mentioned earlier.

The conjecture obviously holds for $n = 1, 2$ and has been proved by J. R. Busch in [Bu] for $n = 3, 4$, and somewhat shorter proofs for these cases have been given by Carnicer and Gasca in [CaG3]. But already for $n = 5$, the conjecture has not been settled so far.

For $d = 2$, $n + 1 = \dim \Pi_{\le n} - \dim \Pi_{<n}$, hence it seems reasonable to generalize the GM conjecture for arbitrary d as

GM$_d$ Conjecture. *Every GC_n-set has an affine maximal, or, equivalently, some hyperplane containing* $\dim \Pi_{\le n} - \dim \Pi_{<n}$ *points of that set.*

In the plane, Carnicer and Gasca have recently strengthened the GM conjecture by proving the following

(11) Theorem ([CaG3]). *If every planar GC_n-set for $n \le \nu$ has an affine maximal, then every such planar GC_n-set has at least three affine maximals.*

This suggests to me the following

CG$_d$ conjecture. *Every GC-set in $\mathbb{R}^d$ has at least $d + 1$ affine maximals.*

Any 1-correct set in $\mathbb{R}^d$ has exactly $d + 1$ affine maximals, namely its $d + 1$ Lagrange polynomials.

As we saw earlier, any *natural* GC_n-set in $\mathbb{R}^d$ has $d + n$ affine maximals.

A second well-known class of GC-sets are the so-called principal lattices, so named by Nicolaides [N] and discussed in [ChY] as another class of GC-sets. A **principal** lattice in $\mathbb{R}^d$ is the 1-1 affine image of the set

$$\{\alpha \in \mathbb{Z}_+^d : |\alpha| \le n\}$$

for some n, hence has exactly $d + 1$ affine maximals, namely the $d + 1$ faces of the simplex spanned by its $d + 1$ extreme points (and "simplex lattice" would have been a better name for such GC-sets).

These conjectures focus attention on the set

$$H_\mathrm{T}^{\max}$$

of all affine maximals for the GC-set T, to be discussed next.

Basic properties of the set of affine maximals

For $d = 2$, the following assertions about affine maximals (but without such terminology and with different proofs) can already be found, e.g., in [CaG1].

(12) Fact. $h \in H^{\max}_{\rm T} \iff h \in \cap_{\tau\in{\rm T}\backslash Z(h)} H_\tau$.

Proof: By (3)(ii), any affine maximal h is a factor of the Lagrange polynomial of any $\tau \in {\rm T}\backslash Z(h)$, while, if the latter holds, then $\#{\rm T}_h = \#({\rm T}\backslash Z(h))$, hence h is an affine maximal, by (10). □

(13) Fact. *If $\{h, k\}$ is a 2-set in $H^{\max}_{\rm T}$, then*
(i) $k|_{Z(h)}$ is an affine maximal for $Z_{\rm T}(h)$ in $Z(h)$;
(ii) k is an affine maximal for ${\rm T}\backslash Z(h)$;
(iii) $Z_{\rm T}(h) \cap Z_{\rm T}(k)$ is a GC$_n$-set in $Z(h) \cap Z(k)$.

Proof: $\#Z_{\rm T}(k) = \dim \Pi_{\le n} - \dim \Pi_{<n} = \dim \Pi_{\le n}(\mathbb{R}^{d-1})$, by the very definition of an affine maximal for a GC$_n$-set in $\mathbb{R}^d$. On the other hand, $Z_{\rm T}(k)$ is the union of two sets,

$$Z_{\rm T}(k) = (Z_{\rm T}(k) \cap Z(h)) \cup (Z_{\rm T}(k)\backslash Z(h)),$$

and, by (2),

$$\#(Z_{\rm T}(k) \cap Z(h)) \le \dim \Pi_{\le n}(\mathbb{R}^{d-1}) - \dim \Pi_{<n}(\mathbb{R}^{d-1})$$

(as the set of points in $Z(h)$ of the GC$_n$-set $Z_{\rm T}(k)$ in $Z(k)$), while, again by (2),

$$\#(Z_{\rm T}(k)\backslash Z(h)) \le \dim \Pi_{<n} - \dim \Pi_{<n-1} = \dim \Pi_{<n}(\mathbb{R}^{d-1})$$

