Error Estimates for Thin Plate Spline Approximation in the Disc

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Abstract

This paper is concerned with approximation properties of linear combinations of scattered translates of the thin-plate spline radial basis function $|\cdot|^2 \log|\cdot|$ where the translates are taken in the unit disc D in $\mathbb{R}^2$. We show that the L_p approximation order for this kind of approximation is $2+1/p$ (for sufficiently smooth functions), which matches Johnson's upper bound and, thus, gives the saturation order. We also show that when one increases the density of the centers at the boundary, approximation order 4 – the best possible order in the absence of a boundary – can be obtained.

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1 Introduction

The theory of radial basis functions has been developed to treat the approximation of multivariate functions. It is especially well suited for functions sampled at scattered locations. Approximants are created by taking linear combinations of scattered translates of a fixed "radial basis" function ϕ. That is, let $\Xi \subset \overline{\Omega}$ (where $\Omega \subset \mathbb{R}^d$) be a finite set of points, hereafter called "centers". For a function $f : \Omega \to \mathbb{C}$, one attempts to find an approximant $s_{f,\Xi}$ in the space

$$S(\phi, \Xi) := S_0(\phi, \Xi) := \operatorname*{span}_{\xi\in\Xi} \phi(\cdot - \xi), \tag{1.1}$$

or, alternatively, in one of the related spaces

$$S_m(\phi, \Xi) := \left\{\sum A_\xi \phi(\cdot - \xi) + p : \ p \in \Pi_{m-1} \text{ and } \sum A_\xi q(\xi) = 0 \text{ for all } q \in \Pi_{m-1}\right\}. \tag{1.2}$$

These approximations should improve as Ξ becomes dense in Ω, and one of the primary goals of this theory is to judge how quickly this convergence takes place. Knowing the rate of convergence can be used to evaluate the quality of a specific method of choosing approximants (a *scheme*), e.g. interpolation, where $s_{f,\Xi} = I_\Xi f$ is chosen to be the unique function in $S_m(\phi, \Xi)$ satisfying $s_{f,\Xi}{}_{|\Xi} = f_{|\Xi}$. It can also be used to evaluate the approximation power of the spaces $S_m(\phi, \Xi)$, that is, to give a rate against which to measure the performance of other multivariate approximation schemes.

Finding the rate of convergence of an approximation scheme generally means estimating the decay of the error $e(\Xi, f)_p := \|s_{f,\Xi} - f\|_{L_p}$ of these approximations as Ξ becomes dense in Ω. This measurement takes the form of gauging the *approximation order* – a number $\gamma > 0$ such that

$$e(\Xi, f)_p = O(h^\gamma)$$

where h, the *fill distance*

$$h := h(\Xi, \Omega) := \sup_{x\in\Omega} \min_{\xi\in\Xi} |x - \xi|,$$

measures the density of Ξ in Ω.

In this paper we investigate thin plate spline approximation in the unit disc – a type of radial basis function approximation, where

$$\phi := |\cdot|^2 \log |\cdot|$$

is the fundamental solution of the 2-fold Laplacian (in $\mathbb{R}^2$), and the functions being approximated are defined on the unit disc $D := \{x \in \mathbb{R}^2 : |x| < 1\}$.

Results concerning approximation order in this (and more general) settings were first obtained by Duchon in [9] and [10] – there it was shown that on domains satisfying an interior cone condition, interpolation of a function possessing square integrable second derivatives delivers approximation order $\gamma_p := \min(2, 1 + 2/p)$. Duchon's approach was extended by Madych and Nelson [18] to treat interpolation by many more radial basis functions – see the surveys [22] or [25] for a detailed history.

Although it is much more restrictive, radial basis function approximation in the shift invariant setting – where centers are assumed to be $h\mathbb{Z}^n$ and the domain of f is all of $\mathbb{R}^n$ – yields drastically improved approximation orders. For instance, [6] treats interpolation in this setting. It is shown that in $\mathbb{R}^2$, interpolation by functions in $S(\phi, h\mathbb{Z}^2)$ (in this case, since Ξ is not finite, and since ϕ has global support, one considers approximants from a suitable space generated by a localization – see [17]) for sufficiently smooth functions delivers approximation order 4, which is best possible in the sense that $d(f, S(\phi, h\mathbb{Z}^2))_p \neq o(h^4)$ for some bandlimited function f (this was obtained in [17, Theorem 3.1]). When $p = 2$, [23, Proposition 4.1] implies that $d(f, S(\phi, h\mathbb{Z}^2))_2 \neq o(h^4)$ holds for all $f \in W_2^4$. Other "free space" results for thin plate spline approximation were obtained in [11], [19], and most recently [7] – these show for various schemes that the approximation order 4 can be attained when $\Omega = \mathbb{R}^2$ and Ξ is scattered.

The improvement in approximation order shows the difficulty posed by the existence of a boundary. Highlighting this is the inverse result of Johnson, which shows that for $\Omega = D$, $\Xi \subset (1 - h)D$ and for any m, $1 \le p \le \infty$, there exists $f \in C^\infty(\overline{D})$ such that

$$d(f, S_m(\phi, \Xi))_p \neq o(h^{2+1/p}) \tag{1.3}$$

Moreover, Matveev [20] and Johnson [16] each showed that $|f(x) - I_\Xi f(x)| = O(h^4)$ for x suitably removed from the boundary, which indicates that the boundary effects of thin plate spline approximation occur in a thin layer around the boundary.

The current state of the art for thin plate spline approximation with scattered centers in bounded domains comes from interpolation by splines in $S_2(\phi, \Xi)$. We separate this into two cases, depending on the parameter p. For $\Omega \subset \mathbb{R}^2$ having sufficiently smooth boundary and for sufficiently smooth f (specifically for f in the Sobolev space $W_2^3(\mathbb{R}^2)$ when $p = 1$ and for f in the Besov space $B_{2,1}^{2+1/p}(\mathbb{R}^2)$ when $1 < p \le 2$)

$$\|f - I_\Xi f\|_p = O(h^{2+1/p}) \tag{1.4}$$

holds for $1 \le p \le 2$ – this is to be found in [16]. Clearly, this is the best possible approximation order. On the other hand, for $p > 2$ and for sufficiently smooth f (for f in the Besov space $B^{5/2}_{2,1}(\mathbb{R}^2)$)

$$\|f - I_\Xi f\|_p = O(h^{\gamma_p + 1/2}) \tag{1.5}$$

holds. This has been shown in [15]. Thus, there is still a gap between the best approximation order for $p > 2$ and Johnson's upper bound (1.3). Moreover, the classes of functions for which (1.5) and (1.4) hold – except when $p = 2$ – are smaller than one would expect.

The primary purpose of this article is to demonstrate that, for sufficiently smooth functions in the unit disc, Johnson's upper bound (1.3) on the approximation order is attained. Specifically, we will prove the following, which is a softened version of the results of Section 4.

Theorem 1. *For $f \in \mathcal{A}_p$, there is $s_f \in S(\phi, \Xi)$ such that*

$$\|f - s_f\|_{L_p(D)} = O(h^{2+1/p}).$$

Here $\mathcal{A}_p := B^{2+1/p}_{p,1}(D)$ for $1 < p < \infty$, and for any $\epsilon > 0$ $\mathcal{A}_\infty = C^{2+\epsilon}(\overline{D})$ and $\mathcal{A}_1 = B^{3+\epsilon}_{1,1}(D)$. The cases $p = 1, \infty$ are exceptional cases in our approach, and we focus extra attention here. We show that if one reduces the smoothness assumptions in these cases (i.e., assumes $f \in W^3_1$ or C^2, resp.) the error estimates are penalized by a factor of $|\log h|$. In addition to showing that the approximation power of the thin plate splines is $2 + 1/p$ (which is novel in the range $2 < p \le \infty$), this also demonstrates that the full approximation order holds for all functions with smoothness of order roughly $2 + 1/p$.

A secondary goal is to show that higher approximation orders can be achieved if one employs approximations utilizing centers with increased density at the boundary. For the sake of easy exposition, we explain this as follows (when the time comes, we will give a more general condition, which involves the density *purely* at the boundary and which the previous condition satisfies – for now we explain *increased density at the boundary* as increased density in the outermost annulus):

Theorem 2. *There is a constant $K > 0$ such that if the centers Ξ satisfy $h = h(\Xi, D)$ and, in the outer annulus, $h(\Xi, D \backslash (1 - Kh^{\sigma_p})D) = h^{\sigma_p}$, where $\sigma_p = 2 - \frac{2}{1+2p}$, then for $f \in W^4_p(D)$, there is $s_{f,\Xi} \in S(\phi, \Xi)$ such that*

$$\|f - s_{f,\Xi}\|_{L_p(D)} = O(h^4).$$

This result is in agreement with Johnson's inverse result (1.3), since there it was assumed that $\Xi \subset (1-h)D$, which precludes any extra density at the boundary.

A happy consequence of this is that by adding marginally few extra centers (with cardinality at most proportional to the original set of centers) near the boundary, the optimal approximation order 4 can be obtained. Specifically we show that by supplementing an initial set of centers Ξ_1 possessing density $h = h(\Xi_1)$ with extra centers Ξ_2 located near to the boundary, and with $\#(\Xi_2)$ proportional to $h^{1/p-2}$ (which is, in turn, no greater than a multiple of $\#(\Xi_1)^{1-1/(2p)}$) to obtain $\Xi = \Xi_1 \cup \Xi_2$ then Theorem 2 follows.

1.1 The Approximation Scheme

The results of this paper derive from a careful error analysis of a new thin plate spline approximation scheme specific to the disc. This scheme consists of three parts, and takes the form:

$$T_\Xi f(x) = \frac{1}{8\pi}\left(\iint_D \Delta^2 f(\alpha)\phi_0(x,\alpha)\,\mathrm{d}\alpha + \int_{\partial D} Nf(\alpha)\phi_1(x,\alpha)\,\mathrm{d}\sigma(\alpha) + \int_{\partial D} Mf(\alpha)\phi_2(x,\alpha)\,\mathrm{d}\sigma(\alpha)\right). \tag{1.6}$$

Each component involves applying a linear operator to the function f (to be approximated) and integrating the result against a kernel ϕ_i. The first integral is taken over the disc and the corresponding operator is the Laplacian squared Δ^2, whereas the second and third integrals are each taken over the boundary and involve operators (M and N) that are not quite – but correspond closely to – third and second order differential operators. Furthermore, for each α, the function $\phi_i(\cdot,\alpha)$ is in $S(\phi,\Xi)$, viz.

$$\phi_i(x,\alpha)=\sum_{\xi\in\Xi}a_i(\alpha,\xi)\phi(x-\xi)\quad i=0,1,2, \tag{1.7}$$

and, hence, by interchanging summation and integration, $T_\Xi f\in S(\phi,\Xi)$. We postpone the discussion of how to select the "error kernels" $(\alpha,\xi)\mapsto a_i(\alpha,\xi)$ until Section 3.1.

We note that in [7] an approximation scheme is developed exploiting the identity

$$f(x)=\frac{1}{8\pi}\phi*\Delta^2 f(x), \tag{1.8}$$

which holds for sufficiently smooth functions having suitable decay. The approximation scheme there is $S_\Xi f=\frac{1}{8\pi}\int_{\mathbb{R}^2}\tilde{\phi}(\cdot,\alpha)\Delta^2 f(\alpha)\,\mathrm{d}\alpha$, where $\tilde{\phi}(\cdot,\alpha)\in S(\phi,\Xi)$. In this paper we do something similar, but we need a representation $f=\phi*\tau_f$ where τ_f is a distribution with support in $\overline{D}$. One example of such a representation is Green's representation. Suppose $f\in C^4(\overline{D})$, then by the Green's representation for f ([2, p. 10]),

$$\begin{aligned}f(x)=\frac{1}{8\pi}\iint_D&\phi(x-\alpha)\Delta^2 f(\alpha)\,\mathrm{d}\alpha\\&-\frac{1}{8\pi}\int_{\partial D}\phi(x-\alpha)D_n\Delta f(\alpha)-D_n\phi(x-\cdot)(\alpha)\Delta f(\alpha)\,\mathrm{d}\sigma(\alpha)\\&\qquad-\frac{1}{8\pi}\int_{\partial D}\Delta\phi(x-\cdot)(\alpha)D_n f(\alpha)-D_n\Delta\phi(x-\cdot)(\alpha)f(\alpha)\,\mathrm{d}\sigma(\alpha)\end{aligned} \tag{1.9}$$

Here $\Delta:=\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}$ is the usual Laplacian and $D_n:C^1(\overline{D})\to C(\partial D)$ is the outer normal derivative. Unfortunately $\Delta\phi$ and $D_n\Delta\phi$ each have singularities at the origin, frustrating the approximation at the boundary.

In [13, Corollary 3.13], Michael Johnson has found self-adjoint operators $U_1,U_2:C^\infty(\partial D)\to C(\partial D)$ and $V_1,V_2:C^\infty(\partial D)\to C(\partial D)$ that, for $x\in D$, satisfy the following relations

$$\mathrm{Tr}\Delta\phi(x-\cdot)\;=\;U_1D_n\phi(x-\cdot)+U_2\mathrm{Tr}\,\phi(x-\cdot) \tag{1.10}$$
$$D_n\Delta\phi(x-\cdot)\;=\;V_1D_n\phi(x-\cdot)+V_2\mathrm{Tr}\,\phi(x-\cdot). \tag{1.11}$$

Here Tr is the "trace" operator, defined initially as the restriction to the boundary. That is, for $f\in C(\overline{D})$, $\mathrm{Tr}f:=f_{|\partial D}$ – the use of trace, rather than restriction, is in anticipation of extending these results, and especially the forthcoming operators N and M to other smoothness spaces.

