

Extremising the L_p -norm of a monic polynomial with roots in a given interval and Hermite interpolation

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Abstract:

Let Θ be a multiset of n points in $[a, b]$, and

$$\omega_\Theta := \prod_{\theta \in \Theta} (\cdot - \theta).$$

In this paper we investigate the extrema of $\Theta \mapsto \|\omega_\Theta\|_p$. Consequences of the results we obtain include: L_p -bounds for Hermite interpolation, error estimates for Gauss quadrature formulæ with multiple nodes, and certain quantitative statements about good and best approximation by polynomials of fixed degree.

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1. Introduction

Let Θ be a multiset of n points in $[a, b]$, and

$$\omega_\Theta := \prod_{\theta \in \Theta} (\cdot - \theta) \in \Pi_n.$$

In this paper we discuss the size of $\|\omega_\Theta\|_p$ as a function of Θ . This constant $\|\omega_\Theta\|_p$ arises naturally in error bounds for Hermite interpolation. For example, if $H_\Theta f \in \Pi_{<n}$ is the Hermite interpolant to f at the points Θ (counting multiplicities), then

$$\|f - H_\Theta f\|_p \leq \frac{\|\omega_\Theta\|_p}{n!} \|D^n f\|_\infty, \quad \forall f \in W_\infty^n, \quad (1.1)$$

with equality iff $f \in \Pi_n$.

In Section 2, we show that if some of the points in Θ are prescribed, then $\|\omega_\Theta\|_p$ is maximised by an appropriate choice of the remaining points from $\{a, b\}$. As an application, we provide L_p -error bounds for Hermite interpolation, in cases where some of the points in Θ are known to be from $\{a, b\}$.

In Section 3, we show that $\|\omega_\Theta\|_p$ is minimised for a certain choice of Θ , consisting of n distinct points in (a, b) . These points are precisely the roots of the error in the unique best L_p -approximation from $\Pi_{<n}$ to any polynomial of (exact) degree n . This result is closely related to Gauss quadrature formulæ with multiple nodes (via s -orthogonal polynomials), for which we are able to give error bounds. Other applications in this section include error bounds for best L_p -approximation by polynomials of fixed degree.

2. Maximising $\|\omega_\Theta\|_p$

Throughout, Θ will be used for a multiset of n points from $[a, b]$. Our functions will be defined on the closed interval $[a, b]$, $b - a > 0$. Thus $\|\cdot\|_p := \|\cdot\|_{L_p[a, b]}$, and $W_p^n := W_p^n[a, b]$ the **Sobolev** space of functions f with $D^{n-1}f$ absolutely continuous on $[a, b]$ and $D^n f \in L_p := L_p[a, b]$. The space of polynomials of degree $\leq n$ will be denoted by Π_n .

(2.1) Theorem. *Let Θ' be a fixed multiset of $\leq n$ points from $[a, b]$. The maximum of*

$$\{\|\omega_\Theta\|_p : \Theta \supset \Theta'\}$$

is attained when $\Theta \setminus \Theta'$ is in $\{a, b\}$.

Proof. Let \mathcal{C} be the convex hull of the compact set

$$\mathcal{W} := \{\omega_\Theta : \Theta \supset \Theta'\} \subset \Pi_n.$$

Since $\mathcal{C} \rightarrow \mathbb{R} : f \mapsto \|f\|_p$ is a continuous convex function, it attains its maximum at an extreme point of \mathcal{C} . Since each point in $\mathcal{C} \setminus \mathcal{W}$ can be written as a (nontrivial) convex combination of two points in \mathcal{C} , the extreme points of \mathcal{C} are contained in \mathcal{W} .

