Multivariate polynomial interpolation: Aitken-Neville sets and generalized principal lattices

Carl de Boor (10aug08)

Abstract. A suitably weakened definition of generalized principal lattices is shown to be equivalent to the recent definition of Aitken-Neville sets.

The recent paper [CGS08] explores the relationship of Aitken-Neville sets, introduced in [SX], to generalized principal lattices, introduced in [CGS06]. Both are subsets X of $\mathbb{F}^d$ (with $\mathbb{F}$ equal to $\mathbb{R}$ or $\mathbb{C}$) that are n-**correct** for some n in the sense that, with

$$\Pi_{\le n}$$

the collection of polynomials on $\mathbb{F}^d$ of (total) degree $\le n$, the restriction map

$$\Pi_{\le n} \to \mathbb{F}^X : p \mapsto p|_X := (p(x) : x \in X) \tag{1}$$

is invertible, hence arbitrary values given at X can be interpolated uniquely by some polynomial of degree $\le n$. In particular, $\#X = \dim \Pi_{\le n}$.

Both kinds of sets have considerably more structure than that (see the definitions below). [CGS08] proves that any generalized principal lattice is an Aitken-Neville set and gives simple examples to show that the converse does not hold. The present note makes more precise how the two notions differ and then proposes an appropriate relaxation of the definition of a generalized principal lattice that makes the two notions equivalent. In the process, some of the arguments from [CGS08] are simplified.

Standard multiindex notation is used. In particular,

$$|\alpha| := \alpha(0) + \cdots + \alpha(d)$$

is the **degree** of the multiindex

$$\alpha = (\alpha(0), \ldots, \alpha(d)) \ \in \mathbb{Z}_+^{0:d},$$

where, in an abuse of standard Matlab notation, $0{:}d$ is the *set* with elements $0, 1, \ldots, d$, i.e.,

$$0{:}d := \{0, 1, \ldots, d\},$$

and $\mathbb{Z}_+ := \{0, 1, 2, \ldots\}$. With that, let

$$\Gamma_n := \{\gamma \in \mathbb{Z}_+^{0:d} : |\gamma| = n\}.$$

Also, let

$$\epsilon_j$$

be the particular multiindex with all entries 0 except for the jth which is 1. Finally, with another abuse of notation,

$$X \backslash x := \{y \in X : y \ne x\}.$$

definitions

(2) Definition. *A* GC_n**-set** *is a set* X *in* $\mathbb{F}^d$ *of cardinality* $\ge \dim \Pi_{\le n}$ *for which, for each* $x \in X$, *there are* $\le n$ *hyperplanes whose union contains* $X \backslash x$ *but not* x.

Since any hyperplane is the zero-set of some polynomial of degree 1, it then follows that, for every x in such a GC_n-set X, there is a product ℓ_x of $\le n$ polynomials of degree 1 that vanishes on all of $X \backslash x$ but not on x, and this implies that the linear map (1) is onto, hence necessarily $\dim \Pi_{\le n} \ge \#X$, therefore $\dim \Pi_{\le n} = \#X$ and the map (1) is invertible, hence X is n-correct, This implies that $\deg \ell_x = n$ for all $x \in X$.

GC_n-sets were introduced in [CY] as those n-correct sets X whose corresponding Lagrange polynomials ℓ_x, $x \in X$, are products of polynomials of degree 1.

[CY] had two particular examples of GC_n-sets in mind, natural lattices and principal lattices.

(3) Definition. *A* **natural lattice of degree** n *in* $\mathbb{F}^d$ *is of the form*

$$X = \{x_{\mathcal{K}} : \mathcal{K} \in \binom{\mathcal{H}}{d}\},$$

with $\mathcal{H}$ *a collection of* $n+d$ *hyperplanes in* $\mathbb{F}^d$ *in general position, meaning that every subset* $\mathcal{K}$ *of* d *hyperplanes in* $\mathcal{H}$ *has exactly one point in common, call it* $x_{\mathcal{K}}$*, with different subsets resulting in different points.*

Such a natural lattice is evidently a GC_n-set, with $X\backslash x_{\mathcal{K}}$ contained in the union of the hyperplanes in $\mathcal{H}\backslash\mathcal{K}$ which does not contain $x_{\mathcal{K}}$.