This allows us to represent $f\in C^\infty(\overline{D})$ using integrals involving translates of ϕ and first order derivatives of ϕ only. To wit, by inserting (1.10) and (1.11) in Green's representation (1.9), invoking the self-adjointness of the operators U_1,U_2,V_1,V_2 and rearranging terms gives Johnson's representation [13, Corollary 3.17]:

$$\begin{aligned}f(x)=\frac{1}{8\pi}\iint_D&\phi(x-\alpha)\Delta^2 f(\alpha)\,\mathrm{d}\alpha\\&+\frac{1}{8\pi}\int_{\partial D}\phi(x-\alpha)[V_2f(\alpha)-U_2D_nf(\alpha)-D_n\Delta f(\alpha)]\,\mathrm{d}\sigma(\alpha)\\&\qquad+\frac{1}{8\pi}\int_{\partial D}D_n\phi(x-\cdot)(\alpha)[V_1f(\alpha)-U_1D_nf(\alpha)+\Delta f(\alpha)]\,\mathrm{d}\sigma(\alpha).\end{aligned} \tag{1.12}$$

Thus, if we define

$$Nf := V_2 f - U_2 D_n f - D_n \Delta f \tag{1.13}$$

and

$$Mf := V_1 f - U_1 D_n f + \mathrm{Tr}\Delta f \tag{1.14}$$

we obtain an integral identity which states that, for $x \in D$

$$f(x) = \frac{1}{8\pi}\left(\iint_D \Delta^2 f(\alpha)\phi(x-\alpha)\,\mathrm{d}\alpha + \int_{\partial D} Nf(\alpha)\phi(x-\alpha)\,\mathrm{d}\sigma(\alpha) + \int_{\partial D} Mf(\alpha)D_n\phi(x-\cdot)(\alpha)\,\mathrm{d}\sigma(\alpha)\right).$$

The error analysis is performed by comparing each term in the identity with its corresponding term in the approximation scheme:

$$\begin{aligned} |f - T_\Xi f| &\lesssim \iint |\Delta^2 f(\alpha)| E_0(\cdot,\alpha)\,\mathrm{d}\alpha \\ &+ \int |Nf(\alpha)| E_1(\cdot,\alpha)\,\mathrm{d}\sigma(\alpha) \\ &+ \int |Mf(\alpha)| E_2(\cdot,\alpha)\,\mathrm{d}\sigma(\alpha) \end{aligned}$$

where

$$\begin{aligned} E_0 &:= |\phi_0(x,\alpha) - \phi(x-\alpha)|,, && (1.15)\\ E_1 &:= |\phi_1(x,\alpha) - \phi(x-\alpha)|,, && (1.16)\\ E_2 &:= |\phi_2(x,\alpha) - D_n\phi(x-\cdot)(\alpha)|. && (1.17) \end{aligned}$$

From this we are lead to error estimates which are optimal for thin plate spline approximation and which, moreover, demonstrate that

1. The second and (especially) the third terms are responsible for the "boundary effects", and
2. One can overcome these boundary effects by suitably increasing the density of centers at the boundary.

1.2 Preliminaries: Notation and Discussion of Function Spaces

We denote by $|t| := (t_1^2 + t_2^2)^{1/2}$, and the open disc of radius ρ centered at $x \in \mathbb{R}^2$ by $B_\rho(x)$, reserving the notation D for $B_1(0)$. In general, the boundary of a set Ω is denoted by $\partial\Omega$, and we identify ∂D with the circle group $\mathbb{T} := \mathbb{R}/2\pi\mathbb{Z}$.

We use multiinteger notation to describe monomials and partial derivatives. Unfortunately , this forces us to break one notational convention: for $\alpha \in \mathbb{Z}^2$, we define $|\alpha| := |\alpha_1| + |\alpha_2|$. Thus $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2}$, $\Pi_k := \mathrm{span}\{x \mapsto x^\alpha \mid |\alpha| \le k\}$ and the basic partial differential operators are written $D^\alpha := \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}}$. Given a unit vector η, we write the directional derivative D_η. Writing the outer normal vector at $\alpha \in \partial D$ as $n(\alpha)$ (and suppressing the argument whenever we can), we obtain the familiar outer normal derivative operator D_n.

In attempting to obtain the "correct" spaces for thin plate spline approximation (the largest space of functions yielding the optimal approximation order) we have employed several smoothness spaces. Some are ubiquitous (the Sobolev and Hölder spaces, denoted W_p^k and C^s and defined below), some are quite common (the Besov spaces $B_{p,q}^s$), and some are relatively uncommon (the Sobolev-Orlicz spaces $W^k L \log L$). We offer [1] as a reference which treats each of these spaces. Before we discuss some of the salient properties of these spaces, we define the following useful notation:

Given normed spaces X and Y, we denote by $B(X,Y)$ the space of bounded linear maps from X to Y equipped with the operator norm

$$\|R\| := \sup_{\substack{x\in X\\ x\neq 0|}} \frac{\|Rx\|_Y}{\|x\|_X}.$$

Furthermore, we use the notation $X \to Y$ to mean X is embedded in Y. That is, $X \subset Y$ and for all $x \in X$, $\|x\|_Y \lesssim \|x\|_X$

For $k \in \mathbb{N}$ and $1 \le p \le \infty$, the Sobolev space $W_p^k(\Omega)$ consists of functions f such that $D^\alpha f \in L_p(\Omega)$ for $|\alpha| \le k$. The $W_p^k(\Omega)$ norm is

$$\|f\|_{W_p^k(\Omega)} := \left(\sum_{\alpha\le k} \|D^\alpha f\|_p^p\right)^{1/p}.$$

Throughout this paper, Ω will be D,$\mathbb{T}$ or $\mathbb{R}^2$ equipped with Lebesgue measure, or ∂D equipped with the induced measure σ. Clearly, $\mathbb{T}$ and ∂D are in one-to-one correspondence, and $W_p^k(\partial D)$ is isomorphic to $W_p^k(\mathbb{T})$.

For $s > 0$, the Hölder space $C^s\left(\overline{\Omega}\right)$ consists of functions in $C^{\lfloor s\rfloor}\left(\overline{\Omega}\right)$ ($\lfloor s\rfloor$ is the greatest integer in s) satisfying $\sup_{x,y} |D^\alpha f(x) - D^\alpha f(y)| \le K|x-y|^{s-\lfloor s\rfloor}$ for some $K < \infty$. The norm in this space is

$$\|f\|_{C^s(\overline{\Omega})} := \|f\|_{C^{\lfloor s\rfloor}(\overline{\Omega})} + \max_{|\alpha|=\lfloor s\rfloor} \sup_{\substack{x,y\in\overline{\Omega}\\ x\neq y}} \frac{|D^\alpha f(x) - D^\alpha f(y)|}{|x-y|^{s-\lfloor s\rfloor}}.$$

When discussing Hölder spaces, Ω will be either D or $\mathbb{R}^2$.

For $k < s < m$ (with k and m being nonnegative integers) and $1 \le p, q \le \infty$, the Besov space $B_{p,q}^s(\Omega)$ comes about as the (θ, q)-interpolation space (with $\theta = (s-k)/(m-k)$) between Sobolev spaces $W_p^k(\Omega)$ and $W_p^m(\Omega)$. Like the Hölder spaces, Besov spaces capture fractional smoothness, which frequently plays a role in approximation schemes with fractional approximation orders. Of course, there are other fractional smoothness spaces – Besov spaces have been used because they provide adequate bounds for the trace operator (viz., $\mathrm{Tr} \in B(B_{p,1}^{1/p}(D), L_p(\partial D))$ which plays an important part in extending the operators N and M. For a discussion of Besov spaces and their trace theorems, we direct the reader to [1, Chapter 7]. For Besov spaces, we will be primarily concerned with $\Omega = D, \mathbb{R}^2$.

Finally, a discussion of the spaces $L_{\exp}(\Omega)$, $L\log L(\Omega)$ and $W^k L\log L(\Omega)$ is postponed until the next section. At this point, we will simply state that these are Banach spaces that are very near to $L_\infty(\Omega)$, $L_1(\Omega)$ and $W_1^k(\Omega)$, respectively.

The rest of this paper is arranged as follows. In Section 2 we discuss further Johnson's identity, the operators N and M as well as the auxiliary operators U_i, V_i and we identify smoothness spaces which N and M map into the spaces $L_p(\partial D)$. Section 3 gives a recipe for obtaining good kernels ϕ_i, estimates the "error kernels" E_i and estimates the norms of their induced integral operators. Section 4 contains the main results. Section 5 demonstrates how to overcome boundary effects by placing extra centers near the boundary, and it shows that this can be done at a marginal cost of additional centers. In Section 6, we modify the approximation scheme T_Ξ to deliver approximants in the space $S_2(\phi, \Xi)$ and we show that the results of Section 5 are extendable to this case. Certain technical details of the smoothness spaces $W^k L\log L$ that are used in our analysis are postponed until Section 7, which can be viewed as an appendix.

2 Analysis of the Boundary Operators N and M

The main purpose of this section is to provide "natural" smoothness spaces $\mathcal{A}_p$ of functions defined on the disc which the operators M and N, introduced in (1.14) and (1.13), map into the L_p spaces. This will allow us to extend the approximation scheme beyond the space $C^\infty(\overline{D})$ on which it is initially defined. Moreover, finding the natural smoothness spaces represents half of the work for obtaining error estimates. (The other half is measuring the norms of the error kernels in the operator norm $B(\mathcal{A}_p, L_p(\mathbb{T}))$ – this is done in the next section.)

The representation (1.12) was developed in [13] to give a precise expression of Duchon's thin plate spline interpolant $I_D f$, where $I_D f$ is the spline matching f on the set D and minimizing the seminorm

$$g \mapsto \left\| |\cdot|^2 \hat{g} \right\|_{L_2(\mathbb{R}^2)}$$

(here $\hat{g}$ signifies the Fourier transform of g). The operators U_1, U_2, V_1, V_2 are (initially) defined on the fundamental set of complex exponentials $e_k : \theta \mapsto e^{ik\theta}$ (identifying $\mathbb{T}$ with ∂D). One can find the original definition in the statement of [13, Corollary 3.17]. The following slightly different, but equivalent, definition consists of two parts. The first part is the action of the operators on linear trigonometric polynomials:

$$\begin{array}{lll} U_1 e_0 = 4e_0; & U_1 e_1 = e_1; & U_1 e_{-1} = e_{-1}; \\ U_2 e_0 = -4e_0; & U_2 e_1 = e_1; & U_2 e_{-1} = e_{-1}; \\ V_1 e_0 = 4e_0; & V_1 e_1 = -e_1; & V_1 e_{-1} = e_{-1}; \\ V_2 e_0 = -4e_0; & V_2 e_1 = -e_1; & V_2 e_{-1} = e_{-1}. \end{array} \tag{2.1}$$

The second part of the definition is for the complementary trigonometric polynomials (i.e. finite linear combinations $\tau = \sum a_j e_j$ where $a_j = 0$ for $|j| \leq 1$).

$$U_1 \tau := 2(W + I)\tau, \tag{2.2}$$
$$U_2 \tau := -2(W^2 + W)\tau, \tag{2.3}$$
$$V_1 \tau := 2(W^2 + W)\tau, \tag{2.4}$$
$$V_2 \tau := -2(W^3 + W^2)\tau. \tag{2.5}$$

where $W := i\mathcal{H}\frac{d}{d\theta}$ and $\mathcal{H}$ is the 2π-periodic Hilbert transform, which maps e_k to $-i\mathrm{sgn}(k)e_k$ (and, hence, commutes with the operator $\frac{d}{d\theta}$). Utilizing the projector P, which has range $\mathrm{span}\{e_{-1}, e_0, e_1\}$ and kernel containing e_k for $|k| \geq 2$, we see that (2.2) -(2.5) hold as operators on trigonometric polynomials modulo addition by a finite rank operator of the form ZP (here $Z : \mathrm{span}\{e_{-1}, e_0, e_1\} \to \mathrm{span}\{e_{-1}, e_0, e_1\}$). Since ZP is in $B(L_p, L_p)$ for each p, most of the task of finding the natural domains for N and M is understanding W as an operator on 2π-periodic functions – specifically, we seek the smoothness space $X_p^j(\mathbb{T})$ which W^j maps (boundedly) into $L_p(\mathbb{T})$. The domain is then the smoothness space which the trace operator, Tr (for V_i), or the normal derivative D_n (for U_i), maps (boundedly) into X_p^j.

L_p bounds for $1 < p < \infty$ We begin by examining this case because by a theorem of M. Riesz ([3]), $\mathcal{H} \in B(L_p(\mathbb{T}), L_p(\mathbb{T}))$. This allows us to view N and M as, essentially, differential operators – the operator $\mathcal{H}$ presents no special obstacle to finding the appropriate domains for N and M.

Proposition 3. *For $1 < p < \infty$, M and N can be extended so that $M \in B(B_{p,1}^{2+1/p}(D), L_p(\partial D))$ and $N \in B(B_{p,1}^{3+1/p}(D), L_p(\partial D))$.*

Proof. We have that $W \in B(W_p^1(\mathbb{T}), L_p(\mathbb{T}))$. Consequently,

$$\begin{aligned} U_1 &\in B(W_p^1(\mathbb{T}), L_p(\mathbb{T})), &(2.6)\\ U_2 &\in B(W_p^2(\mathbb{T}), L_p(\mathbb{T})), &(2.7)\\ V_1 &\in B(W_p^2(\mathbb{T}), L_p(\mathbb{T})), &(2.8)\\ V_2 &\in B(W_p^3(\mathbb{T}), L_p(\mathbb{T})). &(2.9)\end{aligned}$$

Furthermore, Tr is in $B(B_{p,1}^{k+1/p}(D), W_p^k(\partial D))$ for $k = 0, 1, 2, ..$ and D_n is in $B(B_{p,1}^{k+1+1/p}(D), W_p^k(\partial D))$ (see [1, Chapter 6]). Thus,

$$\begin{aligned} \|Mf\|_{L_p(\partial D)} &\leq \text{const}(p)\|f\|_{B_{p,1}^{2+1/p}(D)}, &(2.10)\\ \|Nf\|_{L_p(\partial D)} &\leq \text{const}(p)\|f\|_{B_{p,1}^{3+1/p}(D)}. &(2.11)\end{aligned}$$

□

First L_∞ and L_1 bounds $\mathcal{H}$ is, unfortunately, not bounded on L_1 or L_∞ (or even C), which makes obtaining uniform bounds for Nf and Mf not entirely straightforward. In either case, we can make the sacrifice of requiring a little more smoothness in order to find a space that works.

