COMPACTLY SUPPORTED, PIECEWISE POLYHARMONIC RADIAL FUNCTIONS WITH PRESCRIBED REGULARITY

Michael J. Johnson

Kuwait University
yohnson1963@hotmail.com

Abstract. A compactly supported radially symmetric function $\Phi : \mathbb{R}^d \to \mathbb{R}$ is said to have *Sobolev regularity* k if there exist constants $B \ge A > 0$ such that the Fourier transform of Φ satisfies

$$A(1 + \|\omega\|^2)^{-k} \le \widehat{\Phi}(\omega) \le B(1 + \|\omega\|^2)^{-k}, \qquad \omega \in \mathbb{R}^d.$$

Such functions are useful in radial basis function methods because the resulting native space will correspond to the Sobolev space $W_2^k(\mathbb{R}^d)$. For even dimensions d and integers $k \ge d/4$, we construct piecewise polyharmonic radial functions with Sobolev regularity k. Two families are actually constructed. In the first, the functions have k nontrivial pieces while in the second, exactly one nontrivial piece. We also explain, in terms of regularity, the effect of restricting Φ to a lower dimensional space $\mathbb{R}^{d-2\ell}$ of the same parity.

1. Introduction

At the heart of radial basis function methods (see [5] and [21]), lies a radially symmetric function $\Phi : \mathbb{R}^d \to \mathbb{R}$ whose Fourier transform defines an inner-product space of functions $\mathcal{N}_\Phi$, called the *native space* (see [14]), with norm (or seminorm) $\|\cdot\|_\Phi$. In case $\Phi \in L_1(\mathbb{R}^d)$, which is the case of interest here, the above definitions and resulting theory are almost entirely accessible within the framework of intermediate real analysis (eg. [11] or [12]). The Fourier transform of a function $g \in L_1(\mathbb{R}^d)$ is defined by

$$\widehat{g}(\omega) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} g(\omega) e^{-\imath\omega\cdot x}\, dx,$$

and it is well known that $\widehat{g} \in C(\mathbb{R}^d)$, with $\lim_{\|\omega\|\to\infty} |\widehat{g}(\omega)| = 0$. In case $\widehat{g} \in L_1(\mathbb{R}^d)$, it follows that g is continuous and can be recovered via the inversion formula

$$g(x) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} \widehat{g}(\omega) e^{\imath x\cdot\omega}\, d\omega, \quad x \in \mathbb{R}^d.$$

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If $\widehat{\Phi} \geq 0$ on $\mathbb{R}^d$, then $\mathcal{N}_\Phi$ is defined to be the space of all functions $g \in L_2(\mathbb{R}^d)$ satisfying $\|g\|_\Phi^2 := \int_{\mathbb{R}^d} |\widehat{g}(\omega)|^2 / \widehat{\Phi}(\omega)\, d\omega < \infty$ (see [13] for the definition of the Fourier transform on $L_2(\mathbb{R}^d)$). Schaback and his students Wu and Wendland saw a need for such functions Φ which are compactly supported and easy to evaluate. Wu considered functions of the form $\phi \circ \rho_d$, where $\rho_d(x) := \sqrt{x_1^2 + x_2^2 + \cdots + x_d^2}$ and $\phi(t) = p(t)\chi_{[0,1]}(t)$, p being a polynomial. He constructed (see [23]) a family of such functions having prescribed smoothness and nonnegative Fourier transform. Suspecting that the degree of his polynomials were unnecessarily large, he posed the problem of finding polynomials $p(t)$, of minimal degree, such that $\phi \circ \rho_d$ has a prescribed smoothness and a non-negative Fourier transform. As a solution of this problem, Wendland (see [18]) constructed functions $\phi_{d,\ell} = p_{d,\ell}\chi_{[0,1]}$, for integers $d \geq 1$ and $\ell \geq 0$, such that $\phi_{d,\ell} \circ \rho_d$ has a nonnegative Fourier transform and belongs to $C^{2\ell}(\mathbb{R}^d)$, the degree of the polynomial $p_{d,\ell}$ being minimal. Other noteworthy constructions are those of Buhmann [4], who constructed "single-piece" piecewise functions of the form $\phi \circ \rho_d$, where $\phi = q\chi_{[0,1]}$ with q analytic on $(0, 1]$, as well as several families constructed by Gneiting (see [8] and the references therein). Recently, Al-Rashdan and the author (see [2]) showed that the B-spline ψ_k, having simple knots at $\{\pm 1, \pm 2, \ldots, \pm k\}$ and a double knot at 0, has a positive Fourier transform ($d = 1$).

In applications, it is often desired that Φ be chosen so that the native space will equal (with equivalent norms) the Sobolev space $W_2^k(\mathbb{R}^d)$ (see [1]). When this happens, we will say that Φ has *Sobolev regularity* k; in case $\Phi = \phi \circ \rho_d$, we say that ϕ (which is a univariate function) has *regularity* (d, k). It follows from the definition of $\|\cdot\|_\Phi$, that Φ has Sobolev regularity k if and only if there exist constants $B \geq A > 0$ such that

$$A(1 + \|\omega\|^2)^{-k} \leq \widehat{\Phi}(\omega) \leq B(1 + \|\omega\|^2)^{-k}, \quad \omega \in \mathbb{R}^d. \tag{1.1}$$

In most applications, k is greater than $d/2$ (so that $W_2^k(\mathbb{R}^d)$ is a subspace of $C(\mathbb{R}^d)$), but the case $0 < k \leq d/2$ is also valid, provided one accesses functions $g \in W_2^k(\mathbb{R}^d)$ by local averages, rather than point evaluations. Although Buhmann showed that his functions have a positive Fourier transform, it is not known whether they satisfy (1.1). But Wendland (see [19]) did subsequently prove that his function $\Phi = \phi_{d,\ell} \circ \rho_d$ satisfies (1.1) with $k = \ell + (d+1)/2$ (the case $d = 1, \ell = 0$ is excluded as $A = 0$). It is unfortunate that $k = \ell + (d+1)/2$ is not an integer when d is even, and this motivated Schaback [15] to construct "single-piece" piecewise functions which, in even dimensions, satisfy (1.1) for integers $k > d/2$. As for the B-spline ψ_k, it was shown that it has regularity $(1, k)$ for $k = 1, 2, 3, \ldots$.

Having established several "dimension-walk" identities (see [22] and [23]), Wu has shown that if one has in hand a *base* family of functions ϕ_k, having regularity $(1, k)$ (respectively $(2, k)$) then, provided certain conditions are satisfied, one can easily obtain functions having regularity $(1 + 2j, k)$ (respectively $(2 + 2j, k)$) for $j = 1, 2, 3, \ldots$. The following is a consequence of [23, Th. 3.3] (see also [20, Lemma 6]).