(4) Definition. *A* **generalized principal lattice of degree** n *(or,* $\mathbf{GPL}_n$**-set** *for short) is a set* X *in* $\mathbb{F}^d$ *that can be so indexed as*

$$X = \{x_\alpha : \alpha \in \Gamma_n\}$$

that, for some collection $\mathcal{H} := (H_i^j : i \in 0{:}(n-1), j \in 0{:}d)$ *of hyperplanes and all applicable* $\alpha \in \Gamma_n$*,* r*, and* i*,*

$$\bigcap_{j\neq r} H^j_{\alpha(j)} = \{x_\alpha\} \subset H^r_{\alpha(r)} \tag{5}$$

while

$$x_\alpha \in H_i^j \quad\Longrightarrow\quad \alpha(j) = i. \tag{6}$$

Note that, necessarily, $\#\mathcal{H} = n(d{+}1)$, i.e., *the hyperplanes* H_i^j *are pairwise distinct*: Indeed, if $H_i^j = H_s^r$ for some $i, s < n$, then, by (5), $x_\alpha \in H_i^j = H_s^r$ for any α with $\alpha(j) = i$, hence (6) implies that $\alpha(r) = s$ for any such α, which is nonsense unless $j = r$, in which case it implies that $i = s$.

A GPL_n*-set is a* GC_n*-set*: For, by (5), $X\backslash x_\alpha$ is contained in the union of the $\le \sum_{j=0}^d \alpha(j) = |\alpha| = n$ hyperplanes H_i^j with $i < \alpha(j)$ since, for any $\beta \in \Gamma_n\backslash\alpha$, we must have $\beta(j) < \alpha(j)$ for some j, while, by (6), that union does not contain x_α.

(7) Remark. This conclusion does not use the full power of either (5) or (6). In fact, it only uses

$$\alpha(r) < n \quad\Longrightarrow\quad x_\alpha \in H^r_{\alpha(r)} \tag{8}$$

and

$$x_\alpha \in H_i^j \quad\Longrightarrow\quad \alpha(j) \le i. \tag{9}$$

□

Generalized principal lattices were introduced (and analyzed) in [CGS06], as a generalization from the bivariate situation in [CG05] and [CG06], however in the following different, though equivalent, form: Additional hyperplanes are required to exist, namely, for each $j \in 0{:}d$, a hyperplane H_n^j intersecting X only at $x_{n\epsilon_j}$ is required to exist and, correspondingly, (5) is usually stated

$$\{x_\alpha\} = \bigcap_{j\neq r} H^j_{\alpha(j)} = \bigcap_{j=0}^{d} H^j_{\alpha(j)},$$

where now also $\alpha(j) = n$ can appear. However, since these hyperplanes H_n^j are not uniquely defined by X nor do they play any role in the GC_n-structure of X, it seems unnecessary to bring them in in the first place. Also, (6) is usually stated

$$\forall\alpha \in (0{:}n)^{0:d} \qquad \bigcap_{j=0}^{d} H^j_{\alpha(j)} \cap X \neq \emptyset \quad\Longrightarrow\quad \alpha \in \Gamma_n. \tag{10}$$

However, *(6) and (10) are equivalent in the presence of (5).* First, *(6) implies (10)*: If

$$x_\beta \in \bigcap_{j=0}^{d} H^j_{\alpha(j)}$$

for some $\alpha \in (0{:}n)^{0:d}$ and some $\beta \in \Gamma_n$, then, by (6), $\beta(j) = \alpha(j)$, all j, hence $\alpha = \beta \in \Gamma_n$, at least for $\alpha \in (0{:}(n{-}1))^{0:d}$ and $\beta(j) < n$ for all j; by the choice of the H_n^j, the case $\alpha(j) = n$ can happen only if $\beta = n\epsilon_j$ and, in that case, (6) implies that $\alpha(r) = 0$ for all $r \neq j$, hence again $\alpha = \beta \in \Gamma_n$. Also, as already stated in [CGS06: Remark 2], *(10) implies (6)* in the presence of (8) (hence of (5)): If $x_\alpha \in H_i^j$ then, by (8), $x_\alpha \in \cap_{r=0}^d H^r_{\beta(r)}$ with $\beta := \alpha + (i - \alpha(j))\epsilon_j$, hence, by (10), $\beta \in \Gamma_n$ and so, in particular, $i = \alpha(j)$. □

