On the approximation by γ–polynomials

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1. **Introduction.** As was first pointed out by Hobby and Rice [5], many nonlinear approximation problems – such as approximation by exponential sums or by splines with variable knots – admit of the following abstract description: One has given a real normed linear space X and a map

$$\gamma : T \to X$$

from some subset T of the real line $\mathbb{R}$ to X. One then defines a γ–**polynomial of order** n to be any element of X of the form

$$p = p(\alpha, \tau) = \sum_{i=1}^{n} \alpha_i \gamma(\tau_i),$$

where $\alpha = (\alpha_1, \ldots, \alpha_n)$ is a real vector and $\tau = (\tau_1, \ldots, \tau_n) \subset T^n$ with $\tau_1 < \tau_2 < \cdots < \tau_n$. Let

$$\mathbb{P}_{\gamma,n}$$

denote the set of all γ–polynomials of order n. Then the approximation problem consists in finding, for given $f \in X$, a $p^* \in \mathbb{P}_{\gamma,n}$ such that

$$\|f - p^*\| = \operatorname{dist}(f, \mathbb{P}_{\gamma,n}) = \inf_{p \in \mathbb{P}_{\gamma,n}} \|f - p\|.$$

As usual, p^* is called a best approximation (b.a.) to f in (or, by elements of) $\mathbb{P}_{\gamma,n}$.

To give some examples, let $X = L_p[0,1]$ and set $\gamma(t) = G(\cdot, t)$, where $G(s,t)$ is defined on $[0,1] \times T$. With G Green's function for a k–th order ordinary linear initial value problem on $(0,1]$ and $T = [0,1)$, one has approximation by generalized splines. With $G(s,t) = \mathrm{e}^{st}$ and $T = \mathbb{R}$, one has approximation by exponential sums. With $G(s,t) = (1 + st)^{-1}$ and $T = (-1, \infty)$, one has an approximation problem which shares many features with approximation by rational functions. The last two examples lend themselves easily to an extension of T to the complex plane, and reveal their essential properties only after such an extension has been made [5]. Other examples may be found in [7].

A seemingly different example occurs in Numerical Analysis. Here X is the topological dual Y^* of a normed linear space Y of functions defined on T, and, for $t \in T$, $\gamma(t)$ is the linear functional of point evaluation at t, i.e.,

$$\text{for all} \quad y \in Y, \quad \gamma(t)y = y(t).$$

Best approximation by γ–polynomials of order n to $f \in X$ amounts to the construction of a best approximate rule of the form $\sum_{i=1}^{n} a_i y(t_i)$ for the evaluation of the linear functional f at y.

But it is not difficult to see that many approximation problems by γ–polynomials of fixed order can be considered to be special cases of the last example. For, with X, γ and T given, let Y be the linear space of functions on T whose general element y is given by

$$y(t) = y_\lambda = \lambda\gamma(t), \quad \text{all} \quad t \in T,$$

for some $\lambda \in X^*$. If the linear span of $\{\gamma(t) \mid t \in T\}$ is dense in X, then the map

$$\varphi : X^* \to Y : \lambda \mapsto y_\lambda$$

is one–one, hence Y is normed by

$$\|y_\lambda\| := \|\lambda\|, \quad \text{all} \quad y_\lambda \in Y,$$

and X is mapped linearly and isometrically into Y^* by $(\varphi^*)^{-1}$. In particular, $\gamma(t)$ is mapped by $(\varphi^*)^{-1}$ to the linear functional of point evaluation at t, all $t \in T$.

Hobby and Rice [5] studied the problems of existence, uniqueness, and characterization of best approximations by γ-polynomials when X is an L_p–space on $[0,1]$, for some p with $1 \le p \le \infty$. It is one purpose of this paper to give, in a more abstract setting, simpler proofs for some of their results.

2. **Existence of best approximations.** In most of the examples mentioned in the introduction, $\mathbb{P}_{\gamma,n}$ fails to be an existence set of for $n > 1$, since it fails to be closed. The reason for this is quite clear. If $t_1 \ne t_2$ are points in T, then $\mathbb{P}_{\gamma,n}$ contains the first divided difference

$$\gamma(t_1,t_2) = (t_2 - t_1)^{-1}(\gamma(t_2) - \gamma(t_1))$$

of γ at the points t_1, t_2. Hence, if γ is strongly differentiable at t_1, then $\gamma^{(1)}(t_1) = \lim_{t_2 \to t_1} \gamma(t_1,t_2)$ is in the closure of $\mathbb{P}_{\gamma,n}$, but usually fails to be in $\mathbb{P}_{\gamma,n}$.

To get an existence set, one must at least adjoin to $\mathbb{P}_{\gamma,n}$ all strong limits of the form

$$\lim_{t_0,\dots,t_k \to t} \gamma(t_0,\dots,t_k),$$

where $\gamma(t_0,\dots,t_k)$ denotes the k–th divided difference of γ at $t_0,\dots,t_k$.

We denote by $C_X^{(k)}(T)$ the linear space of all functions on T to X which are k times continuously strongly differentiable on T. If $g \in C_X[a,b]$, where $[a,b]$ is a finite interval, then

$$\omega_{[a,b]}(g,h) = \sup\{\|g(s) - g(t)\| \mid s,t \in [a,b], \quad |s-t| \le h\}$$

denotes the modulus of continuity of g on $[a,b]$.

(1) **Lemma.** Let $k > 0$, $[a,b]$ a finite interval, and $g \in C_X^{(k)}[a,b]$. If

$$a \le t_0 < t_1 < \cdots < t_k \le b$$

and $\widehat{t} \in [t_0,t_k]$, then

$$\|k!\, g(t_0,\dots,t_k) - g^{(k)}(\widehat{t})\| \le \omega_{[a,b]}(g^{(k)}, t_k - t_0).$$

Hence

$$\lim_{t_0,\dots,t_k \to \widehat{t}} g(t_0,\dots,t_k) = g^{(k)}(\widehat{t})/k!.$$

Proof. By Taylor's theorem with integral remainder (cf., e.g., Graves [3]), one has

$$g(t) = \sum_{i=0}^{k-1} [(t-a)^i/i!\,] g^{(i)}(a) + \int_a^b M_k(t;s) g^{(k)}(s)\,\mathrm{d}s,$$

where

$$M_k(t;s) \equiv (t-s)_+^{k-1}/(k-1)!.$$

Hence

$$g(t_0,\dots,t_k) = \int_a^b M_k(t_0,\dots,t_k;s) g^{(k)}(s)\,\mathrm{d}s.$$

Since

$$M_k(t_0,\dots,t_k;s) \ge 0 \quad \text{with equality iff} \quad s \notin (t_0,t_k),$$

and

$$\int_a^b M_k(t_0,\dots,t_k;s)\,\mathrm{d}s = \int_{t_0}^{t_k} M_k(t_0,\dots,t_k;s)\,\mathrm{d}s = 1/k!,$$