Theorem 1.1. *Suppose* $\psi \in C^1[0, \infty)$ *has compact support and regularity* (d, k). *If* $\lim_{r \to 0^+} \frac{1}{r}\psi'(r)$ *exists, then* $\mathcal{D}\psi$ *has regularity* $(d + 2, k)$, *where the operator* $\mathcal{D}$ *is defined by*

$$(\mathcal{D}f)(r) = -\frac{1}{r} f'(r), \quad r > 0.$$

As an illustrative example, consider $\psi(t) = (1 - 10t^2 + 20t^3 - 15t^4 + 4t^5)\chi_{(0,1]}$ (this is Wendland's function $\phi_{3,1}$) which has regularity $(1,2)$. Since $\psi(1) = \psi'(1) = \psi'(0) = 0$, it follows that the hypothesis of Theorem 1.1 is satisfied and therefore $\mathcal{D}\psi = 20(1 - 3t + 3t^2 - t^3)\chi_{(0,1]}$ (this is $\phi_{5,0}$) has regularity $(3,2)$. But we cannot apply Theorem 1.1 again since $(\mathcal{D}\psi)'(0) = -60 \neq 0$. One of the tasks taken up in the present contribution is that of proving Wu's dimension-walk identities under less restrictive assumptions. Using the extended version of Theorem 1.1 (see Corollary 5.5 or Theorem 6.1), it follows that $\mathcal{D}^2\psi(t) = 60(t^{-1} - 2 + t)\chi_{(0,1]}$, $\mathcal{D}^3\psi(t) = 60(t^{-3} - t^{-1})\chi_{(0,1]}$ and $\mathcal{D}^4\psi(t) = 60(3t^{-5} - t^{-3})\chi_{(0,1]}$ have regularity $(5,2)$, $(7,2)$ and $(9,2)$, respectively. Although ψ and $\mathcal{D}\psi$ are piecewise polynomials, the others are not. However, if we look instead at the multivariate radial function, then we recognize that $\psi \circ \rho_1$, $\mathcal{D}\psi \circ \rho_3$, $\mathcal{D}^2\psi \circ \rho_5$, $\mathcal{D}^3\psi \circ \rho_7$ and $\mathcal{D}^4\psi \circ \rho_9$ are all piecewise polyharmonic radial functions. This observation suggests the following modification to Wu and Wendland's framework: rather than search amongst radial functions $\Phi = \phi \circ \rho_d$ whose *profile*, ϕ, is piecewise polynomial, search instead amongst radial functions which are piecewise polyharmonic. When d is odd, this change of framework enlarges the search space because if ϕ is a piecewise polynomial, then $\phi \circ \rho_d$ is piecewise polyharmonic; however, when d is even the search space has been substantially changed.

Definition 1.2. A compactly supported radially symmetric function $\Phi : \mathbb{R}^d \to \mathbb{R}$ is called *piecewise polyharmonic* if there exists a system of nodes $0 = r_0 < r_1 < r_2 < \cdots < r_N < \infty$ and a positive integer n such that $\Phi(x) = 0$ when $\|x\| > r_N$ and $\Delta^n\Phi = 0$ on the annulus $\{x \in \mathbb{R}^d : r_{j-1} < \|x\| < r_j\}$ for $j = 1, 2, \ldots, N$, where Δ denotes the Laplacian operator.

It is known, see eg [9 p.435], that piecewise polyharmonic functions can be written as $\Phi = \phi \circ \rho_d$, where $\phi : (0,\infty) \to \mathbb{R}$ is piecewise in a space Z_d, defined (with t denoting a positive real variable) as follows:

$$Z_1 = span\{1, t, t^2, t^3, \ldots\}, \qquad Z_2 = span\{1, \log t, t^2, t^2\log t, \ldots\},$$
$$Z_3 = span\{t^{-1}, 1, t, t^2, t^3, \ldots\}, \qquad Z_4 = span\{t^{-2}, 1, \log t, t^2, t^2\log t, \ldots\},$$
$$Z_5 = span\{t^{-3}, t^{-1}, 1, t, t^2, t^3, \ldots\}, \qquad Z_6 = span\{t^{-4}, t^{-2}, 1, \log t, t^2, t^2\log t, \ldots\},$$

and in general $Z_{d+2} = span\{t^{-d}\} + Z_d$. Since a radial function $f \circ \rho_d$ belongs to $L_1(\mathbb{R}^d)$ if and only if $\int_0^\infty t^{d-1}\,|f(t)|\,dt < \infty$, it is straightforward to verify that compactly supported piecewise polyharmonic functions always belong to $L_1(\mathbb{R}^d)$.

The primary goal of the present contribution (sections 3,4) is to construct two families of L-splines $\{\eta_k\}$ and $\{\gamma_k\}$ such that $\eta_k \circ \rho_2$ and $\gamma_k \circ \rho_2$ are compactly supported piecewise polyharmonic radial functions with Sobolev regularity k. While η_k has k nontrivial pieces, γ_k has one. Following these constructions we extend Wu's dimension-walk identities (section 5) and then apply them (section 6) to the *base* families $\{\eta_k\}$ and $\{\gamma_k\}$ to obtain larger families $\{\eta_{d,k}\}$ and $\{\gamma_{d,k}\}$, with d even, such that $\eta_{d,k} \circ \rho_d$ and $\gamma_{d,k} \circ \rho_d$ are piecewise polyharmonic radial functions with Sobolev regularity k. We also apply these dimension-walk identities to the base family $\{\phi_{1,k-1}\}$ for odd dimensions d.

A secondary goal is to give an interesting answer to the following question. Suppose we have a radial function $\Phi_d = \phi \circ \rho_d$ having Sobolev regularity k. If $d \geq 3$, what can be said about the radial function $\Phi_{d-2} = \phi \circ \rho_{d-2}$? We will show (section 7) that if Φ_{d-2} belongs to $L_1(\mathbb{R}^{d-2})$, then Φ_{d-2} has a stronger form of Sobolev regularity. This argument can be applied recursively and applies to the families addressed in section 6. Using this stronger notion of Sobolev regularity, we are then able (section 8) to discuss the regularity of the family of B-splines $\{\psi_k\}$ mentioned above.

Throughout the sequel, the natural numbers are denoted by $\mathbb{N} = \{1, 2, 3, \ldots\}$, the nonnegative integers by $\mathbb{N}_0$, and the integers by $\mathbb{Z}$. When convenient, we employ *variables* to define functions. Mathematically, a variable is simply the identity function defined on some set. For example, in the definition of Z_d given above, functions were defined using the positive real variable t. Sometimes the domain of a variable is clear from the context, and so it is not necessary to explicitly state its domain. When working within the Lebesgue theory of functions defined almost everywhere, we adopt the usual convention that when such a function f is equivalent (ie equal a.e.) to a continuous function $\widetilde{f}$, then we assume, without mention, that $f = \widetilde{f}$ everywhere.