Note that any GPL_n-set is the lattice transform (in the sense of [CY]) of the **standard principal lattice**

$$\mathrm{A} := \{\alpha_| : \alpha \in \Gamma_n\},$$

with

$$x_| := (x(j) : j = 1{:}d) \quad \text{for} \quad x \in \mathbb{R}^{0:d},$$

with the collection $\mathcal{K}$ of hyperplanes

$$K_i^j := \{x_| : x \in \mathbb{R}^{0:d}, x(j) = i\}, \quad i \in 0{:}(n-1),\ j \in 0{:}d,$$

and with $\Phi : \mathrm{A} \to X : \alpha_| \mapsto x_\alpha$ and $\Psi : \mathcal{K} \to \mathcal{H} : K_i^j \mapsto H_i^j$. Further, because $\#\mathcal{H} = n(d+1) = \#\mathcal{K}$, Ψ is 1-1, hence ([CGS08]) *any two* GPL_n*-sets in* $\mathbb{F}^d$ *are lattice transforms of each other.*

While [CY] correctly credit [N] with coining the term 'principal lattice' (actually, [N] uses 'principal lattice of the d-simplex'), the recognition that the standard principal lattice just described is n-correct (at least for $d = 2$) goes back at least to [Bi]. Perhaps the major contribution of [N] is to have stimulated [CY].

The following generalization, suggested by (7)Remark, of GPL_n-sets requires much less yet, by (7)Remark, still provides GC_n-sets.

(11) Definition. *A* **fully generalized principal lattice of degree** n *(or,* FGPL_n**-set** *for short) is a set* X *in* $\mathbb{F}^d$ *that can be so indexed as* $X = \{x_\alpha : \alpha \in \Gamma_n\}$ *that (8) and (9) hold for some collection* $(H_i^j : i \in 0{:}(n-1), j \in 0{:}d)$ *of hyperplanes and all applicable* $\alpha \in \Gamma_n$*,* r*, and* i*.*

In what follows, for any $A \subset \mathbb{F}^d$,

$$\operatorname{conv} A \quad \text{and} \quad \flat A$$

denote, respectively, the convex hull of A and the affine space or **flat** spanned by the elements of A.

(12) Definition ([SX]). *An* **Aitken-Neville set** *(or, configuration)* **of degree** n *(or* AN_n**-set***, for short) is a set* X *in* $\mathbb{F}^d$ *that can be so indexed as* $X = \{x_\alpha : \alpha \in \Gamma_n\}$ *that*

$$\{x_{\beta+k\epsilon_j} : j \in 0{:}d\} \text{ is 1-correct}, \qquad \beta \in \Gamma_{n-k},\ k \in 1{:}n, \tag{13}$$

and

$$\alpha \in \operatorname{conv}\{\beta + k\epsilon_j : j \in J\} \quad\Longrightarrow\quad x_\alpha \in \flat\{x_{\beta+k\epsilon_j} : j \in J\}, \qquad \beta \in \Gamma_{n-k},\ k \in 1{:}n,\ J \subset 0{:}d,\ \alpha \in \Gamma_n. \tag{14}$$

Note that the implication in (14) vacuously holds for $J = \emptyset$ and is implied by (13) for $J = 0{:}d$. Aitken-Neville sets were introduced in [SX] as precisely the kind of n-correct sets for which the natural multivariate generalization of the classical Aitken-Neville process is available, as shown in [SX] (and recalled in more detail at the end of this note).