Proposition 4. *For $\epsilon > 0$, M and N can be extended so that $M \in B(C^{2+\epsilon}(\overline{D}), C(\partial D))$ and $N \in B(C^{3+\epsilon}(\overline{D}), C(\partial D)$.*

Proof. For $0 < \epsilon < 1$, $\mathcal{H}$ is a bounded operator from $C^\epsilon(\mathbb{T})$ to $C(\mathbb{T})$ (indeed, to C^ϵ). With this in mind, we can proceed precisely as in the case $1 < p < \infty$ (with trace meaning restriction). Thus, there is a constant, depending only on ϵ so that, for $f \in C^{2+\epsilon}$

$$\|Mf\|_\infty \leq \text{const}\,(\epsilon)\|f\|_{C^{2+\epsilon}(\overline{D})}, \tag{2.12}$$

and, for $f \in C^{3+\epsilon}$

$$\|Nf\|_\infty \leq \text{const}\,(\epsilon)\|f\|_{C^{3+\epsilon}(\overline{D})}. \tag{2.13}$$

□

In the case of L_1, we make use of the estimates already obtained for $1 < p < \infty$. Since L_p is embedded in L_1, we can use (2.10) and (2.11) to obtain, for $f \in B_{p,1}^{2+1/p}$,

$$\|Mf\|_1 \leq (2\pi)^{1-1/p}\|Mf\|_p \leq \text{const}\,(p)\|f\|_{B_{p,1}^{2+1/p}(D)},$$

and, for $f \in B_{p,1}^{3+1/p}$,

$$\|Nf\|_1 \leq (2\pi)^{1-1/p}\|Nf\|_p \leq \text{const}\,(p)\|f\|_{B_{p,1}^{3+1/p}(D)}.$$

Given $\epsilon > 0$, and for p sufficiently small, we have the embedding $B_{1,q}^{j+\epsilon} \to B_{p,1}^{j+1/p}$. So, for f with a little extra smoothness, we have

$$\|Mf\|_1 \leq \text{const}\,(\epsilon, q)\|f\|_{B_{1,q}^{3+\epsilon}(D)},$$

and

$$\|Nf\|_1 \leq \text{const}\,(\epsilon, q)\|f\|_{B_{1,q}^{4+\epsilon}(D)}.$$

Second L_∞ and L_1 bounds We can say more, however, if we utilize the Zygmund spaces, which have arisen in the study of the 2π-periodic Hilbert transform. These are, for a general, finite measure space Ω, $L_{\exp}(\Omega)$ and $L\log L(\Omega)$. $L_{\exp}(\Omega)$ is the Banach space consisting of all integrable functions f which, for some positive λ, satisfy $\int e^{\lambda|f|} < \infty$. $L\log L(\Omega)$ is the Banach space which consists of integrable functions which satisfy $\int |f|\log_+|f| < \infty$, where $\log_+ s := \max(\log s, 0)$. They are given the norms

$$\|f\|_{L\log L} \quad := \quad \inf\left\{\ell > 0 : \int_\Omega \frac{|f(x)|}{\ell}\log_+\frac{|f(x)|}{\ell}\,\mathrm{d}x \le 1\right\}, \tag{2.14}$$

$$\|f\|_{L_{\exp}} \quad := \quad \inf\left\{\ell > 0 : \int_\Omega \frac{|f(x)|}{\ell}\,\mathrm{d}x \le 1\right\}. \tag{2.15}$$

They are closely related to L_∞ and L_1, respectively, as evidenced by the embedding

$$L_\infty \to L_{\exp} \to L_p \to L\log L \to L_1, \quad 1 < p < \infty, \tag{2.16}$$

and are associated in the sense that the Hölder-like inequality

$$\left|\int fg\right| \lesssim \|f\|_{L_{\exp}}\|g\|_{L\log L}$$

holds, as does the converse:

$$\|f\|_{L\log L(\Omega)} \lesssim \sup_{\|g\|_{L_{\exp}}=1}\left|\int fg\right|.$$

In fact, one has $L_{\exp}(\Omega) = (L\log L(\Omega))'$. For good measure, and for use later, we give a third equivalent norm for $L\log L(\Omega)$:

$$\|f\|_{L\log L(\Omega)} \sim \int_0^{|\Omega|} f^*(t)\log_+(|\Omega|/t)\,\mathrm{d}t,$$

where f^* is the decreasing rearrangement of f, defined as $f^*(t) := \sup\left\{\lambda \mid \ t < \left|f^{-1}\left(\,(\lambda,\infty)\,\right)\right|\right\}$.

These spaces are especially useful to us, because $\mathcal{H}$ maps $L_\infty(\mathbb{T})$ boundedly to $L_{\exp}(\mathbb{T})$:

$$\|\mathcal{H}f\|_{L_{\exp}(\mathbb{T})} \lesssim \|f\|_{L_\infty(\mathbb{T})}, \tag{2.17}$$

and it maps $L\log L(\mathbb{T})$ boundedly to $L_1(\mathbb{T})$:

$$\|\mathcal{H}f\|_{L_1(\mathbb{T})} \lesssim \|f\|_{L\log L(\mathbb{T})}. \tag{2.18}$$

An excellent introduction, including definitions and basic properties related to the operator $\mathcal{H}$ which we use here can be found in [5, Ch. 4, Sect.s 6 and 8].

For the L_∞ bounds, we can immediately apply (2.17), which shows us that $W \in B(C^1(\mathbb{T}), L_{\exp}(\mathbb{T}))$. Thus, proceeding as in the case $1 < p < \infty$,

$$\|Mf\|_{L_{\exp}} \lesssim \|f\|_{C^2(\overline{D})} \tag{2.19}$$

and, for $f \in C^3(\overline{D})$,

$$\|Nf\|_{L_{\exp}} \lesssim \|f\|_{C^3(\overline{D})}. \tag{2.20}$$

For the L_1 bounds, we will employ (2.18), but this situation is slightly more complicated because the smoothness space must consist of functions possessing derivatives in $L\log L$. Here we utilize the Sobolev-type space $W^kL\log L(\Omega)$ (actually, this is a Sobolev-Orlicz space, as we remark below), which consists of all functions f possessing weak derivatives of order up to k in $L\log L(\Omega)$. (Obviously, Ω is assumed to be a suitable set. In our case, either a bounded region in $\mathbb{R}^2$ or $\mathbb{T}$.) $W^kL\log L(\Omega)$ is given the norm

$$\|f\|_{W^kL\log L(\Omega)} = \sum_{|\alpha|\le k}\|D^\alpha f\|_{L\log L(\Omega)}.$$

A consequence of the embedding (2.16) is that, for $|\Omega| < \infty$ and $p > 1$, we have the embedding $W_p^k(\Omega) \to W^k L \log L(\Omega)$.

We are particularly interested in the spaces $W^k L \log L(\mathbb{T})$ for $k = 1, 2, 3$, because evidently U_1 maps $W^1 L \log L(\mathbb{T})$ boundedly to $L \log L(\mathbb{T})$, and V_1 maps $W^2 L \log L(\mathbb{T})$ boundedly to $L \log L(\mathbb{T})$. All that remains is a trace theorem which allows us to map functions defined on the disc to their boundary values. This is to be found in Lemma 18, a consequence of which states that Tr maps $W^{k+1} L \log L(D)$ boundedly to $W^k L \log L(\partial D)$. Hence,

$$\|Mf\|_1 \lesssim \|f\|_{W^3 L \log L(D)}. \tag{2.21}$$

Using a similar argument, we have that for $f \in W^4 L \log L(D)$

$$\|Nf\|_1 \lesssim \|f\|_{W^4 L \log L(D)}. \tag{2.22}$$

As a final note regarding the background of $L_{\exp}$ and $L \log L$, we observe that both can be viewed as Orlicz spaces – spaces, each associated with a certain nonnegative function B, consisting of functions which, when scaled and composed with B, are integrable. That is, spaces of functions which consist of all f for which there exist $\lambda > 0$ such that $B(\lambda|f|)$ is integrable. This context, although not strictly necessary for the time being, is useful in Section 7, when we directly employ the theory of Sobolev-Orlicz spaces to obtain the trace theorem for $W^k L \log L(D)$.

Observe that we are now able to extend the domain of the approximation scheme T_Ξ. We have:

Lemma 5. *If the coefficient kernels a_0, a_1 and a_2 (introduced in 1.7) each satisfy the uniform boundedness condition*

$$\sup_\alpha \|a_i(\alpha, \cdot)\|_{\ell_\infty} < \infty,$$

then, for $f \in W^4 L \log L(D)$, the approximant given by (1.6) is well defined.

3 Estimating the Error Kernels

As we will see in the next section, the key to providing good approximations by functions of the form (1.6) is to obtain suitable decay and a sufficiently small uniform bound on the "error kernels" $E_0(x, \alpha) = |\phi(x - \alpha) - \phi_0(x, \alpha)|$, $E_1(x, \alpha) = |\phi(x - \alpha) - \phi_1(x, \alpha)|$ and $E_2(x, \alpha) = |D_n \phi(x - \cdot)(\alpha) - \phi_2(x, \alpha)|$ (with ϕ_i as in (1.7)). In order to obtain appropriate error bounds on the error kernels E_i, we need coefficient kernels that reproduce the functionals of point evaluation and directional derivative on spaces of polynomials of fixed degree, and, especially at the boundary, we want a high degree of polynomial precision: the degree of polynomial precision of the "coefficient kernels" a_i is tied to the rate of decay of the error kernels. In order to diminish boundary effects, it will be important for $E_1(\cdot, \alpha)$ and $E_2(\cdot, \alpha)$ to have a high rate of decay (as $|x - \alpha|$ increases) for α on the boundary.

Our first lemma shows that ϕ (or an arbitrary directional derivative of ϕ) can be approximated well by a few nearby scattered translates producing decreasing and bell-shaped error.

Lemma 6. *Let $\alpha \in \mathbb{R}^2$ and suppose that $c \in \mathbb{R}^\Xi$ satisfies supp$(c) \subset B_\rho(\alpha)$, for some $\rho > 0$.*

If $\sum c(\xi) p(\xi) = p(\alpha)$ for all $p \in \Pi_m$, then, for every $x \in \mathbb{R}^2$,

$$\left| \phi(x - \alpha) - \sum_{\xi \in \Xi} c(\xi) \phi(x - \xi) \right| \le \text{const}\,(m)(1 + \|c\|_{\ell_1}) \rho^2 \left(1 + \frac{|x - \alpha|}{\rho} \right)^{1-m}. \tag{3.1}$$

Alternatively, if, for some $|\eta| = 1$, $\sum c(\xi)p(\xi) = \rho(D_\eta p)(\alpha)$ *for all* $p \in \Pi_m$*, then for every* $x \in \mathbb{R}^2$

$$\left| D_\eta \phi(x-\alpha) - \sum_{\xi\in\Xi} \frac{c(\xi)}{\rho} \phi\,(x-\xi) \right| \leq \mathrm{const}\,(m)(1+\|c\|_{\ell_1})\rho \left(1+\frac{|x-\alpha|}{\rho}\right)^{1-m}. \tag{3.2}$$

Proof. For both estimates, we consider the case when x is far from α separately from the case when x and α are nearby.

We consider the first case, $|x-\alpha| > 2\rho$, because here the sum $\sum c(\xi)\phi\,(\cdot - \xi)$ is in C^∞. Let P be the Taylor polynomial to ϕ of degree m and centered at $x-\alpha$. Then $\phi(x-\xi) = P(x-\xi) + R(x-\xi)$ where R is the remainder term in Taylor's theorem

$$R(x-\xi) = \frac{1}{(m+1)!} |\alpha - \xi|^{m+1} D_\zeta^{m+1} \phi(x - \alpha + \sigma(\alpha-\xi)),$$

where $\zeta := \frac{\alpha-\xi}{|\alpha-\xi|}$ and $0 < \sigma < 1$. Since $|D_\zeta^{m+1}\phi(s)| \leq \mathrm{const}\,(m)|s|^{1-m}$ for $m \geq 2$ (see Lemma 7 below) we obtain, with $\lambda = \delta_{x-\alpha}$ (in the first case) or $\lambda = \rho\delta_{x-\alpha}D_\eta$ (in the second)

$$\begin{aligned} |\lambda\phi - \sum c(\xi)\phi\,(x-\xi)\,| &= |\lambda P - \sum c(\xi)\,(P(x-\xi) + R(x-\xi))\,| = \left|\sum c(\xi) R(x-\xi)\right| \\ &\leq \mathrm{const}\,(m)\|c\|_{\ell_1}\rho^{m+1}\,(|x-\alpha| - \rho)^{1-m} \leq \mathrm{const}\,(m)\|c\|_{\ell_1}\rho^2 \left(1 + \frac{|x-\alpha|}{\rho}\right)^{1-m} \end{aligned}$$

In the second case, (3.2) follows by dividing by ρ.

When $|x-\alpha| \leq 2\rho$, we utilize the fact that ϕ has a zero of order nearly 2 at the origin. Specifically, we make use of the fact that, for any nonzero ρ, $\phi = \rho^2\phi(\cdot/\rho) + |\cdot|^2 \log|\rho|$. In each case, it follows that

$$\lambda\phi - \sum c(\xi)\phi(x-\xi) = \rho^2\lambda\phi(\cdot/\rho) - \rho^2 \sum c(\xi)\phi((x-\xi)/\rho).$$

Thus, in the first case (when $\lambda = \delta_{x-\alpha}$),

$$\left|\lambda\phi - \sum c(\xi)\phi\,(\xi - x)\right| \quad \leq \quad \rho^2 \|\phi\|_{L_\infty(B_3(0))}(1 + \|c\|_{\ell_1})$$

and, in the second case (when $\lambda = \rho\delta_{\alpha-x}D_\eta$)

$$\left|\lambda\phi - \sum c(\xi)\phi\,(\xi - x)\right| \quad \leq \quad \rho^2 \left(\sqrt{2}\,\|\phi\|_{W^1_\infty(B_2(0))} + \|\phi\|_{L_\infty(B_3)(0)}\|c\|_{\ell_1}\right)$$

Again, the result is obtained by dividing through by ρ. □

The following is a technical fact about derivatives of the function ϕ used in Lemma 6.

