

# A sharp upper bound on the approximation order of smooth bivariate pp functions

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## Introduction

It is the purpose of this note to show that the approximation order from the space

$$\Pi_{k,\Delta}^\rho$$

of all piecewise polynomial functions in  $C^\rho$  of polynomial degree  $\leq k$  on a triangulation  $\Delta$  of  $\mathbb{R}^2$  is, in general, no better than  $k$  in case  $k < 3\rho + 2$ . This complements the result of [BH88] that the approximation order from  $\Pi_{k,\Delta}^\rho$  for an arbitrary mesh  $\Delta$  is  $k + 1$  if  $k \geq 3\rho + 2$ .

Here, we define the **approximation order** of a space  $S$  of functions on  $\mathbb{R}^2$  to be the largest real number  $r$  for which

$$\text{dist}(f, \sigma_h S) \leq \text{const}_f h^r$$

for any sufficiently smooth function  $f$ , with the distance measured in the  $L_p$ -norm ( $1 \leq p \leq \infty$ ) on  $\mathbb{R}^2$  (or some suitable subset  $G$  of  $\mathbb{R}^2$ ), and with the **scaling map**  $\sigma_h$  defined by

$$\sigma_h f := f(\cdot/h).$$

In particular, the approximation order from  $\Pi_{k,\Delta}^\rho$  cannot be better than  $k + 1$  regardless of  $\rho$  and is trivially  $k + 1$  in case  $\rho = -1$  or  $0$ . Thus, an upper bound of  $k$  is an indication of the price being paid for having  $\rho$  much larger than  $0$ .

It turns out that the upper bound to be proven here already holds when  $\Delta$  is a very simple triangulation, viz. the **three-direction mesh**, i.e., the mesh

$$\Delta := \bigcup_{i=1}^3 \mathbb{R}e_i + \mathbb{Z}^2$$

with

$$e_1 := (1, 0), e_2 := (0, 1), e_3 := (1, 1) = e_1 + e_2.$$

A first result along these lines was given in [BH83<sub>1</sub>] where it was shown that the approximation order of  $\Pi_{3,\Delta}^1$  (with  $\Delta$  the three-direction mesh) is only  $3$ , which was surprising in view of the fact that all cubic polynomials are contained locally in this space. [J83] showed the corresponding result for  $C^1$ -quartics on the three-direction mesh and [BH83<sub>2</sub>] provided upper and lower bounds for the approximation order of

$$S := \Pi_{k,\Delta}^\rho$$

for arbitrary  $k$  and  $\rho$ .

For  $2k - 3\rho \leq 7$ , the approximation order of  $S$  was completely determined in [J86]. Since it is easy to determine the approximation order of any space spanned by the translates of one box spline ([BH82/3]) with the aid of quasi-interpolants, it is tempting to consider, more generally, **local** approximations from  $S$ , i.e., approximations to the given  $f$  which are linear combinations of box splines in  $S$ , with the restriction that the coefficient of any particular box spline should depend only on the behavior of  $f$  near the support of that box spline. The resulting approximation order has been termed the **local approximation order** of  $S$  in [BJ]. The local approximation order of  $S$  was entirely determined in [J88]. In particular, it is shown there that the local approximation order of  $S$  can never be full, i.e., equal  $k + 1$ . It is also conjectured there that the local approximation order equals the approximation order when  $k < 3\rho + 2$ . In addition, it is shown in [J88] that the approximation order of  $S$  is at least  $k$  when  $k \geq 2\rho + 2$ . This, together with the result to be proved here and the result from [J86], gives the precise approximation order for  $S$  for  $\rho \leq 5$  and all  $k$ . Finally, the fact that the approximation order from  $S$  is only  $k$  when  $k = 3\rho + 1$  was demonstrated in [BH88] for  $\rho = 1, 2, 3$ .

In all of these references cited, only the approximation order with respect to the max-norm was considered.