results

(15) Proposition. *Any* AN_n*-set* $X = \{x_\alpha : \alpha \in \Gamma_n\}$ *is a* FGPL_n*-set, with the hyperplanes given by*

$$H_i^j := \flat\{x_{i\epsilon_j+(n-i)\epsilon_r} : r \neq j\}, \qquad i \in 0{:}(n-1),\ j \in 0{:}d. \tag{16}$$

Proof: For each $i \in 0{:}(n-1)$ and $j \in 0{:}d$, the set $\{x_{i\epsilon_j+(n-i)\epsilon_r} : r \in 0{:}d\}$ is, by assumption, 1-correct, hence H_i^j is, indeed, a hyperplane and, again by assumption, it contains all x_α with $\alpha \in \operatorname{conv}\{i\epsilon_j + (n-i)\epsilon_r : r \neq j\}$. In particular, if $\alpha(j) = i$, then

$$\alpha = \sum_{r\neq j} \frac{\alpha(r)}{n-i}(i\epsilon_j + (n-i)\epsilon_r) \ \in\ \operatorname{conv}\{i\epsilon_j + (n-i)\epsilon_r : r \neq j\},$$

therefore $x_\alpha \in H_i^j$; this proves (8). Further, (8) implies that, for any $\alpha \in \Gamma_n$ with $k := \alpha(j) - i > 0$ for some j, each $x_{(\alpha-k\epsilon_j)+k\epsilon_r}$ with $r \neq j$ is in H_i^j, therefore x_α itself cannot be in H_i^j since, otherwise, H_i^j would contain the entire set $\{x_{(\alpha-k\epsilon_j)+k\epsilon_r} : r \in 0{:}d\}$ which, by assumption, is 1-correct, contradicting the fact that H_i^j is a hyperplane. In short, $x_\alpha \in H_i^j$ implies $\alpha(j) \leq i$, i.e., (9) holds. □

(17) Corollary ([CGS08]). *Any* AN_n*-set* X *is a* GC_n*-set.*

(18) Corollary. *The hyperplanes defined in (16) in terms of the labeling* $X = \{x_\alpha : \alpha \in \Gamma_n\}$ *of an* AN_n*-set satisfy*

$$H^j_{\beta(j)} = \flat\{x_{\beta+k\epsilon_r} : r \neq j\} \tag{19}$$

for all $k \in 1{:}n$ *and all* $\beta \in \Gamma_{n-k}$.

Proof: For any such β, $\gamma := \beta + k\epsilon_j$ is in Γ_n and satisfies $k = \gamma(j) - i$ with $i := \beta(j)$, hence, as we observed in the preceding proof, H^j_i contains the d-set $\{x_{\beta+k\epsilon_r} : r \neq j\}$, and, as this set is affinely independent, its affine hull must be all of H^j_i. □

(20) Proposition. *Any* $X = \{x_\alpha : \alpha \in \Gamma_n\}$ *satisfying (5) (hence (8)) and (9) with respect to some hyperplanes* H^j_i*,* $i \in 0{:}(n-1)$*,* $j \in 0{:}d$*, is an* AN_n*-set, and the* H^j_i *must be as given in (16), hence satisfy (19).*

Proof: Let $k \in 1{:}n$, $\beta \in \Gamma_{n-k}$.

Then $\{x_{\beta+k\epsilon_j} : j \in 0{:}d\}$ *is affinely independent*. Indeed, in the contrary case, there would be some r so that

$$x_{\beta+k\epsilon_r} \in \flat\{x_{\beta+k\epsilon_j} : j \neq r\} \subset H^r_{\beta(r)},$$

the set inclusion since $H^r_{\beta(r)}$ is an affine set and contains, by (8), each $x_{\beta+k\epsilon_j}$ for $j \neq r$, and this would contradict (9) since $(\beta + k\epsilon_r)(r) > \beta(r)$.