Lemma 7. *For any integer* $m \geq 3$*, there exists a constant* $\mathrm{const}\,(m)$ *so that, for any direction* $|\zeta| = 1$ *the iterated directional derivative has polynomial decay:*

$$|D_\zeta^m \phi(s)| \leq \mathrm{const}\,(m)|s|^{2-m}.$$

Proof. First note that, by rotational symmetry of ϕ, it suffices to consider $\zeta = e_1$. By a simple computation, one easily sees that $D_{e_1}^3\phi(s) = 6\frac{s_1}{|s|^2} - 4\frac{s_1^3}{|s|^4} = \frac{s_1}{|s|^2}(6 - 4\frac{s_1^2}{|s|^2})$. By induction, it follows that

$$D_{e_1}^m \phi(s) = \begin{cases} \frac{s_1}{|s|^{m-1}}\, p_m\left(\frac{s_1^2}{|s|^2}\right), & \text{for odd } m, \\ \frac{1}{|s|^{m-2}}\, p_m\left(\frac{s_1^2}{|s|^2}\right), & \text{for even } m. \end{cases}$$

For each m, p_m is a polynomial, and it determines the constant in the statement of the theorem: $\mathrm{const}\,(m) = \|p_m\|_{L_\infty([0,1])}$. □

3.1 Interior and Boundary Conditions

Lemma 6 provides us with some insight how best to choose the coefficient kernels a_i – each must reproduce polynomials using only nearby centers. In this section we give specific conditions on each coefficient kernel.

Interior Center Assumptions. *We say that the kernel $c : D \times \Xi \to \mathbb{C}$ satisfies $IC(m, \rho, k)$ if, for every $\alpha \in D$:*

1. $\sum c(\alpha, \xi)p(\xi) = p(\alpha)$ *for all* $p \in \Pi_m$.
2. $\operatorname{supp}(c(\alpha, \cdot)) \subset B_\rho(\alpha)$.
3. $\sum |c(\alpha, \xi)| \le k$.

Boundary Center Assumptions I. *We say that the kernel $c : \partial D \times \Xi \to \mathbb{C}$ satisfies $BC1(m, \rho, k)$ if for every $\alpha \in \partial D$:*

1. $\sum c(\alpha, \xi)p(\xi) = p(\alpha)$ *for all* $p \in \Pi_m$.
2. $\operatorname{supp}(c(\alpha, \cdot)) \subset B_\rho(\alpha)$.
3. $\sum |c(\alpha, \xi)| \le k$.

Boundary Center Assumptions II. *We say that the kernel $c : \partial D \times \Xi \to \mathbb{C}$ satisfies $BC2(m, \rho, k)$ if for every $\alpha \in \partial D$:*

1. $\sum c(\alpha, \xi)p(\xi) = D_n p(\alpha)$ *for all* $p \in \Pi_m$.
2. $\operatorname{supp}(c(\alpha, \cdot)) \subset B_\rho(\alpha)$.
3. $\sum |c(\alpha, \xi)| \le k/\rho$.

For each set of assumptions, the first argument determines the degree of precision of the coefficients. As we have seen in Lemma 6, this controls the rate of decay of the error

$$|\lambda\phi - \sum c(\alpha, \xi)\phi(x - \xi)| \tag{3.3}$$

as $|x - \alpha|$ grows. It plays a small but important role in the error analysis. The success of the scheme relies on the global integrability of errors such as (3.1) and (3.2). Hence, m should be sufficiently large ($m \ge 4$). A secondary role it plays, is to diminish the effects of the boundary in the interior. The second argument, which corresponds to the size of the support of the coefficients, is a measure of the density of the nearby centers. Although ρ may be selected independently for all three conditions – a fact that we will exploit to good effect in Section 6 – in order to get more conventional results, we wish to find coefficients with ρ proportional to the fill distance $h(\Xi, D)$, while keeping the third coefficient as small as possible. Fortunately, this happens automatically when Ξ is sufficiently dense, as evidenced by the next proposition, which is essentially an application of some basic results from the theory of norming sets, developed in [12, Section 3] for the sphere. A clear exposition (and the one motivating the following lemma) is to be found in [25, Chapter 3].

Lemma 8. *For any degree of polynomial precision $m \ge 1$ and for $h(\Xi, D) \le 1/K_m$ where $K_m := 48m^2$ there exists a kernel $a_0 : D \times \Xi \to \mathbb{R}$ satisfying $IC(m, K_m h, 2)$ and there exist kernels $a_1, a_2 : \partial D \times \Xi \to \mathbb{R}$ satisfying $BC1(m, K_m h, 2)$ and $BC2(m, K_m h, \frac{2}{3}K_m)$ respectively.*

Proof. The result for a_0 and a_1 follow immediately from [25, Theorem 3.14], and the result for a_2, although not explicitly stated in the theorem, is a simple consequence. For the sake of completeness, we paraphrase the argument here.

The proof hinges on finding a nearby set $C_\alpha \subset D$ and a set of centers $\Xi_\alpha := C_\alpha \cap \Xi$, for which the point evaluation functionals $\{\delta_\xi \mid \xi \in \Xi_\alpha\}$ form a *norming set* for the space of polynomials Π_m, meaning that the restriction operator $R_{m,C_\alpha} : p \mapsto p_{|\Xi_\alpha}$ is $1-1$, and hence is invertible on its range. Furthermore, the *norming constant* should be 2, which means that for all $p \in \Pi_m$

$$\|p\|_{L_\infty(C_\alpha)} \leq 2\|R_{m,C_\alpha}p\|_{\ell_\infty(\Xi_\alpha)}.$$

It follows that for any functional μ on Π_m, there is a corresponding functional $(R^{-1}_{m,C_\alpha})^*\mu = \mu \circ R^{-1}_{m,C_\alpha}$ in $((\Pi_m)_{|\Xi_\alpha})'$, which can be extended (with the same norm by the Hahn-Banach theorem) to be in $(\ell_\infty(\Xi_\alpha))'$ and hence has a representation c_μ in $\ell_1(\Xi_\alpha)$ (again, having the same norm). That is,

$$\sum c_\mu(\xi)p(\xi) = \mu p$$

and $\|c_\mu\|_{\ell_1} \leq 2\|\mu\|_{(\Pi_m)'}$.

For $\alpha \in \overline{D}$, the set C_α is identified in [25, Theorem 3.14] as a cone, which we can take to be, for an appropriate choice of θ and r,

$$C_\alpha := \left\{\alpha + \lambda y \;\mid\; |y| = 1,\ |\zeta_\alpha| = -\frac{\alpha}{|\alpha|},\ y^T\zeta_\alpha \geq \cos\theta, 0 \leq \lambda \leq r\right\}.$$

In the interest of easy exposition, we have chosen $\theta = \pi/6$ (although there is some flexibility in selecting this), and $r = 48m^2$. This choice guarantees (by [25, Theorem 3.8] and [25, Proposition 3.13]) that Ξ_α provides a norming set for Π_m on C_α with norming constant 2. By the norming set argument, we have the existence of the kernels a_i, and since C_α is contained in the ball $B_{K_m h}(\alpha)$, it follows that $\rho = K_m h$.

The proof is finished by estimating the norms of the linear functionals $p \mapsto p(\alpha)$ and $p \mapsto D_n p(\alpha)$. Clearly, the first has norm less than 1, and, thus, we have that the kernels a_0, a_1 satisfy their respective conditions. By a multivariate version of the Bernstein inequality [25, Proposition 11.6] (which follows directly from the multivariate Bernstein inequality for cubes developed recently in [21, Section 6]), we have that

$$\|D_n p\|_{L_\infty(C_\alpha)} \leq (3h)^{-1}\|p\|_{L_\infty(C_\alpha)}.$$

It follows, thus, that $\|a_2(\alpha,\cdot)\|_{\ell_1} \leq \frac{2}{3h}$. ☐

3.2 Estimates on the Error Kernels

In light of Lemma 6, we can make some observations about the error kernels $E_i,\ i = 0, 1, 2$ (defined in (1.15), (1.16) and (1.17)), and on the operators they generate: $g \mapsto \int E_i(x,\alpha)g(\alpha)$. These results will be used in the next section to provide error estimates for the approximation scheme T_Ξ. The crucial step is to estimate integrals of $\left(1 + \frac{|x-\alpha|}{\rho}\right)^{1-m}$, which appeared in Lemma 6.

Observe that, employing a simple change to polar coordinates, we obtain

$$\iint_{\mathbb{R}^2} \left(1 + \frac{|y|}{\rho}\right)^{1-m} \mathrm{d}y \leq 2\pi \int_0^\infty \left(1 + \frac{r}{\rho}\right)^{1-m} r\,\mathrm{d}r \leq 2\pi\rho^2 \int_1^\infty t^{1-m}(t-1)\,\mathrm{d}t = \frac{2\pi\rho^2}{(m-3)(m-2)}$$

Thus, if a_0 satisfies $IC(m,\rho,k)$, with $m \geq 4$ then by Lemma 6 we obtain

$$\sup_{x\in D} \|E_0(x,\cdot)\|_1 \quad\leq\quad \mathrm{const}(k,m)\rho^4, \tag{3.4}$$

$$\sup_{\alpha\in D} \|E_0(\cdot,\alpha)\|_1 \quad\leq\quad \mathrm{const}(k,m)\rho^4. \tag{3.5}$$

The first of these inequalities shows that the operator $g \mapsto \int g(\alpha)E_0(\cdot,\alpha)\mathrm{d}\alpha$ is in $B(L_\infty(D), L_\infty(D))$ with norm less than $\mathrm{const}(k,m)\rho^4$, while the second gives an estimate for the norm in $B(L_1(D), L_1(D))$. Interpolating estimates (3.4) and (3.5), it follows that the operator $g \mapsto \int g(\alpha)E_0(x,\alpha)\mathrm{d}\alpha$ is in $B(L_p(D), L_p(D))$ for $1 \leq p \leq \infty$ with operator norm less than $\mathrm{const}(k,m)\rho^4$. That is, for $1 \leq p \leq \infty$,

$$\left\| \iint_D E_0(\cdot,\alpha)g(\alpha)\,\mathrm{d}\alpha \right\|_{L_p(D)} \leq \mathrm{const}\,(k,m)\rho^4\|g\|_{L_p(D)}. \tag{3.6}$$

We note that, although the interpolation can be justified using the Riesz Interpolation Theorem (also known as the Riesz Convexity Theorem [5, Ch. 4, Theorem 1.7]), because the kernel $(x,\alpha) \mapsto E_0(x,\alpha)$ is positive, the result can be obtained from scratch using Hölder's inequality (see [5, Ch. 4, Theorem 1.2]).

Along the same lines, if we assume that a_1 satisfies $BC1(m,\rho,k)$ and that a_2 satisfies $BC2(m,\rho,k/\rho)$ we obtain

$$\sup_{\alpha\in\partial D} \|E_1(\cdot,\alpha)\|_1 \quad\leq\quad \mathrm{const}(k,m)\rho^4, \tag{3.7}$$

$$\sup_{\alpha\in\partial D} \|E_2(\cdot,\alpha)\|_1 \quad\leq\quad \mathrm{const}(k,m)\rho^3. \tag{3.8}$$

Next we focus on estimating the boundary kernels. The difference we encounter in these estimates is that $\|E_1(x,\cdot)\|$ and $\|E_2(x,\cdot)\|$ are sensitive to the position of x. Thus we make separate estimates for x at different distances from the boundary. To this end, we define the "dyadic" annulus A_j, $j \in \mathbb{N}$:

$$A_j := A_j^\rho := \begin{cases} \{x \in D \mid d(x,\partial D) \leq \rho\}, & j = 0, \\ \{x \in D \mid 2^{j-1}\rho \leq d(x,\partial D) \leq 2^j\rho\}, & j \geq 0. \end{cases} \tag{3.9}$$

Lemma 9. *Let $m \geq 4$. For $x \in A_J$*

$$\left\| \left(1 + \frac{|x-\cdot|}{\rho}\right)^{1-m} \right\|_{L_1(\partial D)} \leq \mathrm{const}(m)\ 2^{(2-m)J}\rho,$$

and, for $\rho < 1/2\pi$,

$$\left\| \left(1 + \frac{|x-\cdot|}{\rho}\right)^{1-m} \right\|_{L\log L(\partial D)} \leq \mathrm{const}(m)\ 2^{(2-m)J}\rho\ |\log\rho|.$$

Proof. Both inequalities follow by estimating the decreasing rearrangement F^* of $F := \left(1 + \frac{|x-\cdot|}{\rho}\right)^{1-m}$. To do this, we decompose ∂D into arcs about x. For $j \in \{-1\} \cup \mathbb{N}$, define:

$$\omega_{-1} \quad:=\quad \emptyset \tag{3.10}$$

$$\omega_j \quad:=\quad \partial D \cap B_{2^j\rho}(x) \tag{3.11}$$

and note that the intersection of ∂D with balls of radius $2^j\rho$ are of length less than $2\pi 2^j\rho$. From this it follows that $2^j\rho \leq \sigma(\omega_j) \leq 2\pi 2^j\rho$ for $j > J$, and $0 \leq \sigma(\omega_J) \leq 2\pi 2^J\rho$. For α in the difference of two successive arcs, $\omega_j\backslash\omega_{j-1}$, one has

$$\left(1 + \frac{|x-\alpha|}{\rho}\right)^{1-m} \leq 2^{(j-1)(1-m)}.$$

We have also, that $r \mapsto F(\partial D \cap \partial B_r(x))$ is single valued, decreasing and, hence, for $\sigma(\omega_{j-1}) \leq t \leq \sigma(\omega_j)$, one has $F^*(t) \leq F^*(\sigma(\omega_{j-1})) = F\left(\partial D \cap \partial B_{2^{j-1}\rho}(x)\right) \leq 2^{(j-1)(1-m)}$. This allows us to estimate the above norms:

$$
\begin{aligned}
\left\| \left(1 + \frac{|x - \cdot|}{\rho}\right)^{1-m} \right\|_{L_1} &= \int_0^{2\pi} F^*(t)\,\mathrm{d}t \\
&\leq \sum_{j=J}^{\infty} \int_{\sigma(\omega_{j-1})}^{\sigma(\omega_j)} F^*(t)\,\mathrm{d}t \\
&\leq \sum_{j=J}^{\infty} \text{const } 2^j \ \rho\ 2^{(j-1)(1-m)} \\
&\leq \text{const}\,(m)\ 2^{(2-m)J}\rho.
\end{aligned}
$$