In addition to the notation already defined in the course of the above introduction, we also use the following: We denote by

$$\Pi_k \quad (\Pi_{<k})$$

the collection of all polynomials of total degree  $\leq k$  ( $< k$ ). We denote by

$$\langle y, \cdot \rangle$$

the linear polynomial whose value at  $x \in \mathbb{R}^2$  is the scalar product  $\langle y, x \rangle$  of  $y$  with  $x$ . We write

$$D_y := y(1)D_1 + y(2)D_2$$

for the (unnormalized) directional derivative in the direction  $y$ , with  $D_i$  the partial derivative with respect to the  $i$ th argument,  $i = 1, 2$ . Thus,

$$D_i = D_{e_i},$$

but we use this abbreviation also for  $i = 3$ , and use, correspondingly, the convenient abbreviation

$$D^a := \prod_{i=1}^3 D_i^{a(i)},$$

with  $a \in \mathbb{Z}_+^3$ . For such  $a$ , we write

$$|a| := \sum_i a(i).$$

Correspondingly, we write

$$\tau^a := \prod_{i=1}^3 \tau_i^{a(i)} \quad \text{and} \quad \nabla^a := \prod_{i=1}^3 \nabla_i^{a(i)},$$

with

$$\tau_i f := f(\cdot + e_i) \quad \text{and} \quad \nabla_i := 1 - \tau_i^{-1}.$$

Finally, we denote by  $p(D) := \sum_{\alpha} c(\alpha) D^{\alpha}$  the constant coefficient differential operator associated with the polynomial  $p = \sum_{\alpha} c(\alpha) ()^{\alpha}$ . For example,

$$D_i = \langle e_i, D \rangle.$$

### Main Result

The main result of this note is the following

**Theorem.** *The approximation order of  $S := \Pi_{k,\Delta}^{\rho}$  (in any  $L_p$ ,  $1 \leq p \leq \infty$ ) is at best  $k$  when  $k < 3\rho + 2$ ,  $\rho > 0$  and  $\Delta$  is the three-direction mesh.*

In this section, we outline the proof, leaving the verification of certain technical Lemmata to a subsequent section.

The proof uses the same ideas with which the special cases  $\rho = 1$  and  $2$  were handled in [BH83<sub>1</sub>], [J83], and [BH88], respectively, i.e., the construction of a local linear functional which vanishes on  $\Pi_{k,\Delta}^{\rho}$  but does not vanish on some homogeneous polynomial of degree  $k+1$  and whose integer translates add up to the zero linear functional. But the construction of the specific linear functional follows the rather different lines of [J86].

To begin with, recall from [BH83<sub>2</sub>] that the approximation order of  $S$  equals that of

$$S_{\text{loc}} := \text{span}\{M_{r,s,t}(\cdot - j) : j \in \mathbb{Z}^2, M_{r,s,t} \in S\}.$$

(To be precise, the proof of Proposition 3.1 in [BH83<sub>2</sub>] can be modified to show that if  $r$  is an upper bound on the approximation order of  $S_{\text{loc}}$ , then it is also an upper bound on the approximation order of  $S$ , while the converse is trivial since  $S_{\text{loc}} \subseteq S$ .) Here,  $M_{r,s,t}$  is the box spline  $M(\cdot, \Xi)$ , i.e., the distribution  $f \mapsto \int_{[0..1]^{r+s+t}} f(\Xi t) dt$  (cf., e.g., [BH82/3]), with direction matrix

$$\Xi := \underbrace{[e_1, \dots, e_1]}_{r \text{ times}}, \underbrace{[e_2, \dots, e_2]}_{s \text{ times}}, \underbrace{[e_3, \dots, e_3]}_{t \text{ times}}.$$