Further, let $\emptyset \neq J \subset 0{:}d$. Then

$$\flat\{x_{\beta+k\epsilon_j} : j \in J\} \;=\; \bigcap_{j \notin J} H^j_{\beta(j)}. \tag{21}$$

Indeed, by (8), each $x_{\beta+k\epsilon_j}$ is in every $H^r_{\beta(r)}$ for all $j \neq r$, hence we have the containment $\subset$ in (21). But, by the affine independence just proved, we know the left-hand side to be of dimension $\#J - 1$, while, by (5), we know the hyperplanes on the right-hand side to be in general position (as a subset of a set of d hyperplanes in $\mathbb{F}^d$ having exactly one point in common), hence the intersection has dimension $d - \#((0{:}d)\backslash J) = \#J - 1$, too, therefore must equal the left-hand side.

With that, any $\gamma \in \Gamma_n \cap \mathrm{conv}\{\beta + k\epsilon_j : j \in J\}$ satisfies $\gamma(r) = \beta(r)$ for $r \notin J$, hence, by (8) and (21),

$$x_\gamma \in \bigcap_{r\notin J} H^r_{\gamma(r)} = \bigcap_{r \notin J} H^r_{\beta(r)} = \flat\{x_{\beta+k\epsilon_j} : j \in J\}.$$

□

(22) Remark. The full strength of (5) is used here only at one point, namely to ensure that the $H^j_{\beta(j)}$, $j \neq r$, are in general position. However, since $|\beta| < n$ here, this only requires the condition

$$\alpha(r) > 0 \quad \Longrightarrow \quad \#\bigcap_{j\neq r} H^j_{\alpha(j)} = 1. \tag{23}$$

Even in the presence of (8), which implies that $\cap_{j\neq r} H^j_{\alpha(j)} \supset \{x_\alpha\}$, this condition is weaker than (5) since it does not imply that $\cap_{j\neq r} H^j_{\alpha(j)} = \{x_\alpha\}$ for $\alpha(r) = 0$. A simple example is provided by the natural lattice in (25) below which satisfies (8) and (23) but fails to satisfy (5) for any r and for $\alpha = \epsilon_i + \epsilon_j$ with $\{r, i, j\} = 0{:}2$. □

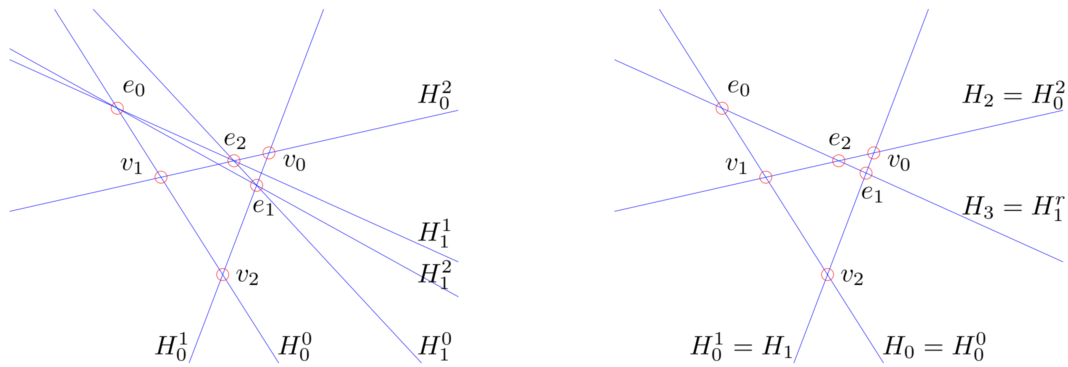

Figure. A natural lattice labeled as an AN_2-set (right), and its perturbation into a GPL_2-set (left).

Since any GPL_n-set satisfies (5) and (9), we have

(24) Corollary ([CGS08]). *Any* GPL_n*-set* X *is an* AN_n*-set.*

(25) Example: Planar AN_2**-sets.**

Let X be a planar AN_2-set. Then $X = \{v_0, v_1, v_2, e_0, e_1, e_2\}$, with

$$v_j := x_{2\epsilon_j}, \quad e_j := x_{\Sigma_{i\neq j}\epsilon_i}, \qquad j \in 0{:}2.$$