(cf., e.g., [1]), one has

$$\begin{aligned}\|k!\,g(t_0,\ldots,t_k)-g^{(k)}(\widehat{t})\| &= \|\int_{t_0}^{t_k} k!\,M_k(t_0,\ldots,t_k;s)[g^{(k)}(s)-g^{(k)}(\widehat{t})]\,\mathrm{d}s\| \\ &\le \int_{t_0}^{t_k} k!\,M_k(t_0,\ldots,t_k;s)\|g^{(k)}(s)-g^{(k)}(\widehat{t})\|\,\mathrm{d}s \\ &\le \max_{t_0\le s\le t_k}\|g^{(k)}(s)-g^{(k)}(\widehat{t})\|k!\int_{t_0}^{t_k} M_k(t_0,\ldots,t_k;s)\,\mathrm{d}s \\ &\le \omega_{[a,b]}(g^{(k)},t_k-t_0); \qquad \text{q.e.d.}\end{aligned}$$

If $\gamma\in C_X^{(k-1)}(T)$, then we define a k–**extended** γ–**polynomial of order** n to be any element of X of the form

$$p=p(\alpha,\tau)=\sum_{i=1}^{r}\sum_{j=0}^{m_i}\alpha_{ij}\gamma^{(i)}(t_i), \tag{2}$$

with

$$m_i+1\le k,\quad t_i\in T,\quad i=1,\ldots,r,$$
$$t_1<t_2<\cdots<t_r,\quad \sum_{i=1}^{r}(m_i+1)=n \quad .$$

Further, we take the τ–vector for p in (2) to be the vector

$$\tau=(t_1,\ldots,t_1,t_2,\ldots,t_2,t_3,\ldots,t_r),$$

with t_i appearing m_i+1 times, $i=1,\ldots,r$. Denote by

$$\mathbb{P}^k_{\gamma,n}$$

the set of all k–extended γ–polynomials of order n.

Remark. This rather narrow definition of extended γ–polynomial suffices for this paper. But one may want to enlarge it at times to include also *weak* limits of $\gamma(t_0,\ldots,t_k)$ as $t_0,\ldots,t_k\to t$. Again, the strong continuity of $\gamma^{(k-1)}$ is not essential, nor does it seem necessary to demand that $\gamma^{(k-1)}$ exist on all of T.

(3) **Theorem.** If (i) $T=[a,b]$ is a finite interval, (ii) $\gamma\in C_X^{(n-1)}(T)$ and (iii) the set

$$\{\gamma^{(j)}(t_i)\mid j=0,\ldots,m_i;\quad i=1,\ldots,r\}$$

is linearly independent whenever $a\le t_1<t_2<\cdots<t_r\le b$ and $\sum_{i=1}^{r}(m_i+1)=n$, then $\mathbb{P}^n_{\gamma,n}$ is the strong closure, $\overline{\mathbb{P}}_{\gamma,n}$, of $\mathbb{P}_{\gamma,n}$. Further, $\mathbb{P}_{\gamma,n}$ is boundedly compact in $\mathbb{P}^n_{\gamma,n}$; hence, $\mathbb{P}^n_{\gamma,n}$ is an existence set.

Proof. Let $\{p_m\}_{m=1}^{\infty}$ be a bounded sequence in $\mathbb{P}_{\gamma,n}$. Since T is compact, we may assume (after going to a subsequence, if necessary) that the sequence $\{\tau^{(m)}\}$ of corresponding τ–vectors converges to some $\tau\in T^n$. Hence, we can write p_m as

$$p_m=\sum_{i=1}^{r}\sum_{j=0}^{m_i}\alpha_{ij}^{(m)}j!\,\gamma(t_{i0}^{(m)},\ldots,t_{ij}^{(m)}),\quad m=1,2,\ldots, \tag{4}$$

where

$$\lim_{m\to\infty}t_{ij}^{(m)}=t_i,\quad j=0,\ldots,m_i;\quad i=1,\ldots,r,$$

with $a\le t_1<\cdots<t_r\le b$ and $\sum_{i=1}^{r}(m_i+1)=n$. Since $\gamma\in C_X^{(n-1)}$ and $m_i+1\le n$, all i, it follows from Lemma (1) that

$$\lim_{m\to\infty}j!\,\gamma(t_{i0}^{(m)},\ldots,t_{ij}^{(m)})=\gamma^{(j)}(t_k),\quad j=0,\ldots,m_i;\quad i=1,\ldots,r, \tag{5}$$

in norm. By assumption (iii), the set $\{\gamma^{(j)}(t_i) \mid j = 0, \ldots, m_i;\ i = 1, \ldots, r\}$ is linearly independent, hence there exists $K > 0$ and m_0 such that $m \geq m_0$ implies

$$\|p_m\| \geq K \max_{i,j} |\alpha_{ij}^{(m)}|.$$

Since $\{p_m\}$ is, by assumption, bounded, this implies that each of the n sequences $\{\alpha_{ij}^{(m)}\}_{m=1}^{\infty}$ is bounded. Hence, after going to a subsequence if necessary, we may assume that

$$\lim_{m\to\infty} \alpha_{ij}^{(m)} = \alpha_{ij}, \quad j = 0, \ldots, m_i; \quad i = 1, \ldots, r.$$

But then

$$\lim_{m\to\infty} p_m = \sum_{i=1}^{r} \sum_{j=0}^{m_i} \alpha_{ij} \gamma^{(j)}(t_i) \in \mathbb{P}_{\gamma,n}^n.$$

This proves that $\mathbb{P}_{\gamma,n}$ is boundedly compact in $\mathbb{P}_{\gamma,n}^n$, hence, that $\overline{\mathbb{P}}_{\gamma,n} \subset \mathbb{P}_{\gamma,n}^n$. As to the converse containment, observe that, by Lemma (1), for $j \leq n-1$ and $t \in T$,

$$\|\gamma^{(j)}(t) - j!\,\gamma(t, t+h, \ldots, t+jh)\| \leq \omega_{[a,b]}(\gamma^{(j)}, jh),$$

so that certainly $\mathbb{P}_{\gamma,n}^n \subset \overline{\mathbb{P}}_{\gamma,n}$. q.e.d.

Theorem (3) by itself has little applicability, since in practice either assumption (i) or assumption (ii) fails. E.g., in the case of approximation by exponential sums in $L_p[0,1]$, T is the whole real line and (i) fails, while assumption (ii) is certainly unjustified when approximating by splines of fixed order with a large enough number of variable knots.