Suppose $\omega_\Theta \in \mathcal{W}$ is an extreme point of \mathcal{C} , with $\{\xi, \Theta'\} \subset \Theta$, for some $\xi \in (a, b)$. Then for small ε

$$\omega_\Theta = \frac{1}{2}(\cdot - (\xi - \varepsilon))\omega_{\Theta \setminus \xi} + \frac{1}{2}(\cdot - (\xi + \varepsilon))\omega_{\Theta \setminus \xi},$$

a convex combination of points in \mathcal{W} , contradicting the fact ω_Θ is an extreme point of \mathcal{C} . Thus the extreme points of \mathcal{C} are given by ω_Θ , where Θ consists of Θ' together with points from $\{a, b\}$. \square

We now use this result to find the maximum of $\Theta \mapsto \|\omega_\Theta\|_p$ over $\mathcal{A}_n(i, j)$, which is, by definition, the set of those Θ containing one endpoint at least i times and the other at least j times, where $i + j \leq n$. Notice that $\mathcal{A}_n(i, j)$ is symmetric in i, j , that $\mathcal{A}_n(0, 0)$ consists of all Θ , and that $\mathcal{A}_n(m, n - m)$ has at most two elements.

Let B be the **beta function**

$$B(x, y) := \int_0^1 t^{x-1}(1-t)^{y-1} dt = \frac{B(x), (y)}{B(x+y)}, \quad \forall x, y > 0.$$

Recall that B is symmetric, and satisfies: $0 < B(x, y) \leq \min\{1, 1/\max\{x, y\}\}$, $\forall x, y > 0$.

(2.2) Corollary. Let $m := \min\{i, j\}$, and $0^0 := 1$. Then

$$\max_{\Theta \in \mathcal{A}_n(i, j)} \|\omega_\Theta\|_p = (b-a)^{n+\frac{1}{p}} \begin{cases} B(pm+1, p(n-m)+1)^{\frac{1}{p}}, & 1 \leq p < \infty \\ m^m(n-m)^{n-m}/n^n, & p = \infty, \end{cases}$$

with the maximum achieved iff $\Theta \in \mathcal{A}_n(m, n-m)$.

Proof. By Theorem (2.1), the maximum occurs when all the points in Θ are from $\{a, b\}$. For $\Theta \in \mathcal{A}_n(k, n-k)$, we compute

$$\|\omega_\Theta\|_p = (b-a)^{n+\frac{1}{p}} \begin{cases} B(pk+1, p(n-k)+1)^{\frac{1}{p}}, & 1 \leq p < \infty \\ k^k(n-k)^{n-k}/n^n, & p = \infty, \end{cases}$$

and then observe that the maximum of $\|\omega_\Theta\|_p$ over $m \leq k \leq n - \max\{i, j\}$ occurs when $k = m$. \square

This improves upon the weaker result of Agarwal [Ag91], that

$$\max_{\Theta \in \mathcal{A}_n(i, j)} \|\omega_\Theta\|_p \leq (b-a)^{n+\frac{1}{p}} (2B_{1/2}(pm+1, p(n-m)+1))^{\frac{1}{p}}, \quad 1 \leq p < \infty. \quad (2.3)$$

Here $B_{1/2}$ is the **incomplete beta function**

$$B_{1/2}(x, y) := \int_0^{1/2} t^{x-1}(1-t)^{y-1} dt, \quad \forall x, y > 0.$$

We observe that $B_{1/2}$ is not symmetric, and satisfies $B(x, y) \leq 2B_{1/2}(x, y)$, $\forall 1 \leq x \leq y$, with strict inequality unless $x = y$. Thus Corollary (2.2) gives better bounds than (2.3) whenever $m \neq n - m$, and the same bounds otherwise.

L_p -Error bounds for Hermite interpolation

Let $1 \leq p, q \leq \infty$, and $H_\Theta f \in \Pi_{<n}$ be the Hermite interpolant to f at Θ (counting multiplicities). Recently, see Waldron [Wa94], the author has shown that:

$$\|f - H_\Theta f\|_p \leq \text{const}_{n,p,q,\Theta} (b-a)^{n+\frac{1}{p}-\frac{1}{q}} \|D^n f\|_q, \quad \forall f \in W_q^n, \quad (2.4)$$

where

$$\text{const}_{n,p,q,\Theta} := \frac{n^{\frac{1}{q}}}{n!} \left\| x \mapsto \frac{\omega_\Theta(x)}{(\text{diam}\{x, \Theta\})^{1/q}} \right\|_p (b-a)^{-(n+\frac{1}{p}-\frac{1}{q})}.$$

Here **diam** denotes the diameter of a (multi)set of points. Using Corollary (2.2), we may estimate the constants $\text{const}_{n,p,q,\Theta}$.