The $L \log L$ norm follows in much the same way:

$$
\begin{aligned}
\left\| \left(1 + \frac{|x - \cdot|}{\rho}\right)^{1-m} \right\|_{L\log L(\partial D)} &= \int_0^{2\pi} F^*(t) \log\left(\frac{2\pi}{t}\right)\,\mathrm{d}t \\
&= \sum_{j=J}^{\infty} \int_{\sigma(\omega_{j-1})}^{\sigma(\omega_j)} F^*(t) \log\left(\frac{2\pi}{t}\right)\,\mathrm{d}t \\
&\leq 2^{(J-1)(1-m)} \int_0^{\sigma(\omega_J)} \log\left(\frac{2\pi}{t}\right)\mathrm{d}t \\
&\quad + \sum_{j=J+1}^{\infty} \text{const}\, 2^j\ \rho\ 2^{(j-1)(1-m)}\ (\log 2\pi - j\log 2 - \log\rho) \\
&\leq \text{const}\,(m)\ 2^{J(2-m)}\rho\,(\log 2\pi + |\log\rho|)\,,
\end{aligned}
$$

where we have made use of the fact that $\int_0^{\sigma(\omega_J)} \log\frac{2\pi}{t}\mathrm{d}t \leq \text{const}\, 2^J\rho\,|\log\rho|$. Thus for $\rho \leq \frac{1}{2\pi}$, the lemma follows. $\square$

These last two estimates allow us to measure the norm of the boundary error kernels. That is, we have two sets of estimates, each complementary to (3.7) and (3.8). Again, assuming that a_1 and a_2 satisfy $BC1(m,\rho,k)$ and $BC2(m,\rho,k)$, respectively, then by Lemma 9 we obtain:

$$
\|E_1(x,\cdot)\|_{L\log L(\partial D)} \leq \text{const}(k,m)\left(1 + \frac{d(x,\partial D)}{\rho}\right)^{2-m} \rho^3\,|\log\rho|, \tag{3.12}
$$

$$
\|E_2(x,\cdot)\|_{L\log L(\partial D)} \leq \text{const}(k,m)\left(1 + \frac{d(x,\partial D)}{\rho}\right)^{2-m} \rho^2\,|\log\rho|, \tag{3.13}
$$

and

$$
\|E_1(x,\cdot)\|_1 \leq \text{const}(k,m)\left(1 + \frac{d(x,\partial D)}{\rho}\right)^{2-m} \rho^3, \tag{3.14}
$$

$$
\|E_2(x,\cdot)\|_1 \leq \text{const}(k,m)\left(1 + \frac{d(x,\partial D)}{\rho}\right)^{2-m} \rho^2. \tag{3.15}
$$

Interpolating between (3.7) and (3.14) shows that the operator $g \mapsto \int E_1(\cdot,\alpha)g(\alpha)\,\mathrm{d}\sigma(\alpha)$ is bounded on L_p:

$$
\left\| \int E_1(\cdot,\alpha)g(\alpha)\,\mathrm{d}\sigma(\alpha) \right\|_{L_p(D)} \leq \text{const}(p,k,m)\rho^{3+1/p}\|g\|_{L_p(\partial D)}. \tag{3.16}
$$

Likewise, interpolating between (3.8) and (3.15) shows that the operator $g \mapsto \int E_2(\cdot,\alpha)g(\alpha)\,\mathrm{d}\sigma(\alpha)$ is bounded on L_p :

$$\left\| \int E_2(\cdot,\alpha)g(\alpha)\,\mathrm{d}\sigma(\alpha)\right\|_{L_p(D)} \le \mathrm{const}(p,k,m)\rho^{2+1/p}\|g\|_{L_p(\partial D)}. \tag{3.17}$$

4 Main Theorem and Corollaries

Having discussed some conditions on the distribution of centers giving rise to error kernels E_i with nicely bounded operator norms, we are now ready to state our main theorem.

Theorem 10. *Suppose $m \ge 4$ is an integer, $\rho_1, \rho_2, k > 0$, and assume the existence of coefficient kernels $a_0 : D \times \Xi \to \mathbb{R}$ and $a_1, a_2 : \partial D \times \Xi \to \mathbb{R}$ that satisfy, respectively, $IC(m,\rho_1,k)$, $BC1(m,\rho_2,k)$ and $BC2(m,\rho_2,k)$. For $f \in C^4\left(\overline{D}\right)$ the approximant $T_\Xi f$ satisfies the pointwise bound,*

$$\begin{aligned} |T_\Xi f(x) - f(x)| \quad &\lesssim \quad \rho_1^4 \|\Delta^2 f\|_{L_\infty(\overline{D})} \\ &\quad + \left(1 + \frac{d(x,\partial D)}{\rho_2}\right)^{2-m} \rho_2^2 |\log \rho_2| \left(\rho_2 \|f\|_{C^3(\overline{D})} + \|f\|_{C^2(\overline{D})}\right). \end{aligned} \tag{4.1}$$

For $\epsilon > 0$, we have the slightly improved estimate

$$\begin{aligned} |T_\Xi f(x) - f(x)| \quad &\lesssim \quad \rho_1^4 \|\Delta^2 f\|_{L_\infty(\overline{D})} \\ &\quad + \mathrm{const}\,(\epsilon) \, \left(1 + \frac{d(x,\partial D)}{\rho_2}\right)^{2-m} \rho_2^2 \left(\rho_2 \|f\|_{C^{3+\epsilon}(\overline{D})} + \|f\|_{C^{2+\epsilon}(\overline{D})}\right). \end{aligned} \tag{4.2}$$

For $f \in W_p^4(D)$, $1 < p < \infty$, we have

$$\|T_\Xi f - f\|_p \lesssim \rho_1^4 \; \|\Delta^2 f\|_{L_p(D)} + \rho_2^{3+1/p} \; \|f\|_{B^{3+1/p}_{p,1}(D)} + \rho_2^{2+1/p} \; \|f\|_{B^{2+1/p}_{p,1}(D)}. \tag{4.3}$$

Finally, for $f \in W^4 L\log L(D)$ we have

$$\|T_\Xi f - f\|_{L_1(D)} \lesssim \rho_1^4 \; \|\Delta^2 f\|_{L_1(D)} + \rho_2^4 \; \|f\|_{W^4 L\log L(D)} + \rho_2^3 \; \|f\|_{W^3 L\log L(D)}. \tag{4.4}$$

Before we embark on the proof, some remarks are in order. First, note that for each estimate in (4.1)-(4.4), the symbol $\lesssim$ implies inequality up to multiplication by a constant depending only on m, k and, in the case of (4.3), p.

Secondly, in light of Proposition 8, we may, if we wish, take $\rho_1 = \rho_2 \sim h(\Xi, D)$, provided Ξ is sufficiently dense. In so doing, we see that Johnson's upper bounds are attained. More will be said about this later in this section, where we give results in terms of the uniform density parameter h for functions of lower smoothness.

Each of the error estimates (4.1)-(4.4) show how the effects of the boundary components of T_Ξ are decoupled from the interior component. As an extreme example, if $Nf = Mf = 0$, (that is, if f is a Newtonian potential) then the error is due solely to the body integral – the boundary effects are absent and the error in approximating this kind of function is $O(\rho_1^4)$.

Conversely, if f is biharmonic (and, hence, determined solely by its boundary values), then the error consists only of boundary effects (and, one needs only to have centers at the boundary). Although approximation at the boundary is (relatively) poor, it is possible to attain extremely fast convergence in the interior by increasing the polynomial precision m. Specifically, on compact subsets of D, $\|T_{\Xi}f - f\|_{L_\infty} = O(\rho_2^m)$. In light of this, we recall Schaback and Wendland's saturation result [24, Theorem 7.4], which seems to be a complementary result (albeit for interpolation). It states that if $\|I_{\Xi}f - f\|_{L_\infty(K)} = o(h^4)$ for every compact subset K of a region Ω with a smooth boundary, then f is biharmonic.

Clearly, to mitigate the effects of the boundary, one has two options: increase the polynomial precision m, which causes the effects to be limited to smaller annuli, or decrease ρ_2 relative to ρ_1. This is the subject of the next section, and will be discussed in more detail there.

Proof. Invoking the definition of T_{Ξ} and Proposition 1.12 we have, for $f \in C^\infty(\overline{D})$,

$$\begin{aligned} |f - T_{\Xi}f| &\leq \iint |\Delta^2 f(\alpha)| E_0(\cdot,\alpha)\,\mathrm{d}\alpha \\ &+ \int |Nf(\alpha)| E_1(\cdot,\alpha)\,\mathrm{d}\sigma(\alpha) \\ &+ \int |Mf(\alpha)| E_2(\cdot,\alpha)\,\mathrm{d}\sigma(\alpha). \end{aligned}$$

(Recall that E_i is defined in (1.15) - (1.17).) This inequality is the starting point for each estimate (4.1)-(4.4).

For the pointwise estimates, we apply Hölder's inequality for each of the three terms, with $\Delta^2 f \in L_\infty$ and $E_0(x,\cdot) \in L_1$; $Nf \in L_\infty$ or $L_{\exp}$ and $E_1(x,\cdot)$ in L_1 or $L\log L$ respectively; and $Mf \in L_\infty$ or $L_{\exp}$ and $E_2(x,\cdot)$ in L_1 or $L\log L$ respectively.

$$\begin{aligned} |T_{\Xi}f(x) - f(x)| &\lesssim \|\Delta^2 f\|_\infty \|E_0(x,\alpha)\|_1 \\ &\quad + \|Nf\|_{L_{\exp}} \|E_1(x,\cdot)\|_{L\log L(\partial D)} + \|Mf\|_{L_{\exp}} \|E_2(x,\cdot)\|_{L\log L(\partial D)} \\ &\lesssim \rho_1^4 \|\Delta^2 f\|_\infty \\ &\quad + \left(1 + \frac{d(x,\partial D)}{\rho_2}\right)^{2-m} \left(\rho_2^3 |\log\rho_2| \|Nf\|_{L_{\exp}} + \rho_2^2 |\log\rho_2| \|Mf\|_{L_{\exp}}\right) \\ &\lesssim \rho_1^4 \|\Delta^2 f\|_\infty \\ &\quad + \left(1 + \frac{d(x,\partial D)}{\rho_2}\right)^{2-m} \rho_2^2 |\log\rho_2| \left(\rho_2 \|f\|_{C^3(\overline{D})} + \|f\|_{C^2(\overline{D})}\right). \end{aligned}$$

The second inequality results from applying the estimates we obtained on the error kernels, viz., (3.4), (3.12) and (3.13). The last inequality is a consequence of the bounds we obtained on N and M, cf. (2.19) and (2.20). From this, (4.1) follows.

For the second pointwise estimate, we merely use a different Hölder's inequality on the boundary integrals.

$$\begin{aligned} |T_{\Xi}f(x) - f(x)| &\leq \|\Delta^2 f\|_\infty \|E_0(x,\alpha)\|_1 \\ &\quad + \|Nf\|_{L_\infty} \|E_1(x,\cdot)\|_{L_1(\partial D)} + \|Mf\|_{L_\infty} \|E_2(x,\cdot)\|_{L_1(\partial D)} \\ &\lesssim \rho_1^4 \|\Delta^2 f\|_\infty \\ &\quad + \left(1 + \frac{d(x,\partial D)}{\rho_2}\right)^{2-m} \left(\rho_2^3 \|Nf\|_{L_\infty} + \rho_2^2 \|Mf\|_{L_\infty}\right) \\ &\lesssim \rho_1^4 \|\Delta^2 f\|_\infty \\ &\quad + \left(1 + \frac{d(x,\partial D)}{\rho_2}\right)^{2-m} \rho_2^2 \left(\rho_2 \|f\|_{C^{3+\epsilon}(\overline{D})} + \|f\|_{C^{2+\epsilon}(\overline{D})}\right). \end{aligned}$$

The second inequality results from applying the estimates we obtained on the error kernels: (3.4), (3.14) and (3.15). The last inequality is a consequence of the bounds we obtained on N and M: (2.12) and (2.13). This settles (4.2).

For the L_p error estimates (4.3), we obtain, for $f \in C^\infty(\overline{D})$

$$\begin{aligned}\|T_\Xi f - f\|_p &\lesssim \rho_1^4 \|\Delta^2 f\|_{L_p(D)} + \rho_2^{3+1/p}\|Nf\|_{L_p(\partial D)} + \rho_2^{2+1/p}\|Mf\|_{L_p(\partial D)} \\ &\lesssim \rho_1^4\|\Delta^2 f\|_{L_p(D)} + \rho_2^{3+1/p}\|f\|_{B^{3+1/p}_{p,1}(D)} + \rho_2^{2+1/p}\|f\|_{B^{2+1/p}_{p,1}(D)}. \qquad (4.5)\end{aligned}$$

The first inequality results from applying the operator norm estimates obtained by interpolation in Section 3, namely (3.6), (3.16) and (3.17). The last inequality is a consequence of the bounds we obtained on N and M: (2.10) and (2.11). Using the density of $C^\infty(\overline{D})$ in $W_p^4(D)$, and the embedding $W_p^4(D) \to B^{3+1/p}_{p,1}(D) \to B^{2+1/p}_{p,1}(D)$, we extend the estimate (4.5) to all of $W_p^4(D)$.