(as the set of points in $Z(k)$ of the GC$_{n-1}$-set ${\rm T}\backslash Z(h)$ in $\mathbb{R}^d$) and, since these two upper bounds add up to the cardinality of the union of these two sets, the inequalities must be equalities. In particular, $h|_{Z(k)}$ is a maximal for $Z_{\rm T}(k)$ in $Z(k)$, and this also proves (i) since h and k enter symmetrically. Also, the equality in the second inequality proves (ii) and, since $Z_{\rm T}(h) \cap Z_{\rm T}(k) = Z_{\rm T}(k) \cap Z(h)$, (ii) implies (iii) by (3)(iv). □

(14) Fact. *If K is an r-set in $H^{\max}_{\rm T}$ and $h \in K$, then*
(i) for $n > 1$, $K\backslash h$ is an $(r-1)$-set of affine maximals for ${\rm T}\backslash Z(h)$, and
(ii) $\{k|_{Z(h)} : k \in K\backslash h\}$ is an $(r-1)$-set of affine maximals for $Z_{\rm T}(h)$. Therefore, by induction on r,
(iii) $\cap_{k\in K} Z_{\rm T}(k)$ is a GC$_n$-set in $S := \cap_{k\in K} Z(k)$. In particular,
(iv) the intersection of any d-set in $H^{\max}_{\rm T}$ consists of exactly one point from T, and
(v) any set of more than d affine maximals has an empty intersection. In other words,
(vi) $H^{\max}_{\rm T}$ is in general position, with any nonempty intersection of some of its elements containing the maximal number of points from T.
(vii) $H^{\max}_{\rm T} \subseteq H_{\rm T}$, with equality iff T *is natural.*
(viii) $\#H^{\max}_{\rm T} \le d + n \le \#H_{\rm T}$, with equality in one iff there is equality in the other iff T *is natural.*
(ix) $\#H^{\max}_{\rm T} > d$ iff, for every $\tau \in {\rm T}$, there is some $h \in H^{\max}_{\rm T}$ not containing τ iff every Lagrange polynomial has a maximal as a factor.

Proof: By (13)(ii), we know that every $k \in K\backslash h$ is a maximal for ${\rm T}\backslash Z(h)$, hence $K\backslash h$ is an $(r-1)$-set of affine maximals for ${\rm T}\backslash Z(h)$ unless ${\rm T}\backslash Z(h)$ is only a singleton. This proves (i).

By (13)(i), we only need to prove for (ii) that, *for any 3-set $\{h, j, k\}$ of maximals, $j|_{Z(h)} \ne k|_{Z(h)}$*. For this, recall from (3) that $Z_{\rm T}(h)$ is a GC$_n$-set in $Z(h)$, with $[\ell_\tau|_{Z(h)} : \tau \in Z_{\rm T}(h)]$ the Lagrange basis. Hence, for any $\tau \in Z_{\rm T}(h)\backslash(Z_{\rm T}(j) \cup Z_{\rm T}(k))$, $\ell_\tau|_{Z(h)}$ has both $j|_{Z(h)}$ and $k|_{Z(h)}$ among its n factors (e.g., by (12)) while, by (6), any two such factors must be distinct. This leaves only the case when $Z_{\rm T}(h)\backslash(Z_{\rm T}(j) \cup Z_{\rm T}(k)) = \emptyset$. But if, in that case, $j|_{Z(h)} = k|_{Z(h)}$, then $\emptyset = Z_{\rm T}(h)\backslash(Z_{\rm T}(j) \cup Z_{\rm T}(k)) = Z_{\rm T}(h)\backslash Z_{\rm T}(j)$ and this would contradict the fact that, by (13), $Z_{\rm T}(h) \cap Z_{\rm T}(j)$ is a proper subset of $Z_{\rm T}(h)$.