2. Operators on profiles of radial functions

A radially symmetric function $\Phi : \mathbb{R}^d \to \mathbb{R}$ can always be written as $\Phi = \phi \circ \rho_d$, where $\rho_d(x) = \sqrt{x_1^2 + x_2^2 + \cdots + x_d^2}$. We will refer to the function $\phi : (0, \infty) \to \mathbb{R}$ as the *profile* of Φ. Let U_{loc} be the space of locally integrable functions $f : (0, \infty) \to \mathbb{R}$ and for $d \in \mathbb{N}$, let U_d be the subspace of U_{loc} given by

$$U_d = \{f \in U_{loc} : \int_0^\infty t^{d-1} |f(t)| \, dt < \infty\}.$$

It is easy to see that a radially symmetric function Φ belongs to $L_1(\mathbb{R}^d)$ if and only if its profile belongs to U_d. It is known (see [17] p.155) that if $\Phi = \phi \circ \rho_d \in L_1(\mathbb{R}^d)$, then the profile of its Fourier transform is the function $F_d\phi$, where the linear operator $F_d : U_d \to C(0, \infty)$ is defined by

$$(F_d\phi)(r) = r^{1-\frac{d}{2}} \int_0^\infty \phi(t) t^{d/2} J_{\frac{d}{2}-1}(rt) \, dt, \quad r > 0. \tag{2.1}$$

Here $J_\nu(t) = \sum_{m=0}^\infty \frac{(-1)^m}{m!\Gamma(m+\nu+1)} (\frac{1}{2}t)^{2m+\nu}$ denotes the Bessel function of the first kind. For $\nu > -1$, $J_\nu \in C^\infty(0, \infty)$ and satisfies $|J_\nu(t)| = O(t^\nu)$ as $t \to 0^+$ and $|J_\nu(t)| = O(t^{-1/2})$ as $t \to \infty$. It follows from these that if $\nu \geq -\frac{1}{2}$, then there exists a constant C_ν such that $|J_\nu(t)| \leq C_\nu t^\nu$, $t \in (0, \infty)$, and hence the integrand in (2.1) is integrable when $\phi \in U_d$. Although $(F_d\phi)(r)$ is only defined for $r > 0$, since $F_d\phi$ is the profile of $\widehat{\Phi} \in C(\mathbb{R}^d)$, it follows that $(F_d f)(0^+) := \lim_{r \to 0^+} (F_d f)(r) = \widehat{\Phi}(0)$. Other useful properties of the Bessel functions are:

$$\begin{aligned}
&\frac{\partial}{\partial t} J_\nu(rt) = -r J_{\nu+1}(rt) + \frac{\nu}{t} J_\nu(rt), \qquad \nu > -1, \\
&\frac{\partial}{\partial t} (t^{-\nu} J_\nu(rt)) = -r t^{-\nu} J_{\nu+1}(rt), \qquad \nu > -1 \\
&\frac{\partial}{\partial t} J_\nu(rt) = r J_{\nu-1}(rt) - \frac{\nu}{t} J_\nu(rt), \qquad \nu > 0, \\
&\frac{\partial}{\partial t} (t^\nu J_\nu(rt)) = r t^\nu J_{\nu-1}(rt), \qquad \nu > 0.
\end{aligned} \tag{2.2}$$

Our definition of Sobolev regularity (1.1), for a radial function $\Phi = \phi\circ\rho_d$, can be formulated in terms of its profile ϕ as follows.

Definition 2.1. Let $d, k \in \mathbb{N}$. A function $\phi : (0,\infty) \to \mathbb{R}$ has *regularity* (d, k) if $\phi \in U_d$ and there exist constants $B \geq A > 0$ such that

$$A(1 + r^2)^{-k} \leq (F_d\phi)(r) \leq B(1 + r^2)^{-k}, \quad r \in (0,\infty).$$

Let U be the subspace of U_{loc} given by $U = \{f \in U_{loc} : \int_1^\infty t\,|f(t)|\,dt < \infty\}$, and let AC_{loc} be the space of functions $f : (0,\infty) \to \mathbb{R}$ which are locally absolutely continuous (ie f is absolutely continuous on $[a, b]$ whenever $0 < a < b < \infty$). The reader is referred to [11, chap. 5] or [12, chap. 7] for the concept of absolute continuity which is needed for a proper statement of integration by parts: If f and g are absolutely continuous on $[a, b]$, then $\int_a^b f(t)g'(t)\,dt = f(b)g(b) - f(a)g(a) - \int_a^b g(t)f'(t)\,dt$. For the functions encountered in this article, it suffices to know that if $f \in C(0,\infty)$ is piecewise C^1 (finitely many pieces), then $f \in AC_{loc}$.

The linear operators $\mathcal{I} : U \to AC_{loc}$ and $\mathcal{D} : AC_{loc} \to U_{loc}$ are defined by

$$(\mathcal{I}f)(r) = \int_r^\infty tf(t)\,dt \text{ and } (\mathcal{D}f)(r) = -\frac{1}{r}f'(r).$$

We note that if $d \geq 2$, then U_d is a subspace of U, and hence $\mathcal{I}$ is define on U_d.

Remark 2.2. The operators $\mathcal{I}$ and $\mathcal{D}$ appear, with a normalizing factor, in [10] where they are called the *montée* and the *descente*.

For $j \in \mathbb{Z}$, let $v_j, w_j \in C^\infty(0,\infty)$ be defined by

$$v_j(t) = \begin{cases} t^j & \text{if } j \geq 0 \\ t^{2j+1} & \text{if } j < 0 \end{cases} \qquad \text{and} \qquad w_j(t) = \begin{cases} t^j & \text{if } j \geq 0 \text{ is even} \\ t^{j-1}\log t & \text{if } j > 0 \text{ is odd} \\ t^{2j} & \text{if } j < 0 \end{cases}$$

Note that the space Z_d, defined in the introduction, can be expressed as $Z_d = span\{v_j : j \geq (1-d)/2\}$, when d is odd, and as $Z_d = span\{w_j : j \geq (2-d)/2\}$, when d is even. The action of the operator $\mathcal{D}$ on these functions is as follows:

$$\mathcal{D}v_j = \begin{cases} -j\,v_{j-2} & \text{if } j > 0 \\ 0 & \text{if } j = 0 \\ -(2j+1)v_{j-1} & \text{if } j < 0 \end{cases}, \ \mathcal{D}w_j = \begin{cases} -w_{-1} & \text{if } j = 1 \\ -j\,w_{j-2} & \text{if } j \geq 2 \text{ is even} \\ -(j-1)w_{j-2} - w_{j-3} & \text{if } j \geq 3 \text{ is odd} \\ -2j\,w_{j-1} & \text{if } j \leq 0 \end{cases}$$

Remark 2.3. Let $d \in \mathbb{N}$. It follows from the above that $\mathcal{D}Z_d = Z_{d+2}$. Moreover, if $\phi \in C(0,\infty)$ is piecewise in Z_d (finitely many pieces) and has bounded support, then $\phi \in AC_{loc}$ and $\mathcal{D}\phi$ is piecewise in Z_{d+2}. Conversely, if $\psi : (0,\infty) \to \mathbb{R}$ is piecewise in Z_{d+2} (finitely many pieces) and has bounded support, then $\psi \in U$ and $\mathcal{I}\psi$ is piecewise in Z_d and is continuous on $(0,\infty)$.