For the L_1 error estimates (4.4), we obtain, for $f \in C^\infty(\overline{D})$.

$$\begin{aligned}\|T_\Xi f - f\|_1 &\lesssim \rho_1^4\|\Delta^2 f\|_1 + \rho_2^4\|Nf\|_{L_1} + \rho_2^3\|Mf\|_{L_1} \\ &\lesssim \rho_1^4\|\Delta^2 f\|_p + \rho_2^4\|f\|_{W^4 L\log L(D)} + \rho_2^3\|f\|_{W^3 L \log L(D)}. \qquad (4.6)\end{aligned}$$

The first inequality results from applying (3.5), (3.7) and (3.8) – the L_1 operator bounds for the kernels E_0, E_1, E_2. The last inequality is a consequence of the bounds we obtained on N and M: (2.21) and (2.22). (4.4) follows by extending (4.6) to all of $W^4 L\log L(D)$ by density of $C^\infty(\overline{D})$, and by the embeddings $W^4L\log L(D) \to W^3 L\log L(D)$ and $W^4 L \log L(D) \to W_1^4(D)$. $\square$

Note: Observe that the last estimate (4.4) also holds for $f \in B^{3+1/p}_{p,1}(D) \cap W_1^4(D)$ for $1 < p < \infty$, and, hence, for $f \in B^{4+\epsilon}_{1,q}(D)$ for $1 \le q \le \infty$ and $\epsilon > 0$.

$$\|T_\Xi f - f\|_{L_1(D)} \lesssim \rho_1^4\ \|\Delta^2 f\|_{L_1(D)} + \rho_2^4\ \|f\|_{B^{3+1/p}_{p,1}(D)} + \rho_2^3\ \|f\|_{B^{2+1/p}_{p,1}(D)}, \qquad (4.7)$$

and

$$\|T_\Xi f - f\|_{L_1(D)} \lesssim \rho_1^4\ \|\Delta^2 f\|_{L_1(D)} + \rho_2^4\ \|f\|_{B^{4+\epsilon}_{1,q}(D)} + \rho_2^3\ \|f\|_{B^{3+\epsilon}_{1,q}(D)}. \qquad (4.8)$$

This is a direct consequence of the fact, noted in Section 2, that $N \in B(B^{3+1/p}_{p,1}(D), L_1(\partial D))$ and $M \in B(B^{2+1/p}_{p,1}(D), L_1(\partial D))$.

We use the preceding theorem to obtain optimal approximation orders – now in terms of a single approximation parameter $\rho = \rho_1 = \rho_2$ (which, by Lemma 8 can be taken proportionate to $h(\Xi, D)$), but for larger classes of functions than previously considered. Specifically, we aim to approximate functions with degree of smoothness roughly $2 + 1/p$ measured in L_p (so, with smoothness commensurate with the approximation order). In each case ($p = \infty$, $1 < p < \infty$, and $p = 1$), this is accomplished by the same procedure. The function to be approximated is split into a smooth part, g, and a part which is appropriately small in L_p, $b = f - g$. The approximant is the operator T_Ξ applied to the smooth part, thus $s_{f,\Xi} = T_\Xi g$. Clearly, g should be in one of the spaces suitable for obtaining error estimates, i.e., those mentioned in Theorem 10, although we will require more than this.

In each case, the split $f = g + b$, is achieved by the same method. To obtain g, first f is extended to $\mathbb{R}^2$. Next, g is the extension convolved with an appropriately dilated, smooth mollifier possessing a Fourier transform which is sufficiently flat near the origin. To this end, we denote by η_h the dilation by h of a function $\eta : \mathbb{R}^2 \to \mathbb{C}$. That is, $\eta_h = h^{-2}\eta(\cdot/h)$. We denote by $\mathcal{E}$ the strong extension operator of order 4, defined in [1, 5.22]. This operator extends functions in $C(\overline{D})$ to functions in $C(\mathbb{R}^2)$ with support in the compact disc $B_K(0)$. Most importantly, $\mathcal{E} \in B(C^j(\overline{D}), C^j(\mathbb{R}^2))$ for $j \le 4$. It extends functions from each smoothness space we employ – Sobolev, Hölder, Besov and Sobolev-Orlicz – to the corresponding smoothness space over $\mathbb{R}^2$. I.e., it is in $B(W_p^j(D), W_p^j(\mathbb{R}^2))$ for integers $j \le 4$ and it is in $B(B^s_{p,q}(D), B^s_{p,q}(\mathbb{R}^2))$ and $B(C^s(\overline{D}), C^s(\mathbb{R}^2))$ for $0 < s < 4$. Moreover, in each case, the crucial properties $\mathrm{supp}(\mathcal{E}f) \subset B_K(0)$ and $(\mathcal{E}f)_{|D} = f$ are preserved. For the spaces $W^k L\log L$, it is straight forward to show that $\mathcal{E} \in B(W^k L \log L(D), W^k L\log L(\Omega))$ where Ω

is any region containing $B_K(0)$. Thus, for h sufficiently small, the convolution $\eta_h * \mathcal{E}f$, for $f \in W^k L \log L(D)$, is well defined as an element of $W^k L \log L(\Omega)$. (Actually, it is only its restriction to D that interests us.) The following proposition captures most of what we need from the split $f = g + b$:

Proposition 11. *Suppose $k \le 4$, $\eta \in C^\infty(\mathbb{R}^2)$, and $\operatorname{supp}(\eta) \subset D$, and that, furthermore, η satisfies the moment conditions: $\int \eta(x)\,\mathrm{d}x = 1$, and $\int x^\alpha \eta(x)\,\mathrm{d}x = 0$ for $|\alpha| = 1, \ldots, k-1$. For $f \in W_p^k(D)$, with $1 \le p \le \infty$ and $j + k \le 4$, the following statements hold*

$$\begin{aligned} \|\eta_h * \mathcal{E}f\|_{W_p^{k+j}(D)} &\lesssim h^{-j}\|f\|_{W_p^k(D)}, && (4.9)\\ \|f - \eta_h * \mathcal{E}f\|_{L_p(D)} &\lesssim h^k \|f\|_{W_p^k(D)}. && (4.10) \end{aligned}$$

Proof. The first set of inequalities follow by estimating $D^\alpha(\eta_h * \mathcal{E}f)$, with $\alpha \le k + j$ putting all but k derivatives on η_h. The second follows by writing $\mathcal{E}f(x-t) = P(x-t;\, \mathcal{E}f, k-1, x) + R(x-t;\, \mathcal{E}f, k-1, x)$, where, $P(\cdot;\, f, \ell, x)$ is the Taylor polynomial centered at x, of degree ℓ. The polynomial annihilation properties of η ensure that

$$\begin{aligned} \|f - \eta_h * \mathcal{E}f\|_{L_p(D)} &\le \left\| \int \eta_h(t) R(\cdot - t;\, \mathcal{E}f, k-1, \cdot)\,\mathrm{d}t \right\|_{L_p(D)} \\ &\le \operatorname{const}(k) \sum_{\alpha = k} \|D^\alpha \mathcal{E}f\|_{L_p((1+h)D)} \int t^\alpha \eta_h(t)\,\mathrm{d}t \\ &\le \operatorname{const}(k)\, h^k \|f\|_{W_p^k(D)}. \end{aligned}$$

□

In what follows, we assume η satisfies the conditions of the previous proposition, with $k = 4$.

Corollary 12. *Suppose $m \ge 4$, $k > 0$ and a_0, a_1 and a_2 satisfy $IC(m, \rho, k)$, $BC1(m, \rho, k)$ and $BC2(m, \rho, k)$, respectively. If $f \in C^2\left(\overline{D}\right)$, there is an approximant $s_{f,\Xi} \in S(\phi, \Xi)$ such that*

$$\|s_{f,\Xi} - f\|_\infty \lesssim \rho^2 |\log \rho|\ \|f\|_{C^2(\overline{D})}. \tag{4.11}$$

If $f \in C^{2+\epsilon}\left(\overline{D}\right)$, $\epsilon > 0$ we have

$$\|s_{f,\Xi} - f\|_\infty \lesssim \rho^2 \|f\|_{C^{2+\epsilon}(\overline{D})}. \tag{4.12}$$

Clearly, by Lemma 8, if Ξ is sufficiently dense (for instance, if $h < \frac{1}{48m^2}$) then the corollary holds with $\rho = h(\Xi, D)$ as the approximation parameter:

$$\|s_{f,\Xi} - f\|_\infty \lesssim h^2 \|f\|_{C^{2+\epsilon}(\overline{D})}.$$

This shows that Johnson's upper bound is attained.

Proof. We split the function $f = g + b$ so that the following hold:

$$\begin{aligned} \|g\|_{C^{2+j}(\overline{D})} &\le \operatorname{const} \rho^{-j} \|f\|_{C^2(\overline{D})}, \quad \text{for } j = 0, 1, 2, \\ \|b\|_\infty &\le \operatorname{const} \rho^2 \|f\|_{C^2(\overline{D})}. \end{aligned}$$

This follows from Proposition 11, with $k = 2$. Now, we set $s_{f,\Xi} := T_\Xi g$ and the result follows from Theorem 10:

$$\begin{aligned} |s_{f,\Xi}(x) - f(x)| &\le |T_\Xi g(x) - g(x)| + |b(x)| \\ &\lesssim \rho^4 \|g\|_{C^4(\overline{D})} + \rho^3 |\log \rho|\ \|g\|_{C^3(\overline{D})} + \rho^2 |\log \rho|\ \|g\|_{C^2(\overline{D})} + \|b\|_\infty \\ &\lesssim \rho^2 |\log \rho|\ \|f\|_{C^2(\overline{D})}. \end{aligned}$$

The second estimate holds if the split satisfies:

$$
\begin{aligned}
\|g\|_{C^4(\overline{D})} &\le \operatorname{const}\rho^{-2}\|f\|_{C^{2+\epsilon}(\overline{D})}, && (4.13)\\
\|g\|_{C^{2+j+\epsilon}(\overline{D})} &\le \operatorname{const}\rho^{-j}\|f\|_{C^{2+\epsilon}(\overline{D})}, \quad \text{for } j=0,1, && (4.14)\\
\|b\|_\infty &\le \operatorname{const}\rho^{2}\|f\|_{C^{2+\epsilon}(\overline{D})}. && (4.15)
\end{aligned}
$$

Of these three statements, the first and last are obvious, by the embedding $C^{2+\epsilon}\to C^2$. The second follows by estimating $|D^\alpha g(x)-D^\alpha g(y)|$, with $|\alpha|=2,3$. Let $\alpha=\alpha_1+\alpha_2$, where $|\alpha_1|=2$, and observe that

$$
\int |D^{\alpha_1}\mathcal{E}f(x-t)-D^{\alpha_1}\mathcal{E}f(y-t)|\,|D^{\alpha_2}(\eta_\rho(t))|\,\mathrm{d}t \le \operatorname{const}\rho^{-|\alpha_2|}|x-y|^\epsilon\|f\|_{C^{2+\epsilon}}
$$

The inequality follows by applying (4.2) of Theorem 10. $\square$

Corollary 13. *Suppose $m\ge 4$ and a_0,a_1 and a_2 satisfy, for some $k>0$, $IC(m,\rho,k)$, $BC1(m,\rho,k)$ and $BC2(m,\rho,k)$, respectively. For f in the Besov space $B^{2+1/p}_{p,1}(D)$, $1<p<\infty$ there exists an approximant $s_f\in S(\phi,\Xi)$ so that*

$$
\|f-s_{f,\Xi}\|_p \lesssim \rho^{2+1/p}\|f\|_{B^{2+1/p}_{p,1}(D)}.
$$

Proof. This follows in much the same way as Corollary 12: we split $f=g+b$ and apply T_Ξ to g. The proof is finished when we show that

$$
\begin{aligned}
\|g\|_{W^4_p} &\le \operatorname{const}\rho^{1/p-2}\|f\|_{B^{2+1/p}_{p,1}}, && (4.16)\\
\|g\|_{B^{2+j+1/p}_{p,1}} &\le \operatorname{const}\rho^{-j}\|f\|_{B^{2+1/p}_{p,1}}, \quad \text{for } j=0,1, && (4.17)\\
\|b\|_p &\le \operatorname{const}\rho^{2+1/p}\|f\|_{B^{2+1/p}_{p,1}}. && (4.18)
\end{aligned}
$$

As in Corollary 12, the result follows by applying Theorem 10.

The inequalities (4.16)-(4.18) are obtained by interpolating the results of Proposition 11: The first comes because the operator $f\mapsto \eta_\rho * f$ is in $B(W^2_p,W^4_p)$ with norm bounded by $\operatorname{const}\rho^{-2}$ and in $B(W^3_p,W^4_p)$, with norm $\operatorname{const}\rho^{-1}$. Since $B^{2+1/p}_{p,1}$ is the $[p,1]$ interpolation space of W^2_p and W^3_p, we have that the operator is in $B(B^{2+1/p}_{p,1},W^4_p)$ with norm bounded by

$$
\operatorname{const}\left(\rho^{-2}\right)^{1-1/p}\left(\rho^{-1}\right)^{1/p} = \operatorname{const}\rho^{1/p-2}.
$$

The second statement is obtained in the same way. Now we observe that, for $j=0,1$, $f\mapsto\eta_\rho * f$ is in $B(W^2_p,W^{2+j}_p)$ with norm bounded by $\operatorname{const}\rho^{-j}$ and in $B(W^3_p,W^{3+j}_p)$, again, with norm less than $\operatorname{const}\rho^{-j}$. Thus the operator $f\mapsto\eta_\rho*\mathcal{E}f$ is in $B(B^{2+1/p}_{p,1},B^{2+j+1/p}_{p,1})$ with norm less than $\operatorname{const}\rho^{-j}$.