Further, the linear functional will be constructed from linear functionals of the form  $f \mapsto \int_T p(D)f$ , with

$$T := \{x \in \mathbb{R}^2 : 0 < x(2) < x(1) < 1\}$$

a triangle in the three-direction mesh  $\Delta$ , and with  $p$  a homogeneous polynomial of degree  $k$ . Such functionals vanish on  $\Pi_{<k}$ , hence also vanish on any  $M_{r,s,t}$  with  $r + s + t - 2 < k$ . It is proved in [BH83<sub>2</sub>] that, for  $k > 2\rho + 1$ ,  $S_{\text{loc}}$  is spanned by the integer translates of the box splines of degree  $< k$  in  $S$  and the box splines  $M_{\alpha}$  with  $\alpha$  in

$$A := A_1 \cup A_2 \cup A_3,$$

where

$$\begin{aligned} A_1 &:= \{(k - \rho + 1 - i, 0, \rho + 1 + i) : i = 1, \dots, k - 2\rho - 1\}, \\ A_2 &:= \{(\rho + 2 - i, i, k - \rho) : i = 1, \dots, \rho + 1\}, \\ A_3 &:= \{(0, \rho + 1 + i, k - \rho + 1 - i) : i = 1, \dots, k - 2\rho - 1\}. \end{aligned}$$

(These are exactly the box splines whose restriction to the line  $e_1 + \mathbb{R}(e_2 - e_1)$  coincide there with a(n appropriately scaled univariate) B-spline of degree  $k$  for the knot sequence in which each of  $0, 1/2, 1$  occurs exactly  $k - \rho$  times.) This implies that it is sufficient to require our linear functional  $\lambda$  to vanish on  $M_\alpha(\cdot - j)$  for  $\alpha \in A$  and  $j \in \mathbb{Z}^2$  in order to ensure that  $\lambda \perp S_{\text{loc}}$ .

**(1) Lemma.** For  $\beta := (1, 1, 0)$ , there exists a set  $B$  of  $\rho + 1$  homogeneous polynomials of degree  $k$  so that, on  $T + \mathbb{Z}^2$ ,

$$(2) \quad p(D)M_\alpha = c_{p,\alpha} \nabla^{\alpha-\beta} M_\beta, \quad p \in B, \alpha \in A,$$

with the constants  $c_{p,\alpha}$  satisfying

$$c_{p,\alpha} = 0, \quad \alpha \in A_3.$$

Here and below, we follow the convenient convention that  $\nabla^\gamma = 0$  if  $\gamma(i) < 0$  for some  $i$ .

**(3) Lemma.** For  $\gamma := (1, 0, 1)$ , there exists a set  $C$  of  $\rho + 1$  homogeneous polynomials of degree  $k$  so that, on  $T + \mathbb{Z}^2$ ,

$$(4) \quad p(D)M_\alpha = c_{p,\alpha} \nabla^{\alpha-\gamma} M_\gamma, \quad p \in C, \alpha \in A,$$

with the constants  $c_{p,\alpha}$  satisfying

$$c_{p,\alpha} = 0, \quad \alpha \in A_3.$$

Now note that  $M_\beta$  and  $M_\gamma$  agree on all of  $T + \mathbb{Z}^2$  with the characteristic function

$$\chi_T$$

of the triangle  $T$ . Thus,

$$p(D)M_\alpha = c_{p,\alpha} \left\{ \begin{array}{l} \nabla^{\alpha-\beta} \\ \nabla^{\alpha-\gamma} \end{array} \right\} \chi_T \quad \text{on } T + \mathbb{Z}^2, \quad \text{for } p \in \left\{ \begin{array}{l} B \\ C \end{array} \right\}.$$

Further,

$$\nabla_2 \nabla^{\alpha-\beta} = \nabla_2 \nabla_3 \nabla^{\alpha-(1,1,1)} = \nabla_3 \nabla^{\alpha-\gamma}.$$

Thus, if

$$(5) \quad \sum_{p \in B \cup C} w(p) c_{p,\alpha} = 0 \quad \text{for all } \alpha \in A_1 \cup A_2,$$

then

$$JM_\alpha = 0 \quad \text{on } T + \mathbb{Z}^2 \text{ for all } \alpha \in A,$$

with

$$(6) \quad J := \sum_{p \in B} w(p) \nabla_2 p(D) + \sum_{p \in C} w(p) \nabla_3 p(D)$$

(since  $c_{p,\alpha} = 0$  for  $p \in B \cup C$  and  $\alpha \in A_3$ ). Here, we may (and do) choose  $w \neq 0$ , since  $\#(B \cup C) = 2\rho + 2 > k - \rho = \#(A_1 \cup A_2)$ .