3. A family of L-splines with k nontrivial pieces

In this section we construct the functions $\{\eta_k\}$, mentioned in the introduction, which are piecewise in Z_2 and have Sobolev regularity $(2, k)$. To get a sense of where things are headed, we display η_1, η_2, η_3, which are defined on their support by: $\eta_1(t) = -(\log t)\chi_{(0,1]}(t)$

$$\eta_2(t) = \frac{1}{3}\begin{cases} 4\log 2 + (\log 2 - 3)t^2 + 3t^2 \log t, & t \in (0,1] \\ (4\log 2 - 4) - 4\log t + (\log 2 + 1)t^2 - t^2 \log t, & t \in (1,2] \end{cases}$$

$$\eta_3(t) = \frac{1}{10}\begin{cases} b_{1,0} + b_{1,2}t^2 + b_{1,4}t^4 - 10t^4 \log t, & t \in (0,1] \\ b_{2,0} + b_{2,1}\log t + b_{2,2}t^2 + b_{2,3}t^2\log t + b_{2,4}t^4 + b_{2,5}t^4 \log t, & t \in (1,2] \\ b_{3,0} + b_{3,1}\log t + b_{3,2}t^2 + b_{3,3}t^2\log t + b_{3,4}t^4 + b_{3,5}t^4 \log t, & t \in (2,3] \end{cases}$$

where $b_{1,0} = -96\log 2 + 81\log 3$, $b_{1,2} = -96\log 2 + 36\log 3$, $b_{1,4} = 15 - 6\log 2 + \log 3$ and $\{b_{2,j}\} = \{45/2 - 96\log 2 + 81\log 3, 15, -96\log 2 + 36\log 3, 60, -15/2 - 6\log 2 + \log 3, 5\}$, $\{b_{3,j}\} = \{-243/2 + 81\log 3, -81, 36\log 3, -36, 3/2 + \log 3, -1\}$.
It is a correct impression that η_k is piecewise in $span\{w_0, w_1, \ldots, w_{2k-1}\}$ and has k nontrivial pieces with nodes $0, 1, 2, \ldots, k$. Note that the first piece in η_2 does not employ $w_1(t) = \log t$ and the first piece in η_3 employs neither w_1 nor $w_3(t) = t^2 \log t$. This too is a correct impression. Defining, for m odd,

$$X^m = span\{w_j : j = 0, 1, , \ldots, m\} \text{ and } \widetilde{X}^m = span\{w_j : j = 0, 2, 4, \ldots, m-1; m\},$$

we can say that the first piece of η_k belongs to $\widetilde{X}^{2k-1}$ while the other pieces belong to X^{2k-1}.

Definition 3.1. For $n, k \in \mathbb{N}$, let $W_{n,k}$ be the space of piecewise functions $f : (0,\infty) \to \mathbb{R}$, with nodes $0, 1, 2, \ldots, k$, such that the first piece of f (supported on $(0,1]$) belongs to $\widetilde{X}^{2n-1}$ and the remaining pieces belong to X^{2n-1}, with $f = 0$ on (k,∞). The coefficient of w_{2n-1} in the first piece of f is called the *singular coefficient* of f.

For example, η_1 belongs to $W_{1,1}$ with singular coefficient -1, η_2 belongs to $W_{2,2}$ with singular coefficient 1 and η_3 belongs to $W_{3,3}$ with singular coefficient -1. It is easy to verify that $\dim W_{n,k} = (n+1)1 + 2n(k-1)$ and, in particular, that $\dim W_{k,k} = 2k^2 - k + 1$. We will be interested in the subspace $W_{k,k} \cap C^{2k-2}(0,\infty)$. Since it is obtained from $W_{k,k}$ by imposing $k(2k-1) = 2k^2 - k$ continuity conditions, it follows from standard linear algebraic considerations that its dimensions is at least 1. We will show, somewhat down the road, that this dimension in fact equals 1, but for the time being we leave open the possibility that the dimension exceeds 1.

For a function $f \in C^1(0,\infty)$, with $f' \in AC_{loc}$, we define the operator L by

$$(Lf)(r) = f''(r) + \frac{1}{r}f'(r), \quad r > 0.$$

The operator L is related to the Laplacian operator, in $\mathbb{R}^2$, in that $\Delta(f \circ \rho_2) = (Lf) \circ \rho_2$. We leave the proof of the following as an exercise in integration by parts.

Theorem 3.2. *Let $f \in C^1[0,\infty)$ vanish outside $[0,M)$ and assume that f' is absolutely continuous on $[0,M]$. Then $(F_2 f)(r) = -\frac{1}{r^2}(F_2 Lf)(r)$, $r>0$.*

Corollary 3.3. *Let $k \ge n \ge 2$ and let $f \in W_{n,k} \cap C^{2(n-1)}(0,\infty)$, with singular coefficient α. Then $L^{n-1}f \in W_{1,k} \cap C(0,\infty)$, with singular coefficient $\alpha 4^{n-1}[(n-1)!]^2$, and*

$$(F_2 f)(r) = \frac{(-1)^{n-1}}{r^{2(n-1)}}(F_2 L^{n-1} f)(r), \quad r>0. \tag{3.1}$$

Proof. We first mention that the effect of L on $\{w_j\}_{j\ge 0}$ is as follows:
$Lw_0 = Lw_1 = 0$, and if $j \in \mathbb{N}$, then $Lw_{2j} = 4j^2 w_{2j-2}$ and $Lw_{2j+1} = 4j^2 w_{2j-1} + 4j\, w_{2j-2}$. Fix $k \ge 2$ and consider the case $n=2$. Then $f \in C^2(0,\infty)$ and vanishes on $[k,\infty)$. The first piece of f can be written as $f_{|(0,1]} = \alpha r^2 \log r + p(r^2)$ for some polynomial p of degree at most 1. Since the function $r^2 \log r$ belongs to $C^1[0,1]$ and its first derivative is absolutely continuous on $[0,1]$, it follows that the hypothesis of the above Theorem is satisfied and hence (3.1) holds. The above described effect of L on $\{w_j\}$ ensures that Lf belongs to $W_{1,k}$ and that Lf has singular coefficient 4α, and therefore the corollary is true when $n=2$. The proof is then completed by induction on n, where the induction step is similar to the case $n=2$. □

Our proof of the following lemma makes use of Corollary 5.5, which is proved (independently) in section 5.

Lemma 3.4. *Let $k \in \mathbb{N}$ and let $f \in W_{1,k} \cap C(0,\infty)$, say $f_{|(j-1,j]} = a_j + b_j \log t$. Then*

$$(F_2 f)(r) = -\frac{1}{r^2}\left(b_1 + \sum_{j=1}^{k}(b_{j+1}-b_j)J_0(jr)\right), \quad r>0.$$

Proof. It follows from Corollary 5.5 that $F_2 f = F_4 \mathcal{D} f$. Noting that $\mathcal{D}f_{|(j-1,j]} = -b_j t^{-2}$, we have

$$\begin{aligned}(F_2 f)(r) = (F_4 \mathcal{D} f)(r) &= \frac{1}{r}\int_0^k (\mathcal{D}f)(t) t^2 J_1(rt)\,dt = -\frac{1}{r}\sum_{j=1}^{k}\int_{j-1}^{j} b_j J_1(rt)\,dt \\ &= -\frac{1}{r}\sum_{j=1}^{k} b_j[-r^{-1}J_0(rt)]\Big|_{t=j-1}^{t=j} = \frac{1}{r^2}\sum_{j=1}^{k} b_j(J_0(rj) - J_0(r(j-1))),\end{aligned}$$

and the desired conclusion now follows since $J_0(0)=1$. □

Combining the above lemma and corollary yields the following.