But one can extend the argument for Theorem (3) to include these and other examples in the following way. Suppose that $f \in X$ is to be approximated by elements in $\mathbb{P}_{\gamma,n}$, and let $\{p_m\}$ be a minimizing sequence for f in $\mathbb{P}_{\gamma,n}$. Now write

$$p_m = q_m + r_m, \quad m = 1, 2, \ldots,$$

where the "nice" part, q_m, involves only those $\tau_i^{(m)}$ which converge to certain $t_i \in T$ and have, in the limit, no more than k coincident, where γ is known to be in $C_X^{(k-1)}(T)$. It can often be shown that the remainder sequence $\{r_m\}$ becomes eventually "orthogonal" to every $x \in X$ in the sense that

(6) $$\text{for all} \quad x \in X, \quad \varliminf_{m\to\infty} \|r_m + x\| \geq \|x\|.$$

This can be shown to imply, together with the boundedness of $\{p_m\}$, that $\{q_m\}$ is bounded, hence Theorem (3) then implies that some subsequence of $\{q_m\}$ converges to some $q \in \mathbb{P}_{\gamma,n}^k$. Further, (6) then implies that this subsequence is a minimizing sequence for f in $\mathbb{P}_{\gamma,n}$, since $\{p_m\}$ is, thus showing q to be a b.a. to f in $\mathbb{P}_{\gamma,n}$.

As a preliminary for a rigorous argument along the lines just indicated, we investigate in the next section the "limit" concept suggested by (6).

3. **Weak convergence concepts and existence sets.** Let S be a subset of the normed linear space X. To show that $f \in X$ has a b.a. in S, one usually proceeds as follows. One picks a minimizing sequence $\{p_m\}$ in S for f, and then attempts to show that some subsequence $\{p_{j(m)}\}$ of $\{p_m\}$ converges to some element p of S in some sense. The weaker the convergence concept used, the easier it should be to establish the compactness of $\{p_m\}$ in S. On the other hand, the convergence concept should be strong enough to imply that

(1) $$\|f - p\| \leq \lim_{m\to\infty} \|f - p_{j(m)}\| = \operatorname{dist}(f, S),$$

which then would finish the argument showing p to be a b.a. to f in S.

To give an example, one might use the following convergence concept: The sequence $\{x_m\}$ in X "converges" to x iff

$$\text{for all} \quad y \in X, \quad \overline{\lim_{m\to\infty}} \|x_m - y\| \geq \|x - y\|.$$

Clearly, if some subsequence of the minimizing sequence $\{p_m\}$ for f in S "converges" to some $p \in S$, then p is a b.a. to f in S. But since such "convergence" is not even preserved when going to a subsequence, we prefer the following slightly stronger notion.

(2) **Definition.** The sequence $\{x_m\}$ in the normed linear space X **comes close to** $x \in X$ iff

$$\text{for all} \quad y \in X, \quad \underline{\lim}_{m\to\infty} \|x_m - y\| \geq \|x - y\|.$$

A subset S of X is **nearly compact** iff every sequence in S has a subsequence which "comes close to" some element of S. With this, the preceding discussion establishes

(3) **Theorem.** Let S be a subset of the normed linear space X. If bounded subsets of S are nearly compact, then S is an existence set.

It seems worthwhile to point out by an example that bounded existence sets need not be nearly compact. For this, let $X = c_0 = \{f : \mathbb{N} \to \mathbb{R} \mid \lim_{n\to\infty} f(n) = 0\}$ be the Banach space of real null sequences with norm

$$\|f\| = \sup_{n\in\mathbb{N}} |f(n)|,$$

and set

$$S = \{f_n \mid n = 1, 2, \ldots\},$$

where

$$f_n(m) = \begin{cases} 1, & m \leq n \\ 0, & m > n, \qquad n, m = 1, 2 \ldots \quad . \end{cases}$$

Then S is an existence set: If $f \in X$, then there is n_0 such that $n \geq n_0$ implies $|f(n)| < 1/2$. But then, for all $n \geq n_0$,

$$|(f - f_n)(m)| = |(f - f_{n-1})(m)|, \quad \text{all} \quad m \neq n,$$

while

$$|(f - f_n)(n)| > |f(n)| = |(f - f_{n-1})(n)|.$$

Hence

$$\text{for all} \quad n \geq n_0, \quad \|f - f_n\| \geq \|f - f_{n-1}\|,$$

therefore,

$$\text{dist}\,(f, S) = \min_{n\leq n_0} \|f - f_n\|.$$

Further, S is bounded. But, S is not nearly compact. For, if $f \in X$, then there exists n_0 such that $|f(n_0)| < 1$. Set

$$g(n) = 2\delta_{n,n_0}, \quad n = 1, 2 \ldots.$$

Then $g \in X$ and

$$\|f - g\| \geq |(f - g)(n_0)| > 1 = \overline{\lim_{n\to\infty}} \|f_n - g\|.$$

Hence, every subsequence of the sequence $\{f_n\}$ in S "comes close to" no $f \in X$, let alone an $f \in S$.

In the remainder of this section, we make some simple remarks, and prove two technical lemmata concerning sequences which "come close to" some element, and, finally, prove an existence theorem.

(4) **Remarks.** (i) If $\{x_m\}$ "comes close to" x, then so does every subsequence of $\{x_m\}$.

(ii) But, a sequence may "come close to" more than one element. Thus, the sequence $\{x_m\}$ in $L_1[0, 1]$ given by

$$x_m(t) = \begin{cases} m, & 0 \leq t \leq 1/m, \\ 0, & 1/m < t \leq 1, \qquad m = 1, 2, \ldots \quad , \end{cases}$$

"comes close to" every $y \in L_1[0,1]$ with $\|y\|_1 \le 1$.

(iii) If $\{x_m\}$ "comes close to" x and $\{y_m\}$ converges in norm to y, then $\{x_m + y_m\}$ "comes close to" $x + y$.

(iv) If $\{x_m\}$ "comes close to" x and is bounded, and the sequence $\{\alpha_m\}$ of scalars converges to α, then $\{\alpha_m x_m\}$ "comes close to" αx.

(v) If $\{x_m\}$ converges in norm to x, then $\{x_m\}$ "comes close to" x and to no other element of X. Hence, if $\{x_m\}$ "comes close to" x, then all strongly convergent subsequences of $\{x_m\}$ converge to x.

(vi) If $\{x_m\}$ converges weakly to x, then $\{x_m\}$ "comes close to" x. This is just a restatement of the fact that the norm is lower semicontinuous with respect to weak sequential convergence.

A slight but important generalization of (4) (vi) concerns convergence with respect to a family of seminorms.

(5) **Definition.** Let X be a linear space, and Φ a family of seminorms on X. The sequence $\{x_m\}$ in X **converges Φ to** $x \in X$ iff

$$\text{for all} \quad \varphi \in \Phi, \quad \lim_{m\to\infty} \varphi(x - x_m) = 0.$$

(6) **Lemma.** Let X be a normed linear space, let Φ be a family of seminorms on X with the property that

$$\text{for all} \quad x \in X, \quad \sup_{\varphi\in\Phi} \varphi x = \|x\|. \tag{7}$$

If the sequence $\{x_m\}$ in X converges Φ to x, then $\{x_m\}$ "comes close to" x.