Further, for any permutation (r, s, t) of $(0, 1, 2)$,

$$H_0^s = \flat\{v_r, e_s, v_t\}, \quad H_1^s = \flat\{e_r, e_t\}.$$

Finally, with $\beta = 0$, we know $\{x_{\beta+2\epsilon_j} : j \in 0{:}2\} = \{v_0, v_1, v_2\}$ to be 1-correct, hence $H_0^r \cap H_0^s = \{v_t\}$. In particular, the H_0^r are pairwise distinct. Also, with $|\beta| = 1$, hence $\beta = \epsilon_r$ say, we know $\{x_{\beta+\epsilon_j} : j \in 0{:}2\} = \{v_r, e_s, e_t\}$ to be 1-correct, hence $H_0^r \cap H_1^s = \{e_r\}$. Therefore, also none of the H_0^r equals any of the H_1^s. But there is, offhand, no such restriction among the H_1^j, except that, if $H_1^r = H_1^s$, then also $H_1^r = H_1^t$. Thus, a planar AN_2-set involves either 6 or 4 (planar) hyperplanes. In the first case, it is a GPL_2-set, in the second, it is not but is (see the Figure) a natural lattice of degree 2. In the second case, it fails to be a GPL_2-set because the hyperplanes H_i^j are not all pairwise distinct, hence (6) must fail, and it does: $e_j \in H_3 = H_1^j$, yet $e_j(j) \neq 1$.

Incidentally, *any planar* GC_2*-set* X *is necessarily an* AN_2*-set* since, for each $x \in X$, at least one of the two hyperplanes containing $X\backslash x$ must be a **maximal**, i.e., must contain three points of X, hence there must be at least three maximals. Pick three maximals. Then the union of these three contains all of X. If x lies on two of these maximals, it is one of the v_j, while any x that lies on only one of these three maximals is one of the e_j. Thus the left-hand picture in the Figure shows the most general planar GC_2-set, – except that the e_j are chosen to be nearly collinear, to make the set nearly a natural lattice (which is the only planar GC_2-set with four maximals). □

(26) Proposition ([CGS08]). *Every natural lattice of degree 2 is an* AN_2*-set, hence the class* AN *is strictly larger than the class* GPL.

Proof: Let $H_0, \ldots, H_{d+1}$ be hyperplanes in $\mathbb{F}^d$ in general position, and, with $y_{i,j}$ the unique point of intersection of the d hyperplanes H_k with $k \neq i, j$ and $i < j$, set

$$x_{\epsilon_i+\epsilon_s} := y_{i,j}, \quad \text{with} \quad s := \begin{cases} i, & j = d+1 \\ j, & \text{otherwise.} \end{cases}$$

Then, for $\beta = 0$, $\{x_{\beta+2\epsilon_j} : j \in 0{:}d\}$ is the natural lattice of degree 1 obtained from $H_0, \ldots, H_d$, hence 1-correct. Further, for $|\beta| = 1$, necessarily $\beta = \epsilon_i$ for some i, and then $\{x_{\beta+\epsilon_j} : j \in 0{:}d\}$ is the natural lattice obtained from the H_k with $k \neq i, j$, therefore also 1-correct. Finally, the only indices in the convex hull of other indices are the indices

$$\epsilon_i + \epsilon_j = (2\epsilon_i + 2\epsilon_j)/2$$

for $i \neq j$ and, by construction, $x_{\epsilon_i+\epsilon_j}$ is, indeed, in the affine hull of $x_{2\epsilon_i}$ and $x_{2\epsilon_j}$ (which is the intersection of the $d-1$ hyperplanes H_k with $k \in (0{:}d)\backslash\{i, j\}$). □

(27) Proposition. *A set X in $\mathbb{F}^d$ is an* AN_n*-set if and only if it is a* FGPL_n*-set satisfying*

$$\alpha(r) > 0 \quad \Longrightarrow \quad \#\bigcap_{j\neq r} H^j_{\alpha(j)} = 1. \tag{23}$$

Proof: By (20)Proposition and (22)Remark, any FGPL_n-set satisfying (23) is an AN_n-set.