As to (vii), the inclusion follows already from (12), with equality in case T is natural while, conversely, equality implies, by (vi), that $H_{\rm T}$ is in general position and ${\rm T} = {\rm T}_{H_{\rm T}}$. Finally, (vi) implies that the map

$$\binom{H^{\max}_{\rm T}}{d} \to \mathbb{R}^d : K \mapsto \tau_K$$

that associates each d-set in $H_{\rm T}^{\max}$ with the unique point in its intersection is a well-defined map into T and 1-1, hence its range has cardinality $\binom{\#H_{\rm T}^{\max}}{d}$, and this cannot exceed $\#{\rm T} = \dim \Pi_{\le n} = \binom{d+n}{n}$. This shows that $\#H_{\rm T}^{\max} \le d+n$ with equality iff T is natural, i.e., the first inequality in (viii), while the second inequality (and its equality) are (8)(iv).

Finally, concerning (ix), if there are $\le d$ affine maximals, then, by (iii), their intersection contains some points from T and, for each of these points, there is no affine maximal not containing it. Conversely, if there is some $\tau \in {\rm T}$ contained in all affine maximals for T, then the intersection of all affine maximals is nontrivial, hence there cannot be more than d affine maximals, by (v). The second iff relies on (12), i.e., on the fact that an affine maximal h is a factor of every ℓ_τ with $h(\tau) \ne 0$. □

As a simple example for (ix), every planar GC$_2$-set T has > 2 affine maximals since, for each $\tau \in {\rm T}$, the other 5 points in T must lie on two lines, hence one line must have 3 points on it, providing an affine maximal not containing τ. The argument in [CaG2] showing that every planar GC$_3$-set has an affine maximal actually shows, similarly, that, for every τ in a planar GC$_3$-set, there is some affine maximal not containing it. Hence, also every planar GC$_3$-set has > 2 maximals. To be sure, since every planar GC$_n$-set with $n < 5$ is known to have an affine maximal, we know from (11)Theorem that every planar GC$_n$-set with $n < 5$ has at least three affine maximals.

Example Figure 1 captures a planar example from [ChY], but with the affine maximals highlighted.

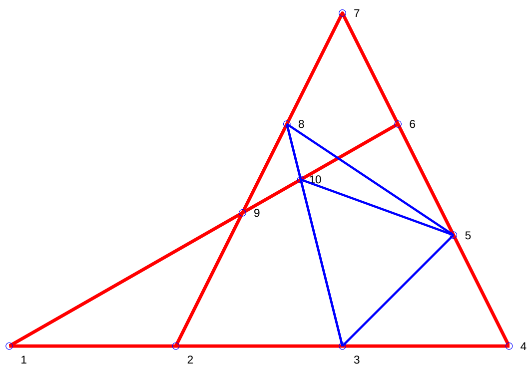

Figure 1. A planar GC$_3$-set with four maximals

There are 4 maximals, evidently in general position, hence four sites on one, an additional three sites on another, an additional 2 sites on a third, and one additional site on a fourth maximal, for altogether $10 = \dim \Pi_{\le 3}$ sites. In particular, every $\tau \in {\rm T}$ lies on at least one maximal, and those on only one maximal have the other three maximals as the three factors in their Lagrange polynomial (hence, on each maximal, there is exactly one such). Those on two maximals have the other two as factors in their Lagrange polynomial but their third factor is not a maximal. Interestingly, in this example, $\#K_\tau$ is minimal (i.e., equals $d = 2$) for every τ contained in two maximals while, for some other τ, K_τ has as many as four elements. Also, $\#H_{\rm T}$ exceeds the minimum $(n + d = 5)$ by 3, while $\#H_{\rm T}^{\max}$ fails to reach its maximum $(n + d = 5)$ by 1. □

The amount by which $\#H_{\rm T}^{\max}$ fails to reach its maximum, $n + 2$, has been termed the **default** [CaG1] or, better yet, the **defect** [CaG3] of a planar GC$_n$-set T, as it can also be interpreted as a measure of the extent to which T fails to be optimal, i.e., natural.

In the planar case, Carnicer and Gasca provide a kind of converse of (14)(i), namely the following

(15) Proposition ([CaG3]). *Let* T *be a planar GC$_n$-set, and h an affine maximal for it. Then, for any r-set, $r > 2$, of affine maximals for the GC$_{n-1}$-set* ${\rm T}\backslash Z(h)$*, at most one of its elements is not an affine maximal for* T.