Proof. Since vector addition is continuous with respect to Φ–convergence, it is sufficient to prove that

$$\varliminf_{m\to\infty} \|x_m\| \ge \|x\|,$$

whenever $\{x_m\}$ converges Φ to x. For this, observe that $\varphi \in \Phi$ and $\lim_{m\to\infty} \varphi(x - x_m) = 0$ implies

$$\varphi x = \lim_{m\to\infty} \varphi x_m.$$

By (7), $\varphi x_m \le \|x_m\|$, therefore,

$$\varphi x = \lim_{m\to\infty} \varphi x_m = \varliminf_{m\to\infty} \|x_m\|,$$

hence, again by (7),

$$\|x\| = \sup_{\varphi\in\Phi} \varphi x \le \varliminf_{m\to\infty} \|x_m\|$$

q.e.d.

(8) **Lemma.** Let X be a normed linear space, and let $\{x_m\}$ be a bounded sequence in X which "comes close to" a certain $x \in X$; if $x = 0$, assume in addition that $\varliminf \|x_m\| > 0$. Further, let $\{y_m\}$ be a sequence in X converging in norm to some $y \in X$ which does not depend linearly on x. If the sequence $\{\alpha_m y_m + \beta_m x_m\}$ is bounded, then so is the sequence $\{\alpha_m y_m\}$.

Proof. Since $\{y_m\}$ converges in norm, it is bounded. Hence, if $\{\alpha_m y_m\}$ is not bounded, then, (after going to a subsequence if necessary) one has

$$\lim_{m\to\infty} |\alpha_m| = \infty.$$

By assumption, there exists C such that

$$\text{for all} \quad m, \quad \|\alpha_m x_m + \beta_m y_m\| \le C.$$

Hence,

$$\|y_m + \frac{\beta_m}{\alpha_m} x_m\| \le C/|\alpha_m| \xrightarrow[m\to\infty]{} 0,$$

showing that $\{\frac{\beta_m}{\alpha_m}x_m\}$ converges in norm to $-y$. In particular, $\{\frac{\beta_m}{\alpha_m}x_m\}$ is bounded, hence, as $\underline{\lim}\,\|x_m\| > 0$ by assumption, $\{\beta_m/\alpha_m\}$ is bounded, therefore, – after going to a subsequence, if necessary, – we may assume that

$$\lim_{m\to\infty} \beta_m/\alpha_m = \alpha.$$

But then, by (4) (iv) above, $\{(\beta_m/\alpha_m)x_m\}$ "comes close to" αx, hence, with (4) (v) above,

$$-y = \alpha x,$$

contradicting the assumption that y does not depend linearly on x. q.e.d.

(9) **Definition.** If $\{x_m\}$ is a sequence in the normed linear space X, then the **normalization of** $\{x_m\}$ is the sequence $\{\widehat{x}_m\}$, given by

$$\widehat{x}_m = \begin{cases} x_m/\|x_m\|, & x_m \neq 0 \\ 0 & , \quad x_m = 0 \end{cases} \quad , \qquad m = 1, 2, \ldots \quad .$$

(10) **Theorem.** Let X be a normed linear space, and let S, $\widehat{S}$ be nonempty subsets of X, closed under scalar multiplication, with the properties: (i) $\widehat{S} \subset \overline{S}$, the (norm) closure of S; (ii) every sequence in S can be written as the sum of two sequences, $\{q_m\}$ and $\{r_m\}$, such that the normalization $\{\widehat{q}_m\}$ of $\{q_m\}$ is compact in $\widehat{S}$, and some subsequence of the normalization $\{\widehat{r}_m\}$ of $\{r_m\}$ "comes close to" zero. Then, $\widehat{S}$ is an existence set.

Proof. Let $f \in X$, and let $\{p_m\}$ be a minimizing sequence for f in $\overline{S}$. We may assume that $\{p_m\}$ is in S. After going to a subsequence, if necessary, we may assume that $\{p_m\}$ is the sum of two sequences $\{q_m\}$ and $\{r_m\}$, such that the normalization $\{\widehat{q}_m\}$ of $\{q_m\}$ converges in norm to some $\widehat{q} \in \widehat{S}$, while the normalization $\{\widehat{r}_m\}$ of $\{r_m\}$ "comes close to" zero. Also,

$$\{p_m\} = \{\|q_m\|\widehat{q}_m + \|r_m\|\widehat{r}_m\}$$

is bounded.

We begin by proving that some subsequence of $\{q_m\}$ converges to an element of $\widehat{S}$. Since $\{\widehat{q}_m\}$ converges to $\widehat{q} \in \widehat{S}$, and $\widehat{S}$ is closed under scalar multiplication, it is sufficient to show that some subsequence of $\{q_m\}$ is bounded. This, in turn, is trivial in case $\underline{\lim}\,\|q_m\| = 0$. It is also trivial in case $\underline{\lim}\,\|r_m\| = 0$, since $\{q_m + r_m\}$ is bounded by assumption. Otherwise, $\|\widehat{q}\| = \lim \|\widehat{q}_m\| = 1$, hence $\widehat{q} \neq 0$, and $\underline{\lim}\,\|\widehat{r}_m\| = 1 > 0$. But then, the boundedness of $\{q_m\}$ follows from (8).

With this, we may assume, after going to a subsequence, if necessary, that $\{q_m\}$ converges in norm to some $q \in \widehat{S}$.

Next, we show that

$$\text{for all} \quad x \in X, \quad \overline{\lim}\,\|r_m - x\| \geq \|x\|.$$

This is trivially true in case $\lim \|r_m\| = \infty$. Otherwise, some subsequence $\{\|r_{j(m)}\|\}$ of $\{\|r_m\|\}$ converges to some scalar α. But then, by (4) (i) and (4) (iv), $\{r_{j(m)}\} = \{\|r_{j(m)}\|\widehat{r}_{j(m)}\}$ "comes close to" $\alpha \cdot 0 = 0$, since the bounded sequence $\{\widehat{r}_m\}$ "comes close to" 0, by assumption. Hence,

$$\text{for all} \quad x \in X, \quad \overline{\lim_{m\to\infty}}\,\|r_m - x\| \geq \underline{\lim_{m\to\infty}}\,\|r_{j(m)} - x\| \geq \|x\|.$$

But this implies that $q \in \widehat{S} \subset \overline{S}$ is a b.a. to f in $\overline{S}$. For, one has

$$\begin{aligned} \operatorname{dist}(f, \overline{S}) = \lim \|p_m - f\| &\geq \overline{\lim}(\|r_m + q - f\| - \|q_m - q\|) \\ &= \overline{\lim}\,\|r_m + q - f\| \\ &\geq \|q - f\| \quad . \end{aligned}$$

Specifically, with $f \in \overline{S}$, it follows that $f = q$ for some $q \in \widehat{S}$, hence, as $\widehat{S} \subset \overline{S}$, $\widehat{S} = \overline{S}$ follows. q.e.d.