The last statement comes by interpolating the results of Proposition 11 for the operator $f\mapsto f-\eta_\rho*f$. We know that it is in $B(W^2_p,L_p)$ and $B(W^3_p,L_p)$ with norms bounded by $\operatorname{const}\rho^2$ and $\operatorname{const}\rho^3$, respectively. $\square$

Corollary 14. *Suppose $m\ge 4$, $k>0$, and a_0,a_1 and a_2 satisfy $IC(m,\rho,k)$, $BC1(m,\rho,k)$ and $BC2(m,\rho,k)$, respectively. For f in the Orlicz-Sobolev space $W^3L\log L(D)$ there exists an approximant $s_{f,\Xi}\in S(\phi,\Xi)$ so that*

$$
\|f-s_{f,\Xi}\|_1 \lesssim \rho^3\|f\|_{W^3L\log L(D)}. \tag{4.19}
$$

Furthermore, if f is merely in W^3_1, there is $s_{f,\Xi}\in S(\phi,\Xi)$ such that

$$
\|f-s_{f,\Xi}\|_1 \lesssim \rho^3|\log\rho|\,\|f\|_{W^3_1(D)}. \tag{4.20}
$$

Proof. (4.19) follows in much the same way as the previous two corollaries. The split $f = g + b$ should satisfy

$$\begin{aligned} \|g\|_{W^{3+j}L\log L(D)} &\leq \operatorname{const}\rho^{-j}\|f\|_{W^3L\log L(D)} \quad \text{for } j=0,1, &(4.21)\\ \|b\|_1 &\leq \operatorname{const}\rho^3\|f\|_{W^3L\log L(D)}. &(4.22)\end{aligned}$$

Since $W^3L\log L \to W_1^3$, the second inequality is obvious by Proposition 11. To obtain the first, we use the fact that $L_{\exp}$ and $L\log L$ are associate, viz.

$$\|F\|_{L\log L} \sim \sup\left\{\int F(x)G(x)\,\mathrm{d}x \mid \|G\|_{L_{\exp}} = 1\right\}.$$

To this end, let $\|r\|_{L_{\exp}(D)} = 1$, $|\alpha| \leq 3$, and $|\beta| \leq 1$. We have

$$\begin{aligned} \left|\int_D r(x)D^{\alpha+\beta}g(x)\,\mathrm{d}x\right| &= \left|\int_D r(x)(D^\alpha\mathcal{E}f) * (D^\beta\eta_\rho)(x)\,\mathrm{d}x\right| \\ &= \rho^{-|\beta|}\left|\int_{\mathbb{R}^2}\int_D r(x)(D^\alpha\mathcal{E}f)(x-t)(D^\beta\eta)_\rho(t)\,\mathrm{d}x\,\mathrm{d}t\right| \\ &\leq \rho^{-|\beta|}\int_{\mathbb{R}^2}\|(D^\alpha\mathcal{E}f)(\cdot - t)\|_{L\log L(D)}\left|(D^\beta\eta)_\rho(t)\right|\,\mathrm{d}t \\ &\leq \operatorname{const}\rho^{-|\beta|}\|f\|_{W^3L\log L(D)}. \end{aligned}$$

The first inequality is a result of Hölder's inequality, and the fact that $|f(\cdot - t)\chi_D| \leq |f(\cdot - t)|$. The second inequality follows from the translation invariance of $L\log L(\mathbb{R}^2)$ and the boundedness of $\mathcal{E}$. (4.19) now follows by applying (4.4).

To prove (4.20), we use the estimate obtained in (4.19), rather than referring to Theorem 10. In this case we split $f \in W_1^3(D)$ into $f = f_1 + f_2$, where $f_1 \in W^3L\log L(D)$ and $f_2 \in L_1(D)$. In this case the split must satisfy:

$$\begin{aligned} \|f_2\|_1 &\leq \operatorname{const}\rho^3\|f\|_{W_1^3(D)}, &(4.23)\\ \|f_1\|_{W^3L\log L(D)} &\leq \operatorname{const}|\log\rho|\,\|f\|_{W_1^3(D)}. &(4.24)\end{aligned}$$

We then apply (4.19) to f_1, to obtain $\|f - s_{f_1,\Xi}\|_1 \leq \|f_1 - s_{f_1,\Xi}\|_1 + \|f_2\|_1$.

To obtain the split, we use the same operator as before: $f_1 = \eta_\rho * \mathcal{E}f$, and from Proposition 11 it is obvious that the first inequality holds. To obtain the second inequality, we utilize the norm for $L\log L$ given by:

$$\|F\|_{L\log L} \sim \inf\left\{K > 0 \mid \int\left|\frac{F(x)}{K}\right|\log_+\left|\frac{F(x)}{K}\right|\,\mathrm{d}x \leq 1\right\}.$$

Thus, we wish to estimate such a K, for $F = D^\alpha f_1$ and $|\alpha| \leq 3$. Letting

$$K = 2\max(\|\eta\|_\infty, \|\eta\|_1)\|\mathcal{E}f\|_{W_1^3}|\log\rho|,$$

we have

$$\begin{aligned} \int\left|\frac{D^\alpha f_1(x)}{K}\right|\log_+\left|\frac{D^\alpha f_1(x)}{K}\right|\,\mathrm{d}x &= \int\frac{|(D^\alpha\mathcal{E}f) * \eta_\rho|}{K}\log_+\frac{|(D^\alpha\mathcal{E}f) * \eta_\rho|}{K} \\ &\leq K^{-1}\log_+\left(\frac{\|\eta_\rho\|_\infty\|D^\alpha\mathcal{E}f\|_1}{2\|\eta\|_\infty\|\mathcal{E}f\|_{W_1^3}|\log\rho|}\right)\int|(D^\alpha\mathcal{E}f) * \eta_\rho(x)|\,\mathrm{d}x \\ &\leq K^{-1}\log_+\left(\frac{\rho^{-2}}{\log\rho^{-2}}\right)\|\eta\|_1\|D^\alpha\mathcal{E}f\|_1 \leq 1. \end{aligned}$$

The first inequality follows by uniformly bounding the $\log_+$ factor – namely by observing that $\|f_1\|_\infty \leq \|\eta_\rho\|_\infty\|\mathcal{E}f\|_1$ – and by pulling this out of the integral. The second integral results from the following fact: $\|\eta_\rho\|_\infty = \rho^{-2}\|\eta\|_\infty$. The final inequality is a consequence of the following: for $\rho < 1$, $\log\rho^{-2} > 1$, so $\log_+\frac{\rho^{-2}}{\log\rho^{-2}} \leq \log_+\rho^{-2} = \log\rho^{-2}$. Thus $\|f_1\|_{W^3L\log L(D)} \leq K \leq \operatorname{const}|\log\rho|\|f\|_{W_1^3(D)}$. ☐

Note: Observe, as before, that estimate (4.19) of Corollary 14 also holds if we have $f \in B^{2+1/p}_{p,1}(D) \cap W^3_1(D)$ or $f \in B^{3+\epsilon}_{1,q}$ (for some $\epsilon > 0$ and any $q \geq 1$), instead of $f \in W^3 L \log L(D)$. Theorem 1 follows by collecting the preceding statement, Corollary 12 and Corollary 13 and applying Lemma 8

5 Overcoming the Boundary Effects

In approximation by scattered translates of ϕ, there is a substantial difference between the upper bound on approximation orders for bounded domains (in case one chooses approximants from $S(\phi, \Xi)$) and the free space approximation order obtained from approximating over $\mathbb{R}^2$. In this section, we discuss how to mitigate these boundary effects by controlling the distribution of the centers.

Observe, first, that by choosing higher polynomial precision at the boundary, we can diminish the region of boundary effects. As in (4.2), for an arbitrary degree of polynomial precision m, and for sufficiently dense Ξ, if $d(x, \partial D) \sim \rho^\alpha$, where $\alpha \leq 1 - \frac{2}{m-2}$, then $|T_\Xi f(x) - f(x)| \lesssim \rho^4 \|f\|_{C^4(\overline{D})}$.

A similar, stronger, observation has been made in [14] for thin plate spline interpolation in general domains. There it is shown that the pointwise error from interpolation possesses the bound $|I_\Xi f(x) - f(x)| \lesssim h^4 \|f\|_{W^4_\infty}$ provided x is in a suitably restricted domain. That is, for $d(x, \partial\Omega) > Kh|\log h|$, where K is a constant depending on Ω. (Of course, this is only for Ω possessing a sufficiently smooth boundary, for sufficiently smooth f and for $h = h(\Xi, \Omega)$ sufficiently small.) Thus the boundary effects in this kind of approximation can be confined to a narrow region along the boundary. A simple consequence of this is that by allowing some centers to fall outside the disc – specifically, by adding enough centers in the annular region $(1 + Kh|\log h|)D \backslash D$ (or any region with sufficiently smooth boundary) the free space approximation orders are obtained (naturally, this requires extending the function to be approximated outside the disc). This can be done at a minimal cost of a multiple of $h|\log h|$ extra centers. A drawback of this is that the region where the error is measured is smaller than the region in which the approximant is actually produced. The situation is somewhat different if one makes the requirement that the extra centers be contained within the disc; this is the problem we address in this section.

Johnson's upper bound was obtained by controlling the density of centers near the boundary of the disc – precisely by limiting the number of centers in the boundary layer. By proper placement of the centers, we can obtain approximation orders equal to the free space case. In what follows, for $1 \leq p < \infty$ let $\sigma_p := 4p/(1+2p)$ and let $\sigma_\infty := 2$ – this constant reflects the increased density near the boundary which is necessary to offset the boundary effects, as the following corollary – a direct application of Theorem 10 with increased density at the boundary – demonstrates.

Corollary 15. *Given a set of centers $\Xi \subset D$ and coefficients satisfying $IC(m, \rho, k)$, $BC1(m, \rho^{\sigma_p}, k)$ and $BC2(m, \rho^{\sigma_p}, k)$ we have the following:*

If $1 < p < \infty$ and $f \in W^4_p(D)$

$$\|T_\Xi f - f\|_{L_p(D)} \lesssim \rho^4 \|f\|_{W^4_p(D)}.$$

If $p = \infty$ and $f \in C^4(\overline{D})$

$$\|T_\Xi f - f\|_{L_\infty(D)} \lesssim \rho^4 \|f\|_{C^4(\overline{D})}.$$

If $p = 1$ and $f \in W^4 L \log L(D)$ then

$$\|T_\Xi f - f\|_{L_1(D)} \lesssim \rho^4 \|f\|_{W^4 L \log L(D)}.$$

Theorem 2 is a result of this corollary, combined with a minor modification of Lemma 8.

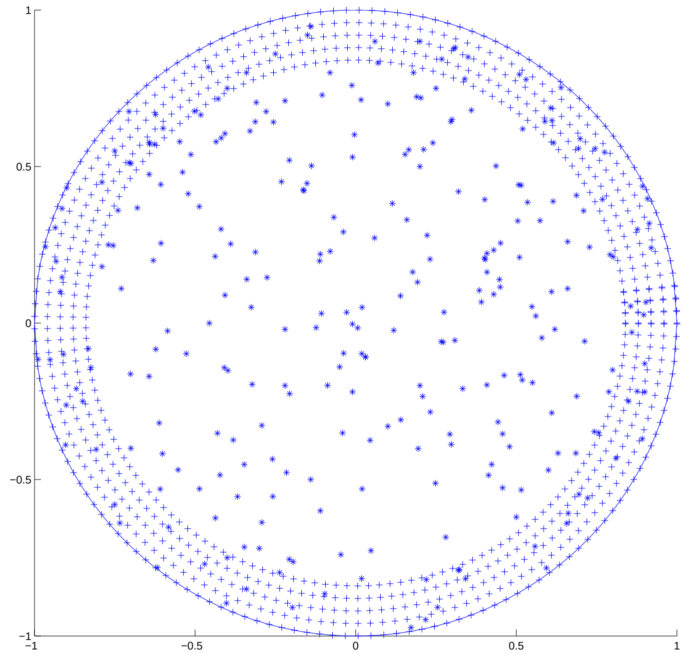

Figure 1: An optimal configuration of centers for measuring error in L_∞, with roughly 290 original (scattered) centers $(*)$ and 800 centers $(+)$ near the boundary.

With Corollary 15 as our guide we are able to construct the following example, in which the boundary effects are overcome.

Example: Observe that if an initial set of centers Ξ_1 is distributed in D with density $h(\Xi_1, D) = h$, then $\#\Xi_1 \geq \mathrm{const}\, h^{-2}$. By uniformly distributing extra centers around the boundary, say $10\pi/H$, we can increase the density there to be H. That is, we add

$$\Xi_2 := \bigcup_{j=0}^{4} \left\{\xi \in D : \xi = (1 - jH)\left(\cos(kH), \sin(kH)\right),\ k \in 1, \cdots, \lceil 2\pi/H^2 \rceil + 2\right\}$$

to obtain $\Xi = \Xi_1 \cup \Xi_2$.

We now show that this configuration admits coefficient kernels satisfying the appropriate boundary conditions. This is accomplished by the following simple continuity argument. The fact that the added centers allow the boundary conditions to be satisfied follows from their nearness to gridded centers, as $H \to 0$. For $t \in \partial D$, there is a nearby set of centers Υ_t^H which can be rotated, translated and rescaled to coincide nearly with the fixed centers $\Upsilon := \{(a, b) \in \mathbb{Z}^2 \mid a \geq 0, a - 2 \leq b \leq 2\}$, which is obviously unisolvent, and has cardinality $15 = \dim \Pi_4$. That is, there is an affine mapping $S : \mathbb{R}^2 \to \mathbb{R}^2$–a composition of a rotation, translation and scaling–such that $S\Upsilon_t^H \sim \Upsilon$. The same affine mapping, S, maps t to some point near to the y-axis. It is easily observed that the distance between the transformed centers $S\Upsilon_t^H$ and gridded centers Υ is $O(H)$. Thus, for H sufficiently small, we can solve the system of 15 equations

$$\lambda p_j = \sum_{\upsilon \in \Upsilon_t^h} a_\upsilon p_j(t), \quad j = 1, \ldots, 15,$$

where $\lambda = \delta_t$ or $\lambda = \frac{1}{h^2}\delta_t D_\nu$ (where $(p_j)_{j=1}^{15}$ is a basis for Π_4). By the invariance of Π_4 under similarity transformations, we can obtain the needed coefficients from the transformed centers and the ℓ_1 bounds on the coefficients follow by continuity.