Next, we construct some  $g \in \Pi_{k+1}$  for which  $Jg = 2$ . For this, note that  $p(D)\Pi_{k+1} \subset \Pi_1$  for any  $p \in B \cup C$ , while  $\nabla_i = D_i$  on  $\Pi_1$ . This implies that

$$J = \sum_{p \in B \cup C} w(p) \tilde{p}(D) \quad \text{on } \Pi_{k+1},$$

with

$$\tilde{p} := p \begin{cases} \langle e_2, \cdot \rangle, & p \in B; \\ \langle e_3, \cdot \rangle, & p \in C. \end{cases}$$

**(7) Lemma.** *If  $k > 2\rho + 1$ , then the sets  $B$  and  $C$  in (1) and (3) can be so chosen that  $\{\tilde{p} : p \in B \cup C\}$  is a linearly independent subset of  $\Pi_{k+1}$ .*

To make use of this lemma, we need to restrict attention to the case  $k > 2\rho + 1$ . We do this by, possibly, *decreasing*  $\rho$  (and, hence increasing  $S$ ) to force the inequality  $k > 2\rho + 1$ . Of course, we must make sure that we still have  $k < 3\rho + 2$ . Assuming that  $\rho'$  is the largest integer for which  $k > 2\rho' + 1$ , we have  $k \leq 2\rho' + 3 < 3\rho' + 2$  except, possibly, when  $\rho' \leq 1$ , hence  $k \leq 5$ . But, for  $k \leq 5$  and  $\rho \geq 1$ , the approximation order of  $S$  is known ([J86], [BH88]) to satisfy our theorem's claim.

Thus, for  $k > 5$ , we may assume without loss of generality that  $k > 2\rho + 1$ , hence use the lemma to conclude, from the fact that  $w \neq 0$ , that  $J = q(D)$  on  $\Pi_{k+1}$  for some *nontrivial* homogeneous polynomial  $q$  of degree  $k + 1$ . This implies that  $J$  maps  $\Pi_{k+1}$  *onto*  $\Pi_0$ , hence  $Jg = 2$  for some  $g \in \Pi_{k+1}$ .

Since  $JM_\alpha = 0$  on  $T + \mathbb{Z}^2$ , and  $J$  commutes with any integer shift, it follows that the linear functional

$$\lambda : f \mapsto \int_T Jf$$

vanishes on  $S_{1\text{oc}}$ , but takes the value 1 on that particular polynomial  $g$ . Further,  $\lambda$  has the form

$$\lambda = \lambda_2 \nabla_2 + \lambda_3 \nabla_3$$

with

$$\lambda_i : f \mapsto \int_T p_i(D)f,$$

for some homogeneous polynomials  $p_i$  of degree  $k$ . This shows that

$$\sum_{j \in \mathbb{Z}^2} \lambda \tau^j = 0,$$

in the sense that, for any compact set, there is some  $n_0$  so that any sum  $\sum_{j \in \mathbb{Z}^2 \cap [-n..n]^2} \lambda \tau^j$  with  $n > n_0$  has no support in that compact set.

We make use of  $\lambda$  in the following more precise fashion. Define

$$H_{i,n} := \sum_{j=1}^n \tau_i^j.$$

Then  $H_{i,n} \nabla_i = \tau_i^n - 1$ . Therefore,

$$\lambda^{(n)} := \lambda \sum_{j \in \mathbb{Z}^3 \cap [1..n]^3} \tau^j = \lambda_2(\tau_2^n - 1)H_{1,n}H_{3,n} + \lambda_3(\tau_3^n - 1)H_{1,n}H_{2,n}$$