This proposition is at the heart of their inductive proof of (11), as follows. Assume we know (as we would for $n \le 3$) that every planar GC$_{n-1}$-set has three affine maximals. Then any GC$_n$-set T having one maximal, h say, would actually have at least three since the GC$_{n-1}$-set ${\rm T}\backslash Z(h)$ has at least three, and two of these, by (15), would actually be affine maximals for T and necessarily different from h.

A counterexample

With the CG_d conjecture known to hold for any GC_1-set, and also to hold for any planar GC_n-set for $n < 5$, it seems reasonable to consider it for GC_2-sets in $\mathbb{R}^3$. And there, it already fails, as the following example shows.

Example Start with the standard quadratic principal lattice in $\mathbb{R}^3$, i.e., $\mathrm{T} = \{\alpha \in \mathbb{Z}_+^3 : |\alpha| \le 2\}$, as shown left-most in Figure 2.

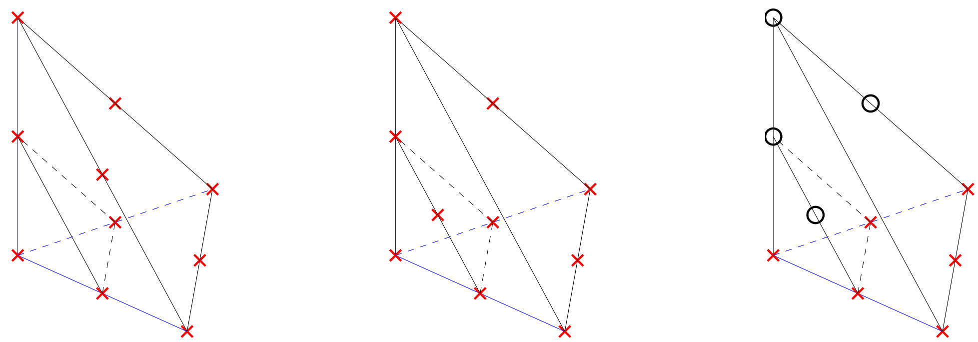

Figure 2. A 1-point variation (b) of a 3D quadratic principal lattice (a) has only 3 maximals but is still a GC-set (see (c)).

Move one of the sites, from the level $|\alpha| = 2$ to the level $|\alpha| = 1$, to obtain the point set T indicated in the middle figure. The set has evidently only 3 affine maximals but is still a GC_2-set. For the latter claim, we have to show that, for each $\tau \in \mathrm{T}$, there are two planes containing all of $\mathrm{T}\backslash\tau$, but not τ. This is evident for the sole point lying on the three remaining affine maximals (the point 0). Any other point is like the four points circled in the rightmost figure: there is one affine maximal not containing these four points, and it serves as one of the planes for each of them. For each of the four, the other plane is the unique one containing the other three of the four points, and this plane will not contain the point since the four points are not coplanar. □

The simplicity of this counterexample to the CG_3 conjecture makes me now doubt even the GC_2 conjecture, hence the original GM conjecture.

Amusingly, by (11), a counterexample to the GM conjecture is already provided by a planar GC-set with only two affine maximals. However, if there were a planar GC_n-set with exactly one or two affine maximals, then there would be a smallest one, T say. By (11), $n > 4$. Suppose T had two affine maximals, h and k, say. Then, by the minimality of T, $\mathrm{T}\backslash Z(h)$ would either have to have > 2 affine maximals or none. But the second possibility would contradict (14)(i), while the first possibility would imply, by (15), that T has more than 2 affine maximals, also a contradiction.

Thus a smallest counterexample, T say, with some affine maximal would have exactly one such maximal, h say, and, for it, $\mathrm{T}\backslash Z(h)$ would necessarily be a GC-set without any affine maximal.

So, there is no hope of using a simple modification, like the one that produced the trivariate counterexample, to get a planar counterexample. To put it positively, an example with just two affine maximals must have $n > 6$.

Since we now know that the CG_d conjecture does not hold in general, I cannot resist making the following

Definition. *A CG-set is a GC-set in $\mathbb{R}^d$ with more than d affine maximals, or, equivalently, with every Lagrange polynomial having a maximal as a factor.*

A final, sobering, remark: According to a personal communication from L. Bos, the (several) interpolation projectors based on GC_n-sets that he was able to study (e.g., the natural lattices for a given triangle) have norms that grow exponentially with n.

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