Assume, conversely, that X is an AN_n-set with respect to a certain labeling $X = \{x_\alpha : \alpha \in \Gamma_n\}$. Then (15)Proposition shows that X is a FGPL_n-set with the hyperplanes H^j_i, $i \in 0{:}(n-1)$ and $j \in 0{:}d$, defined in (16) in terms of that labeling of X. Hence it remains to prove (23). For this, with $k := \alpha(r) > 0$ and $\beta := \alpha - k\epsilon_r$, and by (18)Corollary,

$$\bigcap_{j\neq r} H^j_{\alpha(j)} = \bigcap_{j\neq r} \flat\{x_{\beta+k\epsilon_t} : t \neq j\},$$

and we recognize the right-hand side as the intersection of the d facets, of the nondegenerate simplex with vertices $x_{\beta+k\epsilon_t}$, $t \in 0{:}d$, that contain $x_{\beta+k\epsilon_r} = x_\alpha$, hence that intersection comprises exactly one point, x_α. □

(28) Corollary [CGS08]. *If X is an* AN_n*-set, then X is a* GPL_n*-set if and only if*

$$x_\alpha \in H^j_i \quad \Longrightarrow \quad \alpha(j) \geq i. \tag{29}$$

Proof: Since (6) implies (29), we only have to prove the "if". For this, we note that (29) together with (9) (known to be true for any AN_n-set, by (15)Proposition) implies (6), while (8) (known to be true for any AN_n-set, by (15)Proposition) together with (23) implies (5) except for the claim

$$\alpha(r) = 0 \quad \Longrightarrow \quad \bigcap_{j\neq r} H^j_{\alpha(j)} = \{x_\alpha\} \tag{30}$$

when $\alpha(j) < n$ for all $j \neq r$. But for such α, $k := \alpha(s) > 0$ for some $s \neq r$ and, with $\beta := \alpha - k\epsilon_s$, we conclude from (21) (applicable, by (22)Remark, since we know (23)) that $\cap_{j\neq r,s} H^j_{\alpha(j)} = \flat\{x_\alpha, x_{\alpha-k\epsilon_s+k\epsilon_r}\}$ while (6) implies that $x_{\alpha-k\epsilon_s+k\epsilon_r} \notin H^s_{\alpha(s)}$, and (30) follows. □

My initial attempts at finding some FGPL_n-set that is not an AN_n-set were ultimately defeated because of the following

(31) Theorem. $\mathrm{AN}_n = \mathrm{FGPL}_n$.

Proof: Because of (27)Proposition, we only need to prove that any FGPL_n-set $X = \{x_\alpha : \alpha \in \Gamma_n\}$ with corresponding hyperplanes $(H^j_i : i \in 0{:}(n-1), j \in 0{:}d)$ satisfies

$$\alpha(r) > 0 \quad \Longrightarrow \quad \#\bigcap_{j\neq r} H^j_{\alpha(j)} = 1. \tag{23}$$

This is known to be true when $\alpha(j) = 0$ for all $j \neq r$ since then each $H^j_{\alpha(j)}$ with $j \neq r$ is **maximal** for X in the sense that $\#(X \cap H^j_{\alpha(j)})$ is as large as possible since it equals $\dim \Pi_n(H^j_{\alpha(j)})$, and, according to [B], the maximals for any GC_n-set are in general position.

Hence, to finish the proof, it suffices to prove (23) by induction on n under the additional assumption that $\alpha(j) > 0$ for some $j \neq r$. In that case, x_α is in

$$X_{\backslash j} := \{x_\beta := x_{\beta+\epsilon_j} : \beta \in \Gamma_{n-1}\},$$

and one verifies that this is a FGPL_{n-1}-set in $\mathbb{F}^d$, with

$$K^r_i := \begin{Bmatrix} H^r_i, & r \neq j \\ H^j_{i+1}, & r = j \end{Bmatrix}, \quad i \in 0{:}(n-2), \quad r \in 0{:}d,$$

the corresponding hyperplanes, hence, with $\beta := \alpha - \epsilon_j$, $\#\bigcap_{s\neq r} K^s_{\beta(s)} = 1$ by induction hypothesis while $\bigcap_{s\neq r} K^s_{\beta(s)} = \bigcap_{s\neq r} H^s_{\alpha(s)}$. □

Here, finally, is a brief discussion of the background on Aitken-Neville sets.