Theorem 3.5. *Let $k \in \mathbb{N}$ and let $f \in W_{k,k} \cap C^{2k-2}(0,\infty)$, with singular coefficient α. Put $\beta = \alpha 4^{k-1}[(k-1)!]^2$. Then there exist $c_1, c_2, \dots, c_k \in \mathbb{R}$ such that*

$$(F_2 f)(r) = \frac{(-1)^k}{r^{2k}} \left(\beta + \sum_{j=1}^{k} c_j J_0(jr) \right), \quad r > 0.$$

Now, let f be as in Theorem 3.5. Since $\lim_{r\to 0^+}(F_2 f)(r)$ exists (it equals $\frac{1}{2\pi}\int_{\mathbb{R}^2} f(\|x\|)\, dx$), it must be the case that $\left|\beta + \sum_{j=1}^{k} c_j J_0(jr)\right| = O(r^{2k})$ as $r \to 0^+$. In order to pursue this, we generalize the picture as follows. Let $H(z) = 1 + \sum_{j=1}^{\infty} b_j z^{2j}$ be an even entire function, with $H(0) = 1$, such that $b_j \neq 0$, $j \in \mathbb{N}$, and consider the problem of finding scalars $c_1, c_2, \dots, c_k$ such that

$$\left|\beta + \sum_{\ell=1}^{k} c_\ell H(\ell z)\right| = O(|z|^{2k}) \text{ as } z \to 0. \tag{3.2}$$

It is easy to see that (3.2) is equivalent to $\beta + \sum_{\ell=1}^{k} c_\ell[1 + \sum_{j=1}^{k-1} b_j(\ell z)^{2j}] = 0$, and after expanding the left side as $(\beta + \sum_{\ell=1}^{k} c_\ell) + \sum_{j=1}^{k-1}(\sum_{\ell=1}^{k} c_\ell \ell^{2j}) b_j z^{2j}$, we conclude that (3.2) is equivalent to the equations

$$\sum_{\ell=1}^{k} c_\ell \ell^{2j} = \begin{cases} -\beta, & j = 0 \\ 0, & j = 1, 2, \dots, k-1 \end{cases}$$

Note that this linear system is independent of the values $\{b_j\}$ and can be expressed in matrix form as

$$V\mathbf{c} := \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1^2 & 2^2 & \cdots & k^2 \\ \vdots & \vdots & \vdots & \vdots \\ 1^{2(k-1)} & 2^{2(k-1)} & \cdots & k^{2(k-1)} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ \vdots \\ c_k \end{bmatrix} = \begin{bmatrix} -\beta \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

Since V is (the transpose of) a nonsingular Vandermonde matrix (ie $V(i,j) = (j^2)^{i-1}$), it follows that (3.2) holds if and only if $\mathbf{c} = \beta\mathbf{a}$, where

$$\mathbf{a} := [\, a_1 \quad a_2 \quad a_3 \quad \cdots \quad a_k\,]^T = V^{-1}[-1 \quad 0 \quad 0 \quad \cdots \quad 0]^T \tag{3.3}$$

The upshot of all this is that Theorem 3.5 now read as follows.

Theorem 3.6. *Let $k \in \mathbb{N}$ and let $f \in W_{k,k} \cap C^{2k-2}(0,\infty)$, with singular coefficient α. Put $\beta = \alpha 4^{k-1}[(k-1)!]^2$. Then*

$$(F_2 f)(r) = \frac{(-1)^k \beta}{r^{2k}} \left(1 + \sum_{j=1}^{k} a_j J_0(jr) \right), \quad r > 0,$$

where $a_1, a_2, \dots, a_k$ are as given in (3.3).

Corollary 3.7. *Under the hypothesis of Theorem 3.6, if f is nontrivial, then $\alpha \neq 0$.*

Proof. Suppose $\alpha = 0$. Then it follows from Theorem 3.6 that $F_2 f = 0$. But $F_2 : U_2 \to C(0,\infty)$ is injective; hence $f = 0$. □

Corollary 3.8. *For all $k \in \mathbb{N}$, the subspace $W_{k,k} \cap C^{2k-2}(0,\infty)$ has dimension 1.*

Proof. Suppose not. Then since the dimension is at least 1 (as observed at the beginning of this section), it must be the case that the dimension is greater than 1. But this implies the existence of a nontrivial function $f \in W_{k,k} \cap C^{2k-2}(0,\infty)$ with $\alpha = 0$, which contradicts the above Corollary. □

With the above corollaries in view, we make the following definition.

Definition 3.9. For $k \in \mathbb{N}$, let η_k be the unique function in $W_{k,k} \cap C^{2k-2}(0,\infty)$ which has singular coefficient $(-1)^k$ (ie $\eta_k|_{(0,1]} = (-1)^k t^{2(k-1)} \log t + p(t^2)$ for some polynomial p of degree $\leq k-1$).

Remark 3.10. The action of L on w_j (for $j \geq 0$) was described in the proof of 3.1 and it follows that each piece of η_k is annihilated by L^k; hence η_k is an L-spline.

We now proceed to show that η_k has regularity $(2,k)$. It follows from Theorem 3.6 that

$$(F_2\eta_k)(r) = \frac{4^{k-1}[(k-1)!]^2}{r^{2k}} \left(1 + \sum_{j=1}^{k} a_j J_0(jr)\right), \quad r > 0, \tag{3.4}$$

where $a_1, a_2, \ldots, a_k$ are as given in (3.3). In [2], (3.2) was encountered with $H(z) = \cos z$, and it was shown that $1 + \sum_{j=1}^{k} a_j \cos(jt) = \alpha_k(1-\cos t)^k$, where $\alpha_k > 0$ is defined by $\frac{1}{\alpha_k} = \frac{1}{\pi}\int_0^\pi (1-\cos t)^k\, dt$. Applying the integral representation $J_0(r) = \frac{1}{\pi}\int_0^\pi \cos(r\sin t)\, dt$ to the bracketed factor in (3.4) we obtain

$$\begin{aligned} 1 + \sum_{j=1}^{k} a_j J_0(jr) &= 1 + \sum_{j=1}^{k} a_j \frac{1}{\pi}\int_0^\pi \cos(jr\sin t)\, dt \\ &= \frac{1}{\pi}\int_0^\pi (1 + \sum_{j=1}^{k} a_j \cos(jr\sin t))\, dt = \frac{\alpha_k}{\pi}\int_0^\pi (1-\cos(r\sin t))^k\, dt, \end{aligned}$$

and hence conclude that

$$(F_2\eta_k)(r) = \frac{\alpha_k 4^{k-1}[(k-1)!]^2}{\pi r^{2k}} \int_0^\pi (1-\cos(r\sin t))^k\, dt > 0, \quad r > 0. \tag{3.5}$$