To give a simple example, consider best approximation by exponential sums in $X = C[0,1]$. Here

$$\gamma(t) = \mathrm{e}^{ts}, \quad t \in T = (-\infty, \infty).$$

Since T is not compact, Theorem 2.(3) is not directly applicable. But one verifies that $\mathbb{P}^n_{\gamma,n}$ is in this case an existence set, and $\overline{\mathbb{P}}_{\gamma,n} = \mathbb{P}^n_{\gamma,n}$, by verifying that the assumptions of Theorem (10) are satisfied:

Set $S = \mathbb{P}_{\gamma,n}$, $\widehat{S} = \mathbb{P}^n_{\gamma,n}$. Since γ is infinitely often strongly differentiable, $\widehat{S} \subset \overline{S}$, by Lemma 2.(1). Further, $\{\gamma^{(j)}(t_i) | j = 0, \ldots, m_i;\ i = 1, \ldots, k\}$ is a linearly independent set whenever $t_1 < t_2 < \cdots < t_k$, and for arbitrary integers $m_1, \ldots, m_k$. If now $\{p_m\}$ is a bounded sequence in $S = \mathbb{P}_{\gamma,n}$, then after going to a subsequence if necessary, we can write

$$p_m = q_m + r_m, \quad q_m = \sum_{i=1}^{r} a_i^{(m)} \gamma(t_i^{(m)}), \quad r_m = \sum_{i=r+1}^{n} a_i^{(m)} \gamma(t_i^{(m)}), \quad m = 1, 2 \ldots,$$

where

$$\lim_{m \to \infty} t_i^{(m)} = \begin{cases} t_i \in T, & i \le r \\ \pm\infty, & i > r \end{cases} .$$

By Theorem 2.(3), some subsequence of the normalization $\{\widehat{q}_m\}$ of $\{q_m\}$ converges strongly to some $q \in \widehat{S} = \mathbb{P}^n_{\gamma,n}$. Further, it can be shown [8] that some subsequence of the normalization $\{\widehat{r}_m\}$ of $\{r_m\}$ converges pointwise to zero on $(0,1)$. Since, for $f \in C[0,1]$, $\|f\|_\infty = \sup_{s \in (0,1)} |f(s)|$, this implies, as in (6), that such a subsequence "comes close to" zero.

Remark. Theorem (10) is needed to complete the proof of Theorem 4 in [5].

4. **Existence of best generalized spline approximants.** Let M be the linear differential operator defined by

$$(1) \qquad (Mx)(s) = x^{(k)}(s) + \sum_{i=0}^{k-1} a_i(s) x^{(i)}(s), \quad s \in [0,1],$$

with $a_i \in C^{(i)}[0,1]$, $i = 0, \ldots, k-1$. Let $G(s,t)$ be Green's function for the initial value problem

$$(Mx)(s) = g(s), \quad s \in (0,1]$$
$$x^{(j)}(0) = 0, \quad j = 0, \ldots, k-1 \quad ,$$

and consider the curve

$$\gamma(t) = G(\cdot, t), \quad t \in T = [0,1),$$

in $X = L_p[0,1]$.

For $p < \infty$, γ is in $C_X^{(k-1)}(T)$, but not in $C_X^{(k)}(T)$, with

$$(3) \qquad \gamma^{(j)}(t) = (\partial/\partial t)^j G(\cdot, t), \quad j = 0, \ldots, k-1,$$

independently of p. Hence, $\mathbb{P}^k_{\gamma,n}$ does not change with p, and is a subset of $L_\infty[0,1]$. To emphasize this fact, we will denote this set of functions by $S^e_{M,n}$ in the sequel.

For $p = \infty$, γ is merely in $C_X^{(k-2)}(T)$, since $(\partial/\partial t)^{k-1} G(\cdot, t)$ has a jump discontinuity at t, hence can not be the uniform limit of the continuous function $\gamma^{(k-2)}(t, t+h)$ as $h \to 0$, except for $t = 0$. This fact produces complications in an existence proof, which will be dealt with elsewhere. Here, we will be satisfied with showing that $S^e_{M,n}$ is an existence set in L_∞.

We note that $\gamma^{(j)}(t)$ vanishes identically on $[0,1)$, and is k times continuously differentiable on $[t,1]$. Hence

$$\{\gamma^{(j)}(t_i) \mid j = 0, \ldots, m_i; \quad i = 1, \ldots, r\}$$

is a linearly independent set whenever $0 \le t_1 < \cdots < t_r < 1$ and $m_i + 1 \le k$, $i = 1, \ldots, r$. Each element $\sum_i \sum_j \alpha_{ij}\gamma^{(j)}(t_i)$ of $S^e_{M,n}$ reduces to a function in the k–dimensional null space, $\ker M$, of M on each of the intervals $(0, t_1)$, $(t_1, t_2), \ldots,$ $(t_r, 1)$. Hence, the elements of $S^e_{M,n}$ are generalized splines [4], and are (deficient) L–splines [9] only if $M = L^*L$ for some differential operator L of order $k/2$.

The following observations will be of use later on. Since $\ker M$ is finite–dimensional, all norms on $\ker M$ are equivalent, and bounded sets in $\ker M$ are compact in $\ker M$. Also, no element of $\ker M$ other than zero vanishes identically on any subinterval of $[0, 1]$ of positive length. This implies

(4) **Lemma.** Let $0 \le a < b \le 1$, and let $\{u_m\}$ be a sequence in $\ker M$ which is bounded in $L_1[a.b]$. Then $\{u_m\}$ is uniformly bounded in $[0, 1]$, hence some subsequence of $\{u_m\}$ converges to some $u \in \ker M$ uniformly on $[0, 1]$.

(5) **Definition.** With $0 \le t_1 < t_2 < \cdots < t_r \le 1$, and $1 \le p \le \infty$, let $\Phi_p(t_1, \ldots, t_r)$ denote the collection $\{\varphi_\partial | \partial > 0\}$ of seminorms on $L_p(0, 1)$, with φ_∂ defined by

$$\varphi_\partial f = \|f \cdot \chi_\partial\|_p \quad \text{all} \quad f \in L_p,$$

where

$$\chi_\partial(s) = \begin{cases} 1, & \text{for all } i, \quad |s - t_i| \ge \partial \\ 0, & \text{otherwise.} \end{cases}$$

(6) One observes that

$$\sup\{\varphi f | \varphi \in \Phi_p(t_1, \ldots, t_r)\} = \lim_{\partial \to 0} \|f \cdot \chi_\partial\|_p = \|f\|_p,$$

for all $f \in L_p[0, 1]$. Hence, if $\{f_m\} \subset L_p[0, 1]$ converges $\Phi_p(t_1, \ldots, t_r)$ to some $f \in L_p[0, 1]$, (cf. 3.(5), 3.(6)), then $\{f_m\}$ "comes close to" f in $L_q[0, 1]$, all $q \le p$.