(2.5) Hermite error bounds. *Let $\Theta \in \mathcal{A}_n(i, j)$, with $m := \min\{i, j\} > 0$. Then*

$$\text{const}_{n,p,q,\Theta} \leq \frac{n^{\frac{1}{q}}}{n!} \begin{cases} B(pm+1, p(n-m)+1)^{\frac{1}{p}}, & 1 \leq p < \infty \\ m^m (n-m)^{n-m} / n^n, & p = \infty. \end{cases}$$

Proof. Since $m > 0$, $\text{diam}\{x, \Theta\} = b - a$, and we obtain

$$\text{const}_{n,p,q,\Theta} = \frac{n^{\frac{1}{q}}}{n!} \|\omega_\Theta\|_p (b-a)^{-(n+\frac{1}{p})}.$$

To this, apply Corollary (2.2). □

This improves upon the bounds in [Ag91], which involve $B_{1/2}$. In the case $m = 0$, the above argument can be modified, by observing that

$$\left\| x \mapsto \frac{\omega_\Theta(x)}{(\text{diam}\{x, \Theta\})^{1/q}} \right\|_p \leq \left(\|\omega_\Theta\|_{p(1-\frac{1}{nq})} \right)^{1-\frac{1}{nq}}. \quad (2.6)$$

For a full discussion, including the cases of equality in (2.4), and mention of some related inequalities of Brink [Br72], see [Wa94].

Application to the solution of ordinary differential equations

The Hermite error bounds (2.5) can be applied to the analysis of the boundary value problem: $D^n f = g$, with Hermite multipoint conditions given by $H_\Theta f = 0$. See, e.g., Agarwal and Wong [AW93].

3. Minimising $\|\omega_\Theta\|_p$

To show that $\Theta \mapsto \|\omega_\Theta\|_p$ has a unique minimum, we use the following well-known result, see, e.g., [DL93:Ch.3,§5,§10].

(3.1) Theorem. *If $P \subset C[a, b]$ is an n -dimensional Haar space, then g^* , the unique best L_p -approximation to $f \in C[a, b]$ from P , interpolates f at n distinct points in (a, b) .*

For $1 \leq p < \infty$, by the characterisation theorem for best L_p -approximation (see, e.g., [DL93:p83]) g^* is uniquely determined by

$$\int_a^b |f - g^*|^{p-1} \text{sign}(f - g^*) g = 0, \quad \forall g \in P, \quad (3.2)$$

where **sign** denotes the signum function.

For a more detailed analysis, dealing with the interlacing of the zeros of errors in best L_p -approximations, see Pinkus and Ziegler [PZ76].

Taking $P = \Pi_{<n}$, and $f = (\cdot)^n$, we obtain:

(3.3) Corollary. *There is a unique Θ which minimises $\|\omega_\Theta\|_p$. This Θ consists of n distinct points in (a, b) , which are the roots of $M_{n,p} \in \Pi_n$, which is, by definition, the error in the unique best L_p -approximation to $(\cdot)^n$ from $\Pi_{<n}$. We have*

$$\frac{1}{4^n} (b-a)^{n+\frac{1}{p}} \leq \min_{\Theta} \|\omega_\Theta\|_p = \|M_{n,p}\|_p \leq \frac{2}{4^n} (b-a)^{n+\frac{1}{p}},$$

with equality only when $p = 1, \infty$, respectively. In addition

$$\min_{\Theta} \|\omega_\Theta\|_2 = \|M_{n,2}\|_2 = \frac{(n!)^2}{(2n)! \sqrt{2n+1}} (b-a)^{n+\frac{1}{2}}.$$

Proof. Taking $P = \Pi_{<n}$, and $f = (\cdot)^n$, in Theorem (3.1), we see that $M_{n,p}$, the error in best approximation, is of the form $M_{n,p} = \omega_\Theta$, for a certain Θ consisting of distinct points in (a, b) . Thus, this choice of Θ uniquely minimises $\|\omega_\Theta\|_p$ (even if Θ is not restricted to lie within $[a, b]$).