Theorem 3.11. *For $k \in \mathbb{N}$, η_k has regularity $(2,k)$. That is, there exist constants $B \geq A > 0$ such that*

$$A(1+r^2)^{-k} \leq (F_2\eta_k)(r) \leq B(1+r^2)^{-k}, \quad r > 0.$$

Proof. Since $|J_0(r)| = O(r^{-1/2})$ as $r \to \infty$, it follows that there exists $M > 0$ such that $\frac{1}{2} \le 1 + \sum_{j=1}^{k} a_j J_0(jr) \le 3/2$ for all $r \ge M$, and therefore, with (3.4) in view, there exist $B \ge A > 0$ such that the desired inequality holds for $r \ge M$. Since $F_2\eta_k$ is continuous and positive on $(0, \infty)$, in order to complete the proof, it suffices to show that $(F_2\eta_k)(r)$ has a positive limit as $r \to 0^+$. For $r > 0$, define $g_r(t) = \dfrac{1 - \cos(r \sin t)}{r^2}$, $t \in (0, \pi)$, and note, by (3.5), that $(F_2\eta_k)(r) = c_k \int_0^\pi [g_r(t)]^k \, dt$ for some positive constant c_k. Writing

$$g_r(t) = \frac{2}{r^2} \sin^2\left(\frac{1}{2} r \sin t\right) = \frac{1}{2} \sin^2 t \left(\frac{\sin\left(\frac{1}{2} r \sin t\right)}{\frac{1}{2} r \sin t}\right)^2,$$

we see that $\lim_{r \to 0^+} g_r(t) = \frac{1}{2} \sin^2 t$, $t \in (0, \pi)$, and furthermore that $0 \le g_r(t) \le \frac{1}{2}$, for all $t \in (0, \pi)$, $r > 0$. It therefore follows from the Bounded Convergence Theorem that

$$(F_2\eta_k)(r) = c_k \int_0^\pi [g_r(t)]^k \, dt \longrightarrow \frac{c_k}{2^k} \int_0^\pi \sin^{2k} t \, dt \text{ as } r \to 0^+,$$

and we note that the limiting value $c_k 2^{-k} \int_0^\pi \sin^{2k} t \, dt$ is positive. $\square$

4. A family of L-splines with 1 nontrivial piece

In this section, we construct a family of L-splines $\{\gamma_k\}$, $k \in \mathbb{N}$, which have regularity $(2, k)$. The function γ_k is piecewise in Z_2 and has exactly one nontrivial piece, supported on $(0, 1]$. We display a few of these, showing only the nontrivial piece: $\gamma_1(t) = -\log t$,

$$4\gamma_2(t) = 1 - t^4 + 4t^2 \log t, \qquad 36\gamma_3(t) = 1 - 9t^2 - 9t^4 + 17t^6 - 12t^4(3 + t^2) \log t$$

$$240\gamma_4(t) = 1 - 10t^2 + 60t^4 + 80t^6 - 125t^8 - 6t^{10} + 120t^6(2 + t^2) \log t$$

$$1800\gamma_5(t) = 1 - 12t^2 + 75t^4 - 400t^6 - 825t^8 + 924t^{10} + 237t^{12} - 120t^8(15 + 12t^2 + t^4) \log t$$

For $k \ge 2$, our definition of γ_k (Definition 4.3), which depends on the parity of k, employs an intermediate function Γ_k, defined by

$$\Gamma_{2j}(t) = (t^{-2} - 1)_+^j \text{ and } \Gamma_{2j+1}(t) = t^{-2}(t^{-2} - 1)_+^j, \quad t > 0, \ j \in \mathbb{N},$$

where $x_+ = x$ if $x > 0$ and $x_+ = 0$ if $x \le 0$. Note that the nontrivial piece in Γ_{2j} belongs to Z_{2j+2} and that of Γ_{2j+1} belongs to Z_{2j+4}.

Theorem 4.1. *For $k \geq 2$, Γ_k has regularity (d,k), where $d = 6j$ if $k = 2j$, and $d = 6j+4$ if $k = 2j+1$.*

Our proof of this is broken into three claims:
Claim 1. $(F_d\Gamma_k)(0^+) := \lim_{r\to 0^+}(F_d\Gamma_k)(r) > 0$.
Claim 2. $(F_d\Gamma_k)(r) = \beta_k r^{-2k} + o(r^{-2k})$ as $r \to \infty$, for some positive constant β_k.
Claim 3. $(F_d\Gamma_k)(r) > 0$ for all $r > 0$.

We first address Claim 1 and Claim 2 in the case $k = 2j$, where $d = 6j$ and

$$(F_d\Gamma_k)(r) = r^{1-3j}\int_0^1 (t^{-2}-1)^j t^{3j} J_{3j-1}(rt)\,dt = r^{1-3j}\int_0^1 (1-t^2)^j t^j J_{3j-1}(rt)\,dt, \quad r > 0.$$

The function $f_r(t) = r^{1-3j}J_{3j-1}(rt)$, $t \in [0,1]$, converges uniformly to $f(t) = \frac{t^{3j-1}}{(3j-1)!\,2^{3j-1}}$ as $r \to 0^+$, and therefore $(F_d\Gamma_k)(0^+) = ((3j-1)!\,2^{3j-1})^{-1}\int_0^1(1-t^2)^j t^{4j-1}\,dt > 0$, which establishes Claim 1. Our proof of Claim 2 employs the following.

Lemma 4.2. *Let p be a polynomial and let $\alpha \in \mathbb{N}_0$. Then*

$$\int_0^1 p(t^2) r t^{-\alpha} J_{\alpha+1}(rt)\,dt = \frac{p(0)}{\alpha!\,2^\alpha} r^\alpha - p(1)J_\alpha(r) + \frac{2}{r}\int_0^1 p'(t^2) r t^{1-\alpha} J_\alpha(rt)\,dt, \quad r > 0.$$

Proof. Let v be the entire function $v(t) = -t^{-\alpha}J_\alpha(rt) = -\sum_{m=0}^\infty \frac{(-1)^m r^{2m+\alpha}}{m!\,(m+\alpha)!\,2^{2m+\alpha}} t^{2m}$ and put $u(t) = p(t^2)$. The desired equality is then a straightforward application of integration by parts: $\int_0^1 u(t)v'(t)\,dt = u(1)v(1) - u(0)v(0) - \int_0^1 u'(t)v(t)\,dt$. $\square$

With $q_j(\tau) = (1-\tau)^j \tau^{2j-1}$, we write $(F_d\Gamma_k)(r) = r^{-3j}\int_0^1 q_j(t^2) r t^{-(3j-2)} J_{3j-1}(rt)\,dt$, and applying Lemma 4.2 repeatedly then yields

$$(F_d\Gamma_k)(r) = r^{-3j}\sum_{k=0}^{3j-2} \frac{2^\ell}{r^\ell}\left(\frac{q_j^{(\ell)}(0)}{(3j-\ell-2)!\,2^{3j-\ell-2}} r^{3j-\ell-2} - q_j^{(\ell)}(1) J_{3j-\ell-2}(r)\right) + \frac{2^{3j-1}}{r^{6j-1}}\int_0^1 q_j^{(3j-1)}(t) r t J_0(rt)\,dt.$$

Noting that q_j is a polynomial of degree $3j-1$ with a zero of order $2j-1$ at $\tau = 0$ and a zero of order j at $\tau = 1$, we see that $q_j^{(3j-1)}$ is a constant and that $q_j^{(\ell)}(0) = 0$ for $\ell = 0,1,\ldots,2j-2$ and $q_j^{(\ell)}(1) = 0$ for $\ell = 0,1,\ldots,j-1$. And employing $\int_0^1 rtJ_0(rt)\,dt = J_1(r)$, we conclude that

$$(F_d\Gamma_k)(r) = \sum_{\ell=2j-1}^{3j-2} \frac{2^{2\ell-3j+2}}{(3j-\ell-2)!} q_j^{(\ell)}(0) r^{-(2\ell+2)} - \sum_{\ell=j}^{3j-2} 2^\ell q_j^{(\ell)}(1) r^{-(3j+\ell)} J_{3j-\ell-2}(r) + 2^{3j-1} q_j^{(3j-1)}(0) r^{-(6j-1)} J_1(r), \quad r > 0.$$

Since $|J_\alpha(r)| = O(r^{-1/2})$ as $r \to \infty$, and noting that $q_j^{(2j-1)}(0) = (2j-1)!$, we see that Claim 2 follows from the above with $\beta_k = \frac{(2j-1)!\,2^j}{(j-1)!}$.