If we set $H = h^{\sigma_p}$, then Ξ_2 (and hence Ξ) admits coefficient kernels satisfying $BC1(4, 5h^{\sigma_p}, C)$ and $BC2(4, 5h^{\sigma_p}, C)$. Thus, by Corollary 15, $\|T_\Xi f - f\|_p \le h^4 \|f\|_{W_p^4(D)}$ (making the appropriate changes if $p = 1, \infty$), and to obtain this we have added only roughly $H^{-1} = h^{-\sigma_p} = (h^{-2})^{1-\frac{1}{2p}} \lesssim (\#\Xi_1)^{1-\frac{1}{2p}}$. Specifically, if $p = \infty$ then Ξ_2 consists of at most the same number (up to a constant multiple) of centers as Ξ_1. An illustration of this scenario, is given in **Figure 1.**

6 Approximation from the Space $S_2(\phi, \Xi)$

The purpose of this section is to modify the previous approximation scheme to obtain approximants from $S_2(\phi, \Xi)$, and to show that this scheme satisfies the results of sections 5 and 6. We modify the scheme T_Ξ slightly – specifically, we use the scheme:

$$T_{\Xi,2} f = p + T_\Xi(f - p),$$

for an appropriate choice of $p \in \Pi_1$. To satisfy the side conditions, we exploit the fact that the coefficient kernels a_i each have polynomial reproduction properties. Thus, we can recast the three side conditions $\sum A_\xi q(\xi) = 0$, with $A_\xi = \iint a_0(\cdot, \xi)\Delta^2(f - p) + \int a_1(\cdot, \xi)N(f - p) + a_2(\cdot, \xi)M(f - p)\, d\sigma$, as

$$\begin{aligned}
\int Np(\alpha)d\sigma(\alpha) &= \iint \Delta^2 f(\alpha)d\alpha + \int Nf(\alpha)d\sigma(\alpha), \\
\int (Np(\alpha) + Mp(\alpha))q_1(\alpha)d\sigma(\alpha) &= \iint \Delta^2 f(\alpha)q_1(\alpha)d\alpha + \int (Nf(\alpha) + Mf(\alpha))q_1(\alpha)d\sigma(\alpha), \\
\int (Np(\alpha) + Mp(\alpha))q_{-1}(\alpha)d\sigma(\alpha) &= \iint \Delta^2 f(\alpha)q_{-1}(\alpha)d\alpha + \int (Nf(\alpha) + Mf(\alpha))q_{-1}(\alpha)d\sigma(\alpha).
\end{aligned}$$

Here $q_1(x) := x_1 + ix_2$ and $q_{-1}(x) := x_1 - ix_2$, and for completeness, we set $q_0(x) := 1$. Finding p is a matter of determining each c_j in the expression $p = \sum c_j q_j$. Using the basis q_i has the benefit that $\mathrm{Tr}p$ can be expressed as $\sum c_j e_j$, and, likewise, $D_n p = c_{-1}e_{-1} + c_1 e_1$. Consequently, $Np = Mp = -2c_{-1}e_{-1} - 4c_0 e_0 - 2c_1 e_1$. By the orthogonality of the Fourier basis, the three side conditions yield

$$-8\pi c_0 = \iint \Delta^2 f(\alpha)d\alpha + \int Nf(\alpha)d\sigma(\alpha) \tag{6.1}$$

$$-8\pi c_{-1} = \iint \Delta^2 f(\alpha)q_1(\alpha)d\alpha + \int (Nf(\alpha) + Mf(\alpha))q_1(\alpha)d\sigma(\alpha) \tag{6.2}$$

$$-8\pi c_1 = \iint \Delta^2 f(\alpha)q_{-1}(\alpha)d\alpha + \int (Nf(\alpha) + Mf(\alpha))q_{-1}(\alpha)d\sigma(\alpha). \tag{6.3}$$

We write $\mathrm{Tr}f = \sum \lambda_k e_k$ and $D_n f = \sum \mu_k e_k$. By applying Green's Identity, we can greatly simplify (6.1-6.3). Since $\iint \Delta^2 f(\alpha)d\alpha = \int D_n \Delta f(\alpha)d\sigma(\alpha)$, (6.1) becomes

$$c_0 = -\frac{1}{8\pi}\int V_2 \mathrm{Tr}f - U_2 D_n f = \lambda_0 - \mu_0.$$

Likewise, since $\iint \Delta^2 f(\alpha)q_j(\alpha)d\alpha = \int D_n \Delta f(\alpha)q_j(\alpha) - \Delta f(\alpha)q_j(\alpha)d\sigma(\alpha)$ when $j = -1, 1$, we get

$$c_{-1} = -\frac{1}{8\pi}\int q_1\left(V_2 \mathrm{Tr}f + V_1 \mathrm{Tr}f - U_2 D_n f - U_1 D_n f\right) = \frac{1}{2}(\lambda_{-1} + \mu_{-1})$$

and

$$c_1 = -\frac{1}{8\pi}\int q_{-1}\left(V_2 \mathrm{Tr}f + V_1 \mathrm{Tr}f - U_2 D_n f - U_1 D_n f\right) = \frac{1}{2}(\lambda_1 + \mu_1).$$

Thus, setting

$$Q_j : L_1(\mathbb{T}) \to \Pi_1 : \tau \mapsto \langle \tau, e_j \rangle q_j, \quad j = -1, 0, 1,$$

and

$$p_f := \frac{1}{2} Q_{-1}(\mathrm{Tr} f + D_n f) + Q_0(\mathrm{Tr} f - D_n f) + \frac{1}{2} Q_1(\mathrm{Tr} f + D_n f)$$

we have the modified scheme:

$$T_{\Xi,2} f := p_f + T_\Xi (f - p_f).$$

The fact that the results of the previous two sections hold for $T_{\Xi,2}$ as well, follows from the observation that $\|p_f\| \lesssim \|f\|_{B^{1+p}_{p,1}(D)}, \|f\|_{C^1(\overline{D})}, \|f\|_{W^2 L \log L(D)}$.

7 Appendix

To obtain a trace theorem for $W^k L \log L(D)$ – the goal of this section – we employ the theory of Orlicz and Sobolev-Orlicz spaces, which can be viewed as an expansion of the Lebesgue and Sobolev space theories. Orlicz spaces are Banach function spaces whose elements satisfy simple integrability conditions. Given a left-continuous, nondecreasing, and convex function $B : [0, \infty) \to [0, \infty)$ satisfying $B(0) = 0$, $B'(0) = 0$ and $\lim_{t \to \infty} B(t)/t = \infty$ (such a function is called an N function), and a finite measure space, Ω, the Orlicz space $L_B(\Omega)$ consists of all measurable functions for which there exist $\ell > 0$ such that the integral $\int_\Omega B(\ell |f(x)|)\, dx$ is finite. Although it is possible to give a broader definition of Orlicz spaces – based on a larger class than the N functions (see [5]) and on general (nonfinite) measure spaces – for our purposes, the above definition suffices.

The space L_B has norm

$$\|f\|_{L_B} := \|f\|_B := \inf \left\{ \ell > 0 : \int_\Omega B\left(\frac{|f(x)|}{\ell} \right) dx \le 1 \right\}.$$

Clearly, $B_0(t) := t \log_+ t$ is an N function (according to the definition given in Section 2) and $L \log L(\Omega)$ is an Orlicz space. The following result on embeddings can be found in [1, p.269]:

Proposition 16. *Given N functions A and B, $L_B \to L_A$ if and only if there exists $x_0, k > 0$ such that $A(x) \le B(kx)$ for all $x > x_0$.*

This proposition shows that

$$L \log L(\Omega) \to L_{B_1}(\Omega) \to L_{B_2}(\Omega) \to L \log L(\Omega)$$

for $B_1(t) = t \log(t+1)$ or $B_2(t) = (t+1)\log(t+1) - t$. [5, Ch. Sec.],[1, Ch. 8], and [8] compile the necessary basics of Orlicz spaces, while the second and third of these provide introductions to Sobolev-Orlicz spaces, which we discuss next.

Now suppose $\Omega \subset \mathbb{R}^n$ and the N function, B, satisfies the following condition: there exist constants $k, t_0 > 0$ for which $B(2t) \le k B(t)$ for all $t > t_0$. (Note that B_0, B_1 and B_2 each satisfy this.) The Sobolev-Orlicz space $W^k L_B(\Omega)$ consists of functions $f \in L_B(\Omega)$ for which $D^\alpha f \in L_B(\Omega)$ for all $|\alpha| \le k$. $W^k L_B$ has norm:

$$\|f\|_{W^k L_B} := \sum_{\alpha \le k} \|D^\alpha f\|_{L_B}.$$

Many well known properties of Sobolev spaces also hold for Sobolev-Orlicz spaces. An important property is that $C^\infty(\overline{\Omega})$ is dense in $W^1 L_B(\Omega)$ when Ω has the segment condition [1, p. 68] (which the disc has).

B_*, the Sobolev conjugate to B, is defined–when possible–by

$$B_*^{-1}(t) := \int_0^t \frac{B^{-1}(s)}{s^{(n+1)/n}}. \tag{7.1}$$

Note that even if B is invertible, there is no guarantee that the integral in (7.1) is finite. Indeed, although B_0, B_1 and B_2 generate the same Orlicz space, the first two N functions are inadequate to produce B_* when $n = 2$. The following proposition is essentially [8, Theorem 3.8] (which is also [1, Theorem 8.38]) adapted to our setting:

Proposition 17. *If B is an N function for which $B_*^{-1}(t) < \infty$ for all $0 \le t < \infty$ and $\lim_{t\to\infty} B_*^{-1}(t) = \infty$ then, for $f \in W^1 L_B(D)$,*

$$\|\mathrm{Tr} f\|_{L_{(B_*)^{1/2}}(\partial D)} \le \mathrm{const}\, \|f\|_{W^1 L_B(D)}.$$

The following lemma is an application of this proposition.

Lemma 18. *For $f \in W^1 L \log L(D)$,*

$$\|\mathrm{Tr} f\|_{L\log L(\partial D)} \le \mathrm{const}\, \|f\|_{W^1 L \log L(D)}.$$

Proof. The proof consists of two parts. First, we compute an N function C, such that $C_* = (B_1)^2$. Hence, by Proposition 17, $\|\mathrm{Tr} f\|_{L\log L(\partial D)} \le \|f\|_{W^1 L_C(D)}$. In the second part, we show that there exist positive constants t_0, K such that $C(t) \le B_1(Kt)$ for all $t > t_0$, which provides the embedding $\|f\|_{W^1 L_C(D)} \lesssim \|f\|_{W^1 L \log L(D)}$.

To construct C, we differentiate the relationship (7.1) once, using $B_1(t)^2 = t^2(\log(1+t))^2$ as C_*, and postponing our justification for using C^{-1} until later. Thus $(C_*^{-1})'(t) = C^{-1}(t)/t^{3/2}$. Utilizing the fact that $(C_*^{-1})' = 1/C_*' \circ C_*^{-1}$ and writing $t = C_* \circ C_*^{-1}(t)$ we see that

$$C^{-1}(t) = \frac{C_*^{3/2}}{C_*'} \circ C_*^{-1}(t) = \frac{(B_1)^2}{2(B_1)'} \circ C_*^{-1}(t)$$

Differentiating this once, and applying the chain rule, gives

$$(C^{-1})'(t) = \frac{2((B_1)')^2 - B_1 (B_1)''}{4((B_1)')^3} \circ C_*^{-1}(t)$$

Thus we have that C^{-1} and C are both strictly increasing, since

$$\begin{aligned} 2((B_1)'(t))^2 - B_1(t)(B_1)''(t) &= 2\left(\log(1+t) + \frac{t}{1+t}\right)^2 - \frac{t}{1+t}\log(1+t)\left(1 + \frac{1}{1+t}\right) \\ &\ge (\log(1+t))^2 + \left(\frac{t}{1+t}\right)^2. \end{aligned}$$

It is not difficult to show that C satisfies the other conditions of an N function. That $C(0) = 0$ and $C'(0) = 0$ is evident. Proving that $C(t)/t \to \infty$ as $t \to \infty$ involves changing variables:

$$\lim_{t\to\infty} \frac{C(t)}{t} = \lim_{s\to\infty} \frac{s}{C^{-1}(s)} = \lim_{v\to\infty} \frac{C_*(v)}{C^{-1}(C_*(v))} = \lim_{v\to\infty} 2(B_1)'(v) = \infty$$

We show that C is convex by demonstrating that $(C^{-1})'' \le 0$. To this end, differentiating $(C^{-1})'$ and simplifying, we see that

$$(C^{-1})'' = \frac{3((B_1)'')^2 B_1 - 3((B_1)')^2 (B_1)'' - (B_1)^{(3)} B_1 (B_1)'}{8((B_1)')^5 B_1} \circ C_*^{-1}(t).$$

Clearly, it suffices to show that the numerator of this expression is nonpositive. The first term is

$$
\begin{aligned}
3((B_1)''(s))^2 B_1(s) &= 3\left(\frac{1}{s+1}+\frac{1}{(s+1)^2}\right)^2 s\log(s+1) \\
&\leq 2\left(\frac{1}{1+s}+\frac{1}{(1+s)^2}\right)\left(\log(1+s)+\frac{s}{1+s}\right)^2 . \qquad (7.2)
\end{aligned}
$$

The second term is

$$
-3(B_1)''(s)((B_1)'(s))^2 = -3\left(\frac{1}{1+s}+\frac{1}{(1+s)^2}\right)\left(\log(1+s)+\frac{s}{1+s}\right)^2 . \qquad (7.3)
$$

Estimating the third term,

$$
\begin{aligned}
-(B_1)^{(3)}(s)B_1(s)(B_1)'(s) &= \left(\frac{1}{(1+s)^2}+\frac{2}{(1+s)^3}\right) s\log(s+1)\left(\log(1+s)+\frac{s}{1+s}\right) \\
&\leq \left(\frac{1}{s+1}+\frac{1}{(s+1)^2}\right)\left(\log(1+s)+\frac{s}{s+1}\right)^2 . \qquad (7.4)
\end{aligned}
$$

Combining (7.2),(7.3) and (7.4) we see that the numerator of $(C^{-1})''$ is nonpositive. Hence, C is convex.

Next we show that B_1 dominates C in the sense of Proposition 16. To this end, set $k_j := C_*(2^j)$ and $l_j := C^{-1}(k_j)$. From this it follows that $C(l_j) = k_j$ and $k_j = 4^j(\log(2^j+1))^2$. Estimating k_j:

$$
(\log 2)^2 j^2 4^j \leq k_j \leq 4(\log 2)^2 j^2 4^j
$$

gives us that $k_{j+1} \leq 64(\log 2)^2 j^2 4^j \leq 64 k_j$. We can use the fact that $k_j \sim j^2 4^j$ to estimate l_j:

$$
\frac{(\log 2)^2}{4} j4^j \leq \frac{k_j}{2(\log(2^j+1)+2^j/(1+2^j))} = l_j,
$$

which gives, for j big enough

$$
\frac{(\log 2)^2\log 4}{4} j^2 4^j = \frac{(\log 2)^2}{4} j4^j \log 4^j \leq l_j\log(l_j+1).
$$

Hence, for j big enough, $C(l_{j+1}) \leq cB_1(l_j) \leq B_1(cl_j)$ and, since both B_1 and C are increasing functions, this implies that $C(x) \leq B_1(cx)$. The lemma follows by applying Proposition 16. □

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