From Hölder's inequality, it follows that

$$p \mapsto C_p := \|M_{n,p}\|_p (b-a)^{-(n+\frac{1}{p})} = \min_{\Theta} \|\omega_\Theta\|_p (b-a)^{-(n+\frac{1}{p})}$$

is strictly increasing.

For $p = 1$, $M_{n,p}$ is, up to an affine change of variables equal to U_n , the **Chebyshev polynomial of the second kind**, and we calculate

$$C_1 = \|2^{-n} U_n\|_1 2^{-(n+\frac{1}{1})} = \frac{1}{4^n}.$$

Similarly $M_{n,2}$, $M_{n,\infty}$ are P_n , T_n . i.e., the **Legendre, Chebyshev polynomials**, respectively, and

$$C_2 = \left\| \frac{2^n (n!)^2}{(2n)!} P_n \right\|_2 2^{-(n+\frac{1}{2})} = \frac{(n!)^2}{(2n)! \sqrt{2n+1}},$$

$$C_\infty = \|2^{-(n-1)} T_n\|_\infty 2^{-(n+\frac{1}{\infty})} = \frac{2}{4^n}.$$

The facts about U_n, P_n, T_n that we have used above can be found in any standard book on orthogonal polynomials. \square

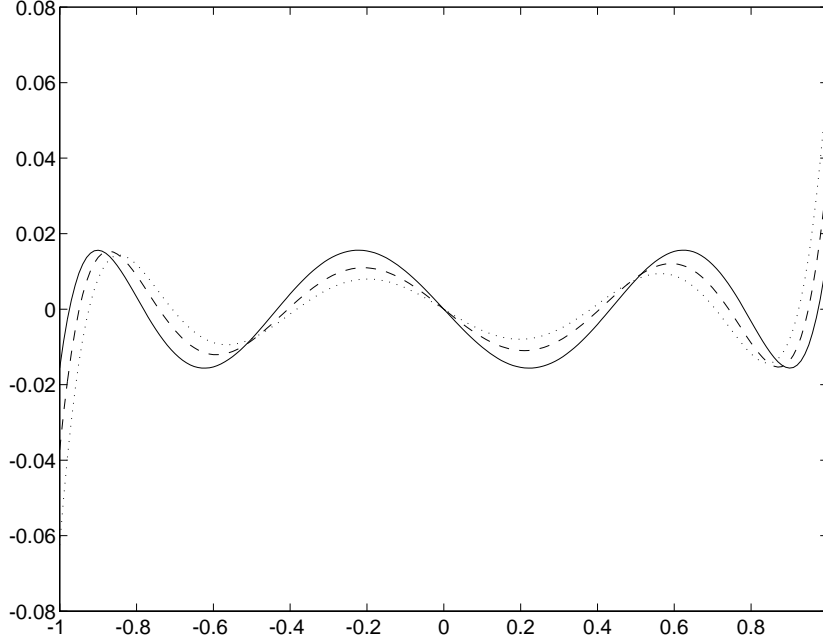


Fig 3.1 Graphs of the polynomials $M_{7,1}$ (dotted), $M_{7,2}$ (dashed), and $M_{7,\infty}$ (line)

Corollary (3.3) is a collection of classical results from the theory of orthogonal polynomials, see, e.g., Szegő [Sz59:p41]. One generalisation of it, of interest to approximation theorists, is Fejér’s convex hull theorem, see Davis [Da75:p244].

As mentioned in the proof, when $p = 1, 2, \infty$, the $M_{n,p}$ are well known orthogonal polynomials. For other values of p , no recurrence relations are known for $M_{n,p}$. By (3.2), for $1 \leq p < \infty$, $M_{n,p}$ is the unique $m \in \Pi_n$ with leading term $(\cdot)^n$ and

$$(3.4) \quad \int_a^b |m|^{p-1} \text{sign}(m) g = 0, \quad \forall g \in \Pi_{<n}.$$

It is possible to view (3.4) as a nonlinear system of equations in the roots of $M_{n,p}$ (with a unique solution), and solve it numerically. For two different iterative schemes, together with sample results, see Burgoyne [Bu67], and Vincenti [Vi86].