has support only in

$$T_n := T + \sum_{j \in \mathbb{Z}^3 \cap [0..n]^3} \sum_i j(i) e_i =: T + I,$$

and is, more explicitly, of the form

$$f \mapsto \sum_{j \in I} \int_{T+j} (b(j)p_2(D) + c(j)p_3(D))f,$$

with  $b(j), c(j) \in \{-1, 0, 1\}$  for all  $j$ . (Put differently, the mesh functions  $b$  and  $c$  are first differences of the discrete box spline associated with the three directions  $e_1, e_2, e_3$ , hence are piecewise constant.) Since  $\tau^j g \in g + \Pi_k$  and  $\lambda^{(n)}$  vanishes on  $\Pi_k$ , this implies that  $\lambda^{(n)}g = n^3$ . Further, as a functional on, say,  $\Pi_{k+1,\Delta}^0 \subset L_1([-1..2n+1]^2)$ ,  $\lambda^{(n)}$  has norm

$$\|\lambda^{(n)}\| \leq \text{const}_k,$$

since, on each  $T + j$ , any  $f$  of interest (i.e., any  $f \in S + \text{span } g$ ) reduces to a polynomial of degree  $\leq k + 1$ , hence

$$\left| \int_{T+j} p_i(D)f \right| \leq \text{const}_k \int_{T+j} |f|$$

with  $\text{const}_k$  derived from Markov's inequality.

Let now  $h := 1/n$  and set  $\sigma : f \mapsto f(\cdot/h)$ . We are interested in a lower bound for the  $L_p(G)$ -distance of  $g$  from  $S_h := \sigma S$ . Since  $\|f\|_1(G') \leq \text{const}_{G'} \|f\|_p(G') \leq \text{const}_{G'} \|f\|_p(G)$  for any bounded subset  $G'$  of  $G$ , it is sufficient to restrict attention to  $p = 1$  and bounded  $G$ . Moreover, after a translation and a scaling, we may assume that the domain  $G$  of interest contains  $[-h..(2n+1)h]^2$ . Then  $\|\lambda^{(n)}\sigma^{-1}\| \leq \text{const}_k h^{-2}$ , and  $\lambda^{(n)}\sigma^{-1} \perp S_h$ , while  $\lambda^{(n)}\sigma^{-1}g = \lambda^{(n)}g(\cdot h) = h^{k+1}\lambda^{(n)}g = h^{k-2}$ . Consequently,

$$\text{dist}_1(g, S_h) \geq \lambda^{(n)}\sigma^{-1}g / \|\lambda^{(n)}\sigma^{-1}\| \geq h^{k-2} / (\text{const}_k h^{-2}) = \text{const } h^k,$$

for some  $h$ -independent positive const. This finishes the proof of the theorem.

## Proof of the technical lemmata

We take  $B$  and  $C$  from the set of polynomials

$$p_a := \prod_{i=1}^3 \langle e_i, \cdot \rangle^{a(i)}$$

with  $a \in \mathbb{Z}_+^3$ ,  $|a| = k$ .

For the computation of  $p_a(D)M_\alpha$ , we rely entirely on the differentiation formula [BH82/3]

$$D_\xi M(\cdot, \Xi) = \nabla_\xi M(\cdot, \Xi \setminus \xi)$$

valid for any particular direction  $\xi$  from the direction set  $\Xi$  for the box spline  $M(\cdot, \Xi)$ , and on the fact that the (closed) support of the box spline  $M(\cdot, \Xi)$  is the set

$$\sum_{\xi \in \Xi} [0..1]\xi.$$

We choose  $B$  to consist of the  $\rho+1$  polynomials  $p_a$  with  $a(3) = k - \rho$ . Then  $a(3) \geq \alpha(3)$  for any  $\alpha \in A$ , hence

$$(8) \quad p_a(D)M_\alpha = \nabla_3^{\alpha(3)} p_{a(1), a(2), a(3) - \alpha(3)}(D)M_{\alpha(1), \alpha(2), 0}.$$