Let X be a set so indexed as $X = \{x_\alpha : \alpha \in \Gamma_n\}$ that (13) holds, hence, for each $k \in 1{:}n$ and each $\beta \in \Gamma_{n-k}$, there is a unique interpolant $P_\beta f$ from $\Pi_{\le 1}$ to arbitrary data values f_α given at the data sites x_α, $\alpha \in \beta + k\Gamma_1$, with the interpolant necessarily writable in Lagrange form as

$$P_\beta f =: \sum_{j=0}^{d} f_{\beta+k\epsilon_j} \ell_{\beta,j}.$$

With this, [SX] introduces the following multivariate **Aitken-Neville algorithm**:

$$\varphi_\beta := \left\{ \begin{array}{ll} f_\beta, & |\beta| = n \\ \sum_{j=0}^{d} \varphi_{\beta+\epsilon_j} \ell_{\beta,j}, & |\beta| < n \end{array} \right\}, \qquad |\beta| = n, n-1, \ldots, 0.$$

Evidently, $\deg \varphi_\beta \le n - |\beta|$ (since each $\ell_{\beta,j}$ is of degree 1). The major result in [SX] concerning this algorithm is

Theorem ([SX]). *Assume that $X = \{x_\alpha : \alpha \in \Gamma_n\}$ satisfies (13), and let φ_β, $\beta \in \Gamma_n$, be the polynomials generated by the Aitken-Neville algorithm. Then*

$$\varphi_\beta(x_\gamma) = f_\gamma, \qquad \gamma \in \beta + \Gamma_k,\ \beta \in \Gamma_{n-k}, \tag{32}$$

for $k \in 1{:}n$ and for arbitrary $f := (f_\alpha : \alpha \in \Gamma_n)$ if and only if X is an AN_n-set, i.e., if and only if X also satisfies (14).

In particular, assuming now $X = \{x_\alpha : \alpha \in \Gamma_n\}$ to be an AN_n-set with this particular labeling, the resulting φ_0 is a polynomial of degree $\le n$ matching the given values on all of X, and, as this holds for arbitrary data values and $\#X \le \#\Gamma_n = \dim \Pi_{\le n}$, it follows that φ_0 is the unique interpolant from $\Pi_{\le n}$ to the data values. More than that, it follows that

$$\varphi_0 = \sum_{\alpha \in \Gamma_n} f_\alpha \ell_\alpha,$$

with

$$\ell_\alpha := \sum_{j \in (0:d)^n,\ \sum_{i=1}^{n} \epsilon_{j(i)} = \alpha} \ \prod_{i=1}^{n} \ell_{\sum_{r<i} \epsilon_{j(r)}, j(i)}\ , \tag{33}$$

which should lead to some interesting identities, given that all the summands in this formula for ℓ_α necessarily are scalar multiples of ℓ_α since they all have the union of $(H_i^j : i < \alpha(j),\ j \in 0{:}d)$ as their zero set (with H_i^j as defined in (16)).

Note that, with the simple change $\ell_{\beta,j} \to \ell_j$, the Aitken-Neville algorithm becomes the **de Casteljau algorithm**:

$$\varphi_\beta := \left\{ \begin{array}{ll} f_\beta, & |\beta| = n \\ \sum_{j=0}^{d} \varphi_{\beta+\epsilon_j} \ell_j, & |\beta| < n \end{array} \right\}, \qquad |\beta| = n, n-1, \ldots, 0,$$

in which the ℓ_j are the Lagrange polynomials for interpolation from $\Pi_{\le 1}$ at the $d+1$ vertices of a simplex in general position in $\mathbb{R}^d$ and

$$\varphi_0 = \sum_{\beta \in \Gamma_n} f_\beta \binom{n}{\beta} \ell^\beta$$

is the **Bernstein-Bézier form** of the polynomial φ_0 with respect to that set of vertices, i.e., the f_β being the coefficients and $\ell^\beta := \ell_0^{\beta(0)} \cdots \ell_d^{\beta(d)}$.

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