For $p < \infty$, and $f \in L_p[0, 1]$, one has

$$\lim_{\partial \to \infty} \|f \cdot (1 - \chi_\partial)\|_p = 0. \tag{7}$$

This implies

(8) **Lemma.** Let $p < \infty$, $\{f_m\} \subset L_p[0, 1]$ converging $\Phi_p(t_1, \ldots t_r)$ to some $f \in L_p[0, 1]$. If $\lim_{m\to\infty} \|f_m\|_p = \|f\|_p$, then $\{f_m\}$ converges L_p to f.

Proof. One has

$$\begin{aligned} \|f_m\|_p^p =& \ \|f_m\chi_\partial\|_p^p + \|f_m(1 - \chi_\partial)\|_p^p \\ \ge& \ |\|f\chi_\partial\|_p - \|(f_m - f)\chi_\partial\|_p|^p + |\|(f_m - f)(1 - \chi_\partial)\|_p - \|f(1 - \chi_\partial)\|_p|^p. \end{aligned}$$

Let $\mu > 0$ be given. Then, by (7), there exists $\partial > 0$ such that

$$\|f \cdot (1 - \chi_\partial)\|_p < \eta,$$

hence

$$\|f\chi_\partial\|_p \ge \|f\|_p - \eta.$$

For this ∂, there exists $\overline{m}$ such that $m \ge \overline{m}$ implies

$$\|(f_m - f)\chi_\partial\|_p \le \eta,$$

hence

$$\|(f_m - f)(1 - \chi_\partial)\|_p \ge \|f_m - f\|_p - \eta.$$

But then, for all $m \ge \overline{m}$,

$$\|f_m\|_p^p \ge |\|f\|_p - 2\eta|^p + |\|f - f_m\|_p - 2\eta|^p,$$

therefore,

$$\|f\|_p^p = \lim_{m\to\infty} \|f_m\|_p^p \ge |\|f\|_p - 2\eta|^p + |\overline{\lim_{m\to\infty}} \|f - f_m\|_p - 2\eta|^p.$$

Since η is arbitrary, $\overline{\lim}_{m\to\infty} \|f - f_m\|_p = 0$ follows; q.e.d.

(9) **Lemma.** Let $0 \le t_1 < t_2 < \cdots < t_r < 1$, let $\varepsilon > 0$ be small enough so that $t_r - 1 < t_r - \varepsilon$, $t_r + \varepsilon \le 1$, and let $\{p_m\}$ be a sequence in $S^e_{M,n}$ which is bounded in $L_p[0, t_r + \varepsilon]$ for some $1 \le p \le \infty$, and whose corresponding sequence $\{\tau^{(m)}\}$ of τ–vectors converges to some vector

$$\tau = (t_1, \ldots, t_1, t_2, \ldots, t_2, t_3, \ldots, t_r).$$

Then some subsequence of $\{p_m\}$ converges $\Phi_\infty(t_1, \ldots, t_r)$ to an element of $S^e_{M,n}$.

Proof by induction on r, it being vacuously true for $r = 0$. Assume $r > 0$ and assume the correctness of the statement for $r - 1$. Let h be the number of components of τ which equal t_r. Then, after going to a subsequence if necessary, we may assume that

$$|\tau_i^{(m)} - t_r| \le \varepsilon/2, \quad i = n - h + 1, \ldots, n; \quad m = 1, 2, \ldots.$$

For $m = 1, 2, \ldots$, write

$$p_m = q_m + u_m,$$

where u_m involves only the last h terms of p_m. Then

$$p_m(s) = q_m(s), \quad \text{all} \quad s \in [0, t_r - \varepsilon], \quad m = 1, 2, \ldots,$$

since $\gamma^{(j)}(t)$ vanishes on $[0, t)$. It follows that $\{q_m\}$ is bounded in $L_p[0, t_r - \varepsilon]$, and is in $S^e_{M,n-h}$, hence, by induction hypothesis, we may assume (after going to a subsequence if necessary) that $\{q_m\}$ converges $\Phi_\infty(t_1, \ldots, t_{r-1})$ to some element q of $S^e_{M,n-h}$. This implies that for some constant c, and all large enough m,

$$|q_m(s)| \le c, \quad \text{all} \quad s \in [t_r - \varepsilon, 1],$$

hence, $\{u_m\}$ is bounded in $L_p[0, t_r + \varepsilon]$.

For $m = 1, 2, \ldots$, let u^+_m be the element of $\ker M$ for which

$$u^+_m(s) = u_m(s), \quad \text{all} \quad s \in [t_r + \varepsilon/2, 1].$$

By Lemma (4) and the boundedness of $\{u_m\}$ in $L_p[0, t_r + \varepsilon]$, we may assume, after going to a subsequence if necessary, that $\{u^+_m\}$ converges uniformly on $[0, 1]$ to an element $u^+ \in \ker M$. Set

$$u(s) = \begin{cases} 0, & s < t_r \\ u^+(s), & s \ge t_r \end{cases}.$$

Then $u \in S^e_{M,k}$, and $\{u_m\}$ converges $\Phi_\infty(t_r)$ to u. Hence, $\{p_m\}$ converges $\Phi_\infty(t_1, \ldots, t_r)$ to $q + u$, and $q + u \in S^e_{M,n-h+k}$. If $h \ge k$, we are done. Otherwise, use Theorem 2.(3) together with the fact that γ is $(k-1)$–times continuously differentiable in L_p and $\{\gamma^{(j)}(t_r)\}_{j=0}^{k-1}$ is linearly independent to conclude from the boundedness of $\{u_m\}$ in $L_p[0, t_r + \varepsilon]$ that some subsequence of $\{u_m\}$ converges L_p to an element $\widehat{u}$ of $\mathbb{P}^h_{\gamma,h}$. Since $\{u_m\}$ converges $\Phi_\infty(t_r)$ to u, it follows then by 3.(4)(v), Lemma 3.(6), and by (6) above, that $u = \widehat{u}$, hence $p = q + u \in S^e_{M,n}$; q.e.d.

(10) **Theorem.** Let $S^e_{M,n} = \mathbb{P}^k_{\gamma,n}$ be the set of k–extended γ–polynomials of order n in $L_1[0, 1]$, with γ given by (2). Then $S^e_{M,n}$ is an existence set in $L - p[0, 1]$, $1 \le p \le \infty$. For $p < \infty$, $S^e_{M,n}$ is approximatively compact, and is the strong closure of $\mathbb{P}_{\gamma,n}$.