The proof of Claim 1 and 2 in case $k = 2j + 1$, where $d = 6j + 4$, is similar to the above: First one obtains $(F_d\Gamma_k)(r) = r^{-(3j+1)} \int_0^1 (1 - t^2)^j t^j J_{3j+1}(rt)\, dt$ and deduces that $(F_d\Gamma_k)(0^+) = ((3j+1)!\, 2^{3j+1})^{-1} \int_0^1 (1-t^2)^j t^{4j+1}\, dt > 0$, which proves Claim 1. With $p_j(\tau) = (1-\tau)^j \tau^{2j}$, we have $(F_d\Gamma_k)(r) = r^{-(3j+2)} \int_0^1 p_j(t^2) r t^{-3j} J_{3j+1}\, dt$ and then applying Lemma 4.2 and simplifying yields

$$(F_d\Gamma_k)(r) = \sum_{\ell=2j}^{3j} \frac{2^{2\ell-3j} p_j^{(\ell)}(0)}{(3j-\ell)!} r^{-(2\ell+2)} - \sum_{\ell=j}^{3j} 2^\ell p_j^{(\ell)}(1) r^{-(3j+\ell+2)} J_{3j-\ell}(r), \quad r > 0$$

From this one then obtains Claim 2 with $\beta_k = \frac{(2j)!\, 2^j}{j!}$.

Turning now to Claim 3, we again consider first the case $k = 2j$. Following Wendland [19], we express $(F_d\Gamma_k)(r)$ in the form

$$(F_d\Gamma_k)(r) = r^{1-3j} \int_0^1 (t^{-2} - 1)^j t^{3j} J_{3j-1}(rt)\, dt = r^{-6j} \int_0^r (r^2 - t^2)^j t^j J_{3j-1}(t)\, dt.$$

With $\lambda = j - \frac{1}{2}$, $\mu = j$ and $\alpha = 3j - 1$, Gasper [6, p.874,875] has shown that $\int_0^r (r^2 - t^2)^\lambda t^\mu J_\alpha(t)\, dt > 0$ for all $r > 0$, and then with $\gamma = \frac{1}{2}$, $\delta = \varepsilon = 0$, it follows [6, p.878] that

$$\int_0^r (r^2 - t^2)^j t^j J_{3j-1}(t)\, dt = \int_0^r (r^2 - t^2)^{\lambda+\gamma+\varepsilon} t^{\mu-2\varepsilon-\delta} J_{\alpha+\delta}(t)\, dt > 0, \quad r > 0,$$

which establishes Claim 3 for the case $k = 2j$. The proof of Claim 3 in case $k = 2j+1$ is the same except that $(F_d\Gamma_k)(r) = r^{-(6j+2)} \int_0^r (r^2 - t^2)^j t^j J_{3j+1}(t)\, dt$ and $\alpha = 3j + 1$. This completes the proof of Theorem 4.1.

Definition 4.3. Let $\gamma_1 = \eta_1$ and for $j \in \mathbb{N}$, we define

$$\gamma_{2j} = c_{2j} \mathcal{I}^{3j-1} \Gamma_{2j} \quad \text{and} \quad \gamma_{2j+1} = c_{2j+1} \mathcal{I}^{3j+1} \Gamma_{2j+1},$$

where $c_{2j} = 2^{3j-2}\, (2j-1)!\, (j-1)!$ and $c_{2j+1} = 2^{3j}\, (2j)!\, j!$ (this choice of c_k ensures that the coefficient of $t^{2k-2} \log t$, in $\gamma_k(t)$, equals $(-1)^k$).

Our proof of the following result employs Theorem 5.3 which is proved (independently) in section 5.

Theorem 4.4. *For $k \in \mathbb{N}$, the following hold.*
(i) γ_k is piecewise in Z_2.
(ii) $\gamma_k \in C^{2k-2}(0, \infty)$.
(iii) γ_k has regularity $(2, k)$.

Proof. The case $k = 1$ is proved in section 3, since $\gamma_1 = \eta_1$. Let $j \in \mathbb{N}$. Then, as noted above, Γ_{2j} is piecewise in $Z_{2j+2} \subset Z_{6j}$, and it follows by Remark 2.3 that γ_{2j} is piecewise in $Z_{6j-2(3j-1)} = Z_2$. Since $\Gamma_{2j} \in C^{j-1}(0, \infty)$, if follows that $\gamma_{2j} \in C^{4j-2}$. This proves (i) and (ii) for the case $k = 2j$, and the proof in case $k = 2j + 1$ is similar. We turn now to (iii). Let d be as defined in Theorem 4.1. Since $\Gamma_k \in U_d$, it follows by repeated application of Theorem 5.3 that $\gamma_k \in U_2$ and $(F_2\gamma_k)(r) = c_k (F_d\Gamma_k)(r)$, $r > 0$. And since Γ_k has regularity (d, k), it now follows that γ_k has regularity $(2, k)$. $\square$

5. Extended dimension-walk identities

In this section, we prove two fundamental identities involving the operators $\mathcal{D}$, $\mathcal{I}$ and F_d. These "dimension walk" identities were first proved by Wu [23] (see also [20, Lemma 6]), under overly restrictive conditions.

Lemma 5.1. *Let $f \in AC_{loc}$ be such that $\lim_{t\to\infty} f(t) = 0$ and $\mathcal{D}f \in U$. Then $f = \mathcal{I}\mathcal{D}f$.*

Proof. Since $\mathcal{D}f \in U$, it follows that $\lim_{t\to\infty}(\mathcal{I}\mathcal{D}f)(t) = 0$ and that

$$(\mathcal{I}\mathcal{D}f)(r) - (\mathcal{I}\mathcal{D}f)(t) = \int_r^t s(\mathcal{D}f)(s)\,ds = -\int_r^t f'(s)\,ds = f(r) - f(t), \quad 0 < r < t.$$

Taking the limit as $t \to \infty$ then yields $(\mathcal{I}\mathcal{D}f)(r) = f(r)$. $\square$

Lemma 5.2. *For $d \geq 3$ and $f \in U_d$, the following hold:*
(i) $\lim\limits_{r\to 0^+} r^{d-2}(\mathcal{I}f)(r) = 0$, (ii) $\lim\limits_{r\to\infty} r^{d-2}(\mathcal{I}f)(r) = 0$, (iii) $\mathcal{I}f \in U_{d-2}$.