Since  $\alpha(2) = 0$  for  $\alpha \in A_1$  and  $\alpha(1) = 0$  for  $\alpha \in A_3$ , this shows that  $p_a(D)M_\alpha$  has no support in  $T + \mathbb{Z}^2$  when  $\alpha \in A_1 \cup A_3$ , hence (2) holds for this case with  $c_{p, \alpha} = 0$ . For the remaining case,  $\alpha \in A_2$ , we have  $\alpha(3) = k - \rho = a(3)$ , and therefore, more explicitly than (8),

$$p_a(D)M_\alpha = \nabla_3^{\alpha(3)} D_1^{a(1)} D_2^{a(2)} M_{\alpha(1), \alpha(2), 0},$$

and this has support in  $T + \mathbb{Z}^2$  if and only if  $a(i) < \alpha(i)$  for  $i = 1, 2$ . Since  $a(1) + a(2) = \alpha(1) + \alpha(2) - 2$ , this condition is met if and only if  $\alpha = a + \beta$  with  $\beta = (1, 1, 0)$ , and in that case we get

$$p_a(D)M_\alpha = \nabla^{\alpha - \beta} M_\beta.$$

This finishes the proof of (1)Lemma.

The verification of (3)Lemma proceeds analogously. We choose  $C$  to consist of the  $\rho+1$  polynomials  $p_a$  with  $a(2) = k - \rho$ . Then  $a(2) \geq \alpha(2)$  for any  $\alpha \in A$ , hence

$$(9) \quad p_a(D)M_\alpha = \nabla_2^{\alpha(2)} p_{a(1), a(2) - \alpha(2), a(3)}(D)M_{\alpha(1), 0, \alpha(3)}.$$

Since  $\alpha(1) = 0$  for  $\alpha \in A_3$ , this shows that  $p_a(D)M_\alpha$  has no support in  $T + \mathbb{Z}^2$  when  $\alpha \in A_3$ , hence (4) holds for this case with  $c_{p, \alpha} = 0$ . For the remaining case, i.e., for  $\alpha \in A_1 \cup A_2$ , we make use of the fact that  $D_2 = D_3 - D_1$  to write (9) in the form

$$p_a(D)M_\alpha = \nabla_2^{\alpha(2)} \sum_j c_j D_1^{j(1)} D_3^{j(3)} M_{\alpha(1), 0, \alpha(3)},$$

with the sum over all  $j$  of the form  $(a(1) + r, 0, a(3) + t)$  with  $r + t = a(2) - \alpha(2)$ . Thus,  $j(1) + j(3) = \alpha(1) + \alpha(3) - 2$ , hence the only terms with some support in  $T + \mathbb{Z}^2$  are of the form  $j(i) = \alpha(i) - 1$  for  $i = 1, 3$ , and in that case,

$$D_1^{j(1)} D_3^{j(3)} M_{\alpha(1), 0, \alpha(3)} = \nabla^{\alpha(1)-1, 0, \alpha(3)-1} M_\gamma.$$

As to (7)Lemma, we note first that  $\tilde{B} := \{\tilde{p} : p \in B\}$  is linearly independent since it consists of the sequence

$$\langle e_2, \cdot \rangle \langle e_3, \cdot \rangle^{k-\rho} \{ \langle e_1, \cdot \rangle^j \langle e_2, \cdot \rangle^{\rho-j} : j = 0, \dots, \rho \},$$

and  $e_1, e_2$  form a basis for  $\mathbb{R}^2$ . Analogously,  $\tilde{C} := \{\tilde{p} : p \in C\}$  is linearly independent since it consists of the sequence

$$\langle e_2, \cdot \rangle^{k-\rho} \langle e_3, \cdot \rangle \{ \langle e_1, \cdot \rangle^j \langle e_3, \cdot \rangle^{\rho-j} : j = 0, \dots, \rho \},$$

and  $e_1, e_3$  form a basis for  $\mathbb{R}^2$ . Thus it is sufficient to prove that  $\text{span } \tilde{B}$  has only trivial intersection with  $\text{span } \tilde{C}$ . But this follows from the facts (obtainable by substituting  $e_3 - e_2$  for  $e_1$  and collecting terms) that

$$\tilde{B} \subset \text{span} \{ \langle e_2, \cdot \rangle^{1+j} \langle e_3, \cdot \rangle^{k-j} : j = 0, \dots, \rho \}$$

and

$$\tilde{C} \subset \text{span} \{ \langle e_2, \cdot \rangle^{k-j} \langle e_3, \cdot \rangle^{1+j} : j = 0, \dots, \rho \},$$

since  $k - \rho > \rho + 1$ , by assumption.

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