Proof. Let $f \in L_p[0, 1]$, and let $\{p_m\}$ be a minimizing sequence for f in $S^e_{M,n}$. If $\{\tau^{(m)}\}$ is the corresponding sequence of τ–vectors, then, after going to a subsequence if necessary, we may assume that $\{\tau^{(m)}\}$ converges to some $\tau \in [0, 1]^n$. Let $t_1 < t_2 < \cdots < t_r$ be the distinct ones among the components of τ. Then $0 \le t_1 < t_2 \cdots < t_r \le 1$. Since $\{p_m\}$ is bounded in $L_p[0, 1]$, we may assume (after going to a subsequence if necessary) that $\{p_m\}$ converges $\Phi_\infty(t_1, \ldots, t_r)$ to some element $\widehat{p} \in S^e_{M,n}$: For, if $t_r < 1$, this follows directly from Lemma (9). If $t_r = 1$, then those terms of p_m which involve $\tau_i^{(m)}$ with $\lim_{m\to\infty} \tau_i^{(m)} = t_r$ converge trivially $\Phi_\infty(t_r)$ to zero, hence using Lemma (9) for the sequence of remaining terms, one reaches the same conclusion in this case.

By (6), it then follows that $\{p^{(m)}\}$ "comes close to" $\widehat{p}$ in $L_p[0,1]$, hence

$$\operatorname{dist}(f, S^e_{M,n}) = \lim \|p_m - f\|_p = \underline{\lim} \|p_m - f\|_p \geq \|\widehat{p} - f\|_p.$$

This shows $S^e_{M,n}$ to be an existence set. But, it also follows that

$$\lim \|p_m - f\|_p = \|\widehat{p} - f\|_p.$$

Hence, if $p < \infty$, then, by Lemma (8), $\{p_m - f\}$ converges in norm to $\widehat{p} - f$, therefore, $\{p_m\}$ converges in norm to $\widehat{p}$, showing $S^e_{M,n}$ to be approximatively compact; q.e.d.

(11) **Corollary.** If $1 < p < \infty$, then some $f \in L_p[0,1]$ has more than one b.a. in $S^e_{M,n}$.

Proof. By [2: Theorem 3], an approximatively compact subset S of $L_p[0,1]$, $1 < p < \infty$, is a uniqueness set iff S is convex. Since $S^e_{M,n}$ is closed under scalar multiplication, convexity of $S^e_{M,n}$ would imply that $S^e_{M,n}$ is a linear subspace of $L_p[0,1]$, contradicting the fact that $\{\gamma^{(j)}(t_i) \mid j = 0, \ldots, m_i;\ i = 1, \ldots, r\}$ is linearly independent for $0 \leq t_1 < t_2 < t_r < 1$ and $m_i + 1 \leq k$, $i = 1, \ldots, r$, with arbitrary r.

Remark, If $\{u_i \mid i = 1, \ldots, k\}$ is a basis for $\ker M$, then

$$G(s,t) = \begin{cases} h(s,t), & s \geq t \\ \\ 0, & s < t \end{cases},$$

where

$$h(s,t) = \sum_{i=1}^{k} u_i(s) v_i(t),$$

$\{v_i \mid i = 1, \ldots, k\}$ being the set of **adjunct** functions for $\{u_i \mid i = 1, \ldots, k\}$. This means [6:p. 669] that

$$\text{for all} \quad t \in [0,1], \quad \text{for} \quad j = 0, \ldots, k-1, \quad \sum_{i=1}^{k} u_i^{(j)}(t) v_i(t) = \delta_{j,k-1}.$$

The argument for Theorem (10) uses that $v_i \in C^{(k-1)}[0,1]$, $i = 1, \ldots, k$, and that, for each $t \in [0,1]$, the set of functions $\{\widehat{u}_j \mid j = 1, \ldots, k\}$, given by

$$\widehat{u}_{j+1}(s) = \sum_{i=1}^{k} u_i(s) v^{(j)}(t), \quad j = 0, \ldots, k-1,$$

is a basis for $\ker M$. This is ensured by the assumption that the coefficients of M (cf. (1)) satisfy $a_i \in C^{(i)}[0,1]$, $i = 0, \ldots, k-1$.

In particular, best approximation by the set of generalized spline functions with respect to $\ker M$ with m joints (in the sense of Greville [4]) is best approximation by $\{p(\alpha, \tau) \in S^e_{M,m+k} \mid \tau_1 = \cdots = \tau_k = 0\}$, and is covered by Theorem (10) with minor and obvious modifications.

5. **Strict monotonicity of the error.** In almost all of the examples given in the introduction, the linear span of $\{\gamma(t) \mid t \in T\}$ is dense in X. If X is smooth, this has the perhaps surprising consequence that, for all $f \in X$, $\operatorname{dist}(f, \mathbb{P}_{\gamma,n})$ is strictly desceasing as a function of n. Precisely, one has,

(1) **Theorem.** If X is smooth, and the linear span of $\{\gamma(t) \mid t \in T\}$ is dense in X, then, for all $f \in X$, $\operatorname{dist}(f, \mathbb{P}_{\gamma,1}) < \|f\|$ unless $f = 0$.

Proof. If $\operatorname{dist}(f, \mathbb{P}_{\gamma,1}) = \|f\|$ then, for all $t \in T$, 0 is a b.a. to f in the linear span of $\gamma(t)$, hence there exists $\lambda_+ \in X^*$ such that

$$\lambda_t f = \|f\|, \quad \|\lambda_t\| = 1 \tag{2}$$

and

$$\lambda_t \gamma(t) = 0. \tag{3}$$

If $f \neq 0$, then, by the smoothness of X, λ_t is uniquely determined by (2), i.e., λ_t does not depend on t. It then follows from (3), that some nonzero continuous linear functional on X vanishes on the linear span of $\{\gamma(t) \mid t \in T\}$, contradicting the denseness of the linear span of $\{\gamma(t) \mid t \in T\}$ in X. q.e.d.

(4) **Corollary.** If X is smooth, the linear span of $\{\gamma(t) \mid t \in T\}$ is dense in X, and $\mathbb{P}^k_{\gamma,n}$ is an existence set, then for all $f \in X$,

$$\operatorname{dist}(f, \mathbb{P}_{\gamma,n+1}) < \operatorname{dist}(f, \mathbb{P}_{\gamma,n}) \quad \text{or} \quad f \in \mathbb{P}^k_{\gamma,n}.$$

Thus, for $1 < p < \infty$, the distance of $f \in L_p[0,1]$ from $S^e_{M,n}$ is strictly decreasing with n unless and until $f \in S^e_{M,n}$ for some n.