Good approximation by polynomials

Combining Corollaries (2.2) and (3.3), we obtain:

$$\frac{1}{2} \frac{4^n}{(np + 1)^{1/p}} \leq \frac{\max_{\Theta} \|\omega_{\Theta}\|_p}{\min_{\Theta} \|\omega_{\Theta}\|_p} \leq \frac{4^n}{(np + 1)^{1/p}},$$

where $(np + 1)^{1/p} := 1$, when $p = \infty$. Thus, a good choice of Θ can greatly improve the size of the constant $\|\omega_{\Theta}\|_p$ occurring in (1.1), over that for a poor choice.

For example, with Θ_{Eq} consisting of points with equal spacing $h := (b - a)/(n - 1)$, and Θ_{Ch} the Chebyshev points, Isaacson and Keller [IK66:p267] provide the estimate:

$$\frac{\|\omega_{\Theta_{\text{Eq}}}\|_{\infty}}{\|\omega_{\Theta_{\text{Ch}}}\|_{\infty}} > \frac{\sqrt{2}}{n-1} \left(\frac{4}{e}\right)^{n-1},$$

for large n , in support of doing Lagrange interpolation at the Chebyshev points.

Best approximation by polynomials

By Theorem (3.1), the unique best L_p -approximation to $f \in C[a, b]$ from $\Pi_{<n}$ is obtained by Lagrange interpolation at n points in (a, b) . Thus, in view of (1.1), we expect some relation between $\min_{\Theta} \|\omega_{\Theta}\|_p$, and the error

$$E_{n,p}(f) := \inf_{g \in \Pi_{<n}} \|f - g\|_p$$

in best L_p -approximation. The main result in this direction, which is due to Phillips, is the following.

(3.5) Theorem ([Ph70]). *If $f \in C^n[a, b]$, then $\exists \xi \in [a, b]$, such that*

$$E_{n,p}(f) = \frac{\|M_{n,p}\|_p}{n!} |D^n f(\xi)| \leq \frac{\|M_{n,p}\|_p}{n!} \|D^n f\|_{\infty},$$

with equality iff $f \in \Pi_n$.

Along the same lines, Fink [Fi77], defines $B(n, p, q)$ as the smallest constant such that

$$E_{n,p}(f) \leq B(n, p, q)(b - a)^{n + \frac{1}{p} - \frac{1}{q}} \|D^n f\|_q, \quad \forall f \in W_q^n,$$

and gives some equivalent definitions.

Since best approximations are given by Lagrange interpolation, we might hope to estimate $B(n, p, q)$ by interpolating f at some Θ , as does Phillips in Theorem (3.5), where he shows:

$$B(n, p, \infty) = \frac{\|M_{n,p}\|_p}{n!} (b - a)^{-(n + \frac{1}{p})}. \quad (3.6)$$

Pursuing this idea, we are able to estimate $B(n, p, q)$ to within a factor of $8n$.

(3.7) Estimate for Fink's constant.

$$\frac{1}{n!} \frac{1}{4^n} \leq B(n, p, q) \leq \frac{n^{\frac{1}{q}}}{n!} \left(\frac{2}{4^n}\right)^{1 - \frac{1}{nq}} \leq 8n \frac{1}{n!} \frac{1}{4^n}.$$

Proof. Let $b - a = 1$. First the lower bound. Since $M_{n,p}$ is the error in approximating $f = (\cdot)^n$, which has $D^n f = n!$, we must have

$$B(n, p, q) \geq \frac{\|M_{n,p}\|_p}{n!} \geq \frac{1}{n!} \frac{1}{4^n}.$$

By (2.4) and (2.6):

$$B(n, p, q) \leq \frac{n^{\frac{1}{q}}}{n!} \|\omega_{\Theta}^{1-\frac{1}{nq}}\|_p = \frac{n^{\frac{1}{q}}}{n!} \left(\|\omega_{\Theta}\|_{p(1-\frac{1}{nq})} \right)^{1-\frac{1}{nq}}. \quad (3.8)$$

Choosing $\omega_{\Theta} = M_{n,p(1-1/nq)}$, then applying Corollary (3.3) to (3.8), we obtain the upper bound. \square

Gauss quadrature formulæ with multiple nodes

The polynomials $M_{n,p}$ have the following interesting connection with quadrature, see Turan [Tu50], also Ghizzetti and Ossicini [GO70:p74].

If $p = 2s + 2$, $s = 0, 1, 2, \dots$, then (3.2) reduces to

$$\int_a^b m^{2s+1} g = 0, \quad \forall g \in \Pi_{<n}.$$

The corresponding m ($= M_{n,2s+2}$) is called **s -orthogonal** (with weight dx).

There is a quadrature formula of the form

$$Q(f) := \sum_{i=0}^{2s} \sum_{v \in \Theta} w(i, v) D^i f(v), \quad (3.9)$$

for the integral $I(f) := \int_a^b f$, of precision $(2s + 2)n - 1$, iff Θ is the zeros of $M_{n,2s+2}$. In keeping with the special case $s = 0$, such a Q is referred to as a **Gauss formulæ with multiple nodes**, or simply as a **s -Gauss** formula, and $M_{n,2s+2}$ is called a **Legendre s -polynomial**.

The s -Gauss formulæ are **interpolatory**, i.e. $Q(f) = I(H_{\Theta^*} f)$, where Θ^* is any set of $\leq n(2s + 2)$ points, which contains each zero of $M_{n,2s+2}$ with multiplicity at least $2s + 1$. This allows us to estimate the error for these formulæ.

(3.10) Error bound for s -Gauss formulæ. *Let Θ be the zeros of $M_{n,2s+2}$. Then*

$$|I(f) - Q(f)| \leq \frac{1}{(n(2s + 2))!} \left(\|\omega_{\Theta}\|_{2s+2} \right)^{2s+2} \|D^{n(2s+2)} f\|_{\infty}, \quad \forall f \in W_{\infty}^{n(2s+2)},$$

with equality for all $f \in \Pi_{n(2s+2)}$. In addition

$$\left(\|\omega_{\Theta}\|_{2s+2} \right)^{2s+2} < \left(\frac{2}{4^n} \right)^{2s+2} (b - a)^{n(2s+2)+1},$$

which differs from equality by a factor of $< 2^{2s+2}$.

Proof. With Θ^* as above, by (1.1)

$$|I(f) - Q(f)| = |I(f - H_{\Theta^*} f)| \leq \|f - H_{\Theta^*} f\|_1 \leq \frac{1}{(n(2s + 2))!} \|\omega_{\Theta^*}\|_1 \|D^{n(2s+2)} f\|_{\infty}. \quad (3.11)$$

Let Θ^* consist of the points Θ , each with multiplicity $2s + 2$. For this choice,

$$\|\omega_{\Theta^*}\|_1 = \left(\|\omega_{\Theta}\|_{2s+2}\right)^{2s+2}.$$

Further, if $f \in \Pi_{n(2s+2)}$, then $f - H_{\Theta^*}f$ is a scalar multiple of ω_{Θ}^{2s+2} , which is nonnegative, and so equality holds in (3.11). Finally by Corollary (3.3)

$$\left(\|\omega_{\Theta}\|_{2s+2}\right)^{2s+2} < \left(\frac{2}{4^n}(b-a)^{n+\frac{1}{2s+2}}\right)^{2s+2} = \left(\frac{2}{4^n}\right)^{2s+2} (b-a)^{n(2s+2)+1},$$

which differs from equality by a factor of $< 2^{2s+2}$. □

Only when $s = 0$ is this result known; see, e.g., Davis and Rabinowitz [DR75:p98]. In this case $\|\omega_{\Theta}\|_2$ is the L_2 -norm of a Legendre polynomial, and can be computed exactly. For a full account of s -Gauss formulæ, including other error estimates, see the survey article of Gautschi [Ga81].

By using (2.4) and (2.6), it is possible to run through the above argument, to get error bounds for s -Gauss formulæ in terms of $\|D^m f\|_q$, where $n(2s + 1) \leq m \leq n(2s + 2)$.

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