6. **Characterization.** A first step toward the characterization of a b.a. to $f \in X$ in $\mathbb{P}^k_{\gamma,n}$ is the following

(1) **Theorem.** Assume that $\gamma \in C^{(k)}_X(T)$ and that T is an interval with endpoints a, b. Let

$$p = \sum_{i=1}^{r} \sum_{j=0}^{m_i} a_{ij} \gamma^{(j)}(t_i), \quad \text{with} \quad a < t_i < b, \quad a_{im_i} \neq 0, \quad i = 1, \ldots, r,$$

be an element of $\mathbb{P}^k_{\gamma,n}$, and let S be the linear span of

$$\{\gamma^{(j)}(t_i) \mid j = 0, \ldots, m_i + 1; i = 1, \ldots, r\} \cup \{\gamma(t_{r+i}) \mid i = 1, \ldots, h\}, \tag{2}$$

where $h = n - \sum_{i=1}^r (m_i + 1)$, and the additional points $t_{r+1}, \ldots, t_{r+h}$ (if any) are arbitrary in T.

If p is a b.a. to $f \in X$ in $\mathbb{P}^k_{\gamma,n}$, then p is a b.a. to f in S.

Proof. Assume by way of contradiction that, for some $q \in S$,

$$\|f - q\| < \|f - p\|. \tag{3}$$

Then q is of the form

$$q = \sum_{i=1}^{r} \sum_{j=0}^{m_i+1} b_{ij} \gamma^{(j)}(t_i) + \sum_{i=r+1}^{h} b_i \gamma(t_{r+i}).$$

Assume without loss of generality that $b_{i,m_i+1} \neq 0$, $i = 1, \ldots, m$, while $b_{i,m_i+1} = 0$ for $i > m$. Then we can write q as

$$q = \sum_{i=1}^{m} b_{i,m_i+1} \gamma^{(m_i+1)}(t_i) + \widehat{q},$$

where $\widehat{q}$ has the property that

$$p + \alpha \widehat{q} \in \mathbb{P}^k_{\gamma,n}, \quad \text{for all scalars} \quad \alpha.$$

Since $m_i + 1 \leq k$, all i, and γ is k times continuously differentiable, it follows from (3) and from Lemma 2.(1), that for some $\varepsilon > 0$ and all ε_i with $|\varepsilon_i| \leq \varepsilon$, $i = 1, \ldots, m$,

$$\|f - \widehat{q} - \sum_{i=1}^{m} b_{i,m_i+1} (m_i + 1)!\, \gamma(t_i, \ldots, t_i, t_i + \varepsilon_i)\| < \|f - p\|.$$

Therefore, for all ε_i with $|\varepsilon_i| \leq \varepsilon$, and all $\theta \in (0, 1]$,

$$\|f - u(\theta, \varepsilon_1, \ldots, \varepsilon_m)\| < \|f - p\|,$$

where

$$u(\theta, \varepsilon_1, \ldots, \varepsilon_m) = (1 - \theta) p + \theta \widehat{q} + \theta \sum_{i=1}^{m} b_{i,m_i+1} (m_i + 1)!\, \gamma(t_i, \ldots, t_i + \varepsilon_i).$$

In order to reach a contradiction, it is sufficient to show that $\theta, \varepsilon_1, \ldots, \varepsilon_m$ can be so chosen that $u(\theta, \varepsilon_1, \ldots, \varepsilon_m) \in \mathbb{P}^k_{\gamma,n}$. One observes that for $\varepsilon_i \neq 0$, the $(m_i+1)st$ divided difference of γ at $t_i, \ldots, t_i$, $t_i + \varepsilon_i$ can be written as

$$\gamma(t_i, \ldots, t_i, t_i + \varepsilon_i) = \varepsilon_i^{-m_i-1}[\gamma(t_i + \varepsilon_i) - \sum_{j=0}^{m_i} \varepsilon_i^j \frac{1}{j!} \gamma^{(j)}(t_i)].$$

Hence,

$$u(\theta, \varepsilon_1, \ldots, \varepsilon_m) = \sum_{i=1}^{m} c_i \gamma^{(m_i)}(t_i) + v,$$

where $v \in \mathbb{P}^k_{\gamma,n}$, and

$$c_i = (1-\theta)a_{i,m_i} + \theta[b_{i,m_i} - \varepsilon_i^{-1}(m_i+1)b_{i,m_i+1}], \quad i = 1, \ldots, m.$$

Hence, to get $c_i = 0$, all i, one needs that

$$\varepsilon_i = \theta(m_i+1)b_{i,m_i+1}/(\theta b_{i,m_i} + (1-\theta)a_{i,m_i}), \quad i = 1, \ldots, m. \tag{4}$$

Since b_{i,m_i+1}, a_{i,m_i} are not zero, $i = 1, \ldots, m$, it is clearly possible to find $\theta \in (0, 1]$, and $\varepsilon_1, \ldots, \varepsilon_m$ with $0 < |\varepsilon_i| \leq \varepsilon$, all i, such that (4) is satisfied; q.e.d.

Remark. The linear span of (2) is, at best, of dimension $n+r$ even though the general element of $\mathbb{P}^k_{\gamma,n}$ depends on $2n$ parameters. This apparent loss of degrees of freedom (when $r < n$) is, in part, due to the fact that the vanishing of a linear parameter eliminates also the corresponding (non-linear) t–parameter. The phenomenon of varisolvence of exponential sums and other γ–polynomials originates this way. But, this loss also occurs when two or more of the τ_i's coalesce without the vanishing of any of the linear parameters. The implications of this fact for the geometry of the set $\mathbb{P}^k_{\gamma,n}$ are quite striking and should make $\mathbb{P}^k_{\gamma,n}$ worthy of the attention of accomplished topologists.

References

[1] H. B. Curry and I. J. Schoenberg, On Pólya frequency functions IV: the fundamental spline functions and their limits, J. Analyse Math.**17**(1966), 71–107.

[2] N V. Efimov and S. B. Stechkin, Approximative compactness and Tchebycheff sets, Dokl. Akad. Nauk SSSR**140**(1961), 522–524.

[3] R. E. Graves, Riemann integration and Taylor's theorem in general analysis, Trans. Amer. Math. Soc.**29**(1927), 163–177.

[4] T. N. E. Greville (1964), "Interpolation by generalized spline functions", MRC 476.

[5] C. R. Hobby and J. R. Rice, Approximation from a curve of functions, Arch. Rat. Mech. Anal.**27**(1967), 91–106.

[6] T. Muir, *A treatise on the theory of determinants*, Longmans, Green and Co. (London), 1933.

[7] J. R. Rice, *The Approximation of Functions Vol. II*, Addison-Wesley (Reading, Mass.), 1969.

[8] J. R. Rice, Chebyshev approximation by exponentials, J. Soc. Indust. Math.**10**(1962), 149–161.

[9] M. H. Schultz and R. S. Varga, *L*-splines, Numer. Math.**10**(1967), 345–369.

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