Proof. Let $\varepsilon > 0$. There exists $a > 0$ such that $\int_0^a t^{d-1}|f(t)|\,dt < \varepsilon$. For $0 < r < a$, we have $r^{d-2}|(\mathcal{I}f)(r)| \leq r^{d-2}\int_r^a t|f(t)|\,dt + r^{d-2}\int_a^\infty t|f(t)|\,dt$. Since a is fixed, it is clear that the latter term on the right tends to 0 as $r \to 0^+$, while for the first term, we have

$$r^{d-2}\int_r^a t|f(t)|\,dt = \int_r^a (r/t)^{d-2}t^{d-1}|f(t)|\,dt \leq \int_r^a t^{d-1}|f(t)| < \varepsilon,$$

whence follows (i). For (ii), we have

$$|(\mathcal{I}f)(r)| \leq \int_r^\infty t|f(t)|\,dt = \int_r^\infty \frac{t^{d-1}}{t^{d-2}}|f(t)|\,dt \leq \frac{1}{r^{d-2}}\int_r^\infty t^{d-1}|f(t)|\,dt.$$

Hence, $r^{d-2}|(\mathcal{I}f)(r)| \leq \int_r^\infty t^{d-1}|f(t)|\,dt \to 0$ as $r \to \infty$, which proves (ii). And finally,

$$\begin{aligned}\int_0^\infty r^{d-3}|(\mathcal{I}f)(r)|\,dr &\leq \int_0^\infty r^{d-3}\Big(\int_r^\infty t|f(t)|\,dt\Big)\,dr \\ &= \int_0^\infty \Big(\int_0^t r^{d-3}\,dr\Big)t|f(t)|\,dt = \frac{1}{d-2}\int_0^\infty t^{d-1}|f(t)|\,dt < \infty,\end{aligned}$$

which proves (iii). $\square$

Theorem 5.3. *Let $d \geq 3$ and $f \in U_d$. Then $\mathcal{I}f \in U_{d-2}$ and $F_{d-2}\mathcal{I}f = F_d f$.*

Proof. By Lemma 5.2, $\mathcal{I}f \in U_{d-2}$ and hence $F_{d-2}\mathcal{I}f$ is defined. Fix $r > 0$. We first write $(F_d f)(r)$ as

$$(F_d f)(r) = r^{1-\frac{d}{2}}\int_0^\infty f(t)t^{d/2}J_{d/2-1}(rt)\,dt = r^{1-\frac{d}{2}}\lim_{(\delta,T)\to(0^+,\infty)}\int_\delta^T t^{d/2-1}J_{d/2-1}(rt)tf(t)\,dt.$$

Noting that $-(\mathcal{I}f)(t)$ is an antiderivative of $tf(t)$, and with (2.2) in view, we apply integration by parts to obtain

$$\int_\delta^T t^{d/2-1} J_{d/2-1}(rt) t f(t)\,dt = -(\mathcal{I}f)(T)T^{d/2-1}J_{d/2-1}(rT) + (\mathcal{I}f)(\delta)\delta^{d/2-1}J_{d/2-1}(r\delta) + \int_\delta^T (\mathcal{I}f)(t)[rt^{(d-2)/2}J_{(d-2)/2-1}(rt)]\,dt.$$

Since $\left|J_{d/2-1}(t)\right| \le C_{d/2-1}t^{d/2-1}$, it follows from (i) and (ii) of Lemma 5.2 that the first two terms have limit 0 as $(\delta, T) \to (0^+, \infty)$. As for the remaining term, since $\mathcal{I}f \in U_{d-2}$, it follows that the integrand is integrable over $(0, \infty)$, and hence it converges, as $(\delta, T) \to (0^+, \infty)$, to the full integral over $(0, \infty)$. It follows therefore that $(F_d f)(r) = r^{1-\frac{d}{2}} \int_0^\infty (\mathcal{I}f)(t)[rt^{(d-2)/2}J_{(d-2)/2-1}(rt)]\,dt = (F_{d-2}\mathcal{I}f)(r)$. □

Remark 5.4. The conclusion $F_{d-2}\mathcal{I}\phi = F_d\phi$ was obtained by Wu (see [23, Th. 3.3]) assuming that $\phi \in C[0, \infty)$ is compactly supported.

Corollary 5.5. *Let $d \ge 1$ and let $f \in AC_{loc}$ be such that $\lim_{t\to\infty} f(t) = 0$ and $\mathcal{D}f \in U_{d+2}$. Then $f \in U_d$ and $F_{d+2}\mathcal{D}f = F_d f$.*

Proof. Put $g = \mathcal{D}f$. Since $g \in U_{d+2}$, it follows from Theorem 5.3 that $\mathcal{I}g \in U_d$ and $F_d\mathcal{I}g = F_{d+2}g$. But $\mathcal{I}g = \mathcal{I}\mathcal{D}f = f$, by Lemma 5.1, and therefore, $F_d f = F_d\mathcal{I}g = F_{d+2}g = F_{d+2}\mathcal{D}f$. □

Remark 5.6. As noted in the introduction, the conclusion $F_{d+2}\mathcal{D}\psi = F_d\psi$ was obtained by Wu (see [23, Th. 3.3]) assuming that $\psi \in C^1[0, \infty)$ is compactly supported with $\mathcal{D}\psi \in C[0, \infty)$.

6. Walking piecewise polyharmonic radial functions into higher dimensions

In the following theorem we specialize Corollary 5.5 to the particular case when the function $\phi : (0, \infty) \to \mathbb{R}$ is piecewise in Z_d (finitely many pieces) with bounded support. Recall that such functions necessarily belong to U_d, so $F_d\phi$ is defined.

Theorem 6.1. *Let $d \in \mathbb{N}$ and suppose $\phi : (0, \infty) \to \mathbb{R}$ is piecewise in Z_d (finitely many pieces) with bounded support. If ϕ is continuous on $(0, \infty)$, then the following hold:*
(i) $\phi \in AC_{loc}$.
(ii) $\mathcal{D}\phi$ is piecewise in Z_{d+2} with bounded support.
(iii) $F_{d+2}\mathcal{D}\phi = F_d\phi$.
(iv) If ϕ has Sobolev regularity (d, k), then $\mathcal{D}\phi$ has Sobolev regularity $(d + 2, k)$.

Proof. Suppose $\phi \in C(0, \infty)$. Since Z_d is a subspace of $C^\infty(0, \infty)$, it follows that ϕ is absolutely continuous on $[a, b]$ whenever $0 < a < b < \infty$; this establishes (i). Condition (ii) now follows from the observation (made in section 2) that $\mathcal{D}Z_d = \mathcal{D}Z_{d+2}$. It is now clear that (iii) is a consequence of Corollary 5.5, and now (iv) is an immediate consequence of (iii). □

Theorem 6.1 can be applied recursively to obtain the following.

Corollary 6.2. *Let $d \in \mathbb{N}$ and suppose $\phi : (0,\infty) \to \mathbb{R}$ is piecewise in Z_d (finitely many pieces), with bounded support. If, for some $k, n \in \mathbb{N}$, ϕ has Sobolev regularity k and belongs to $C^{n-1}(0,\infty)$, then $\mathcal{D}^j\phi$ is piecewise in Z_{d+2j} and has Sobolev regularity $(d+2j, k)$, for $j = 1, 2, \ldots, n$.*

We now apply this corollary to the base families $\{\eta_k\}$, $\{\gamma_k\}$ and $\{\phi_{1,k-1}\}$. For $k \in \mathbb{N}$, recall that both η_k and γ_k are piecewise in Z_2, have Sobolev regularity $(2, k)$ and belong to $C^{2k-2}(0,\infty)$. It follows from Corollary 6.2 that for $j = 1, 2, \ldots, 2k-1$, $\mathcal{D}^j\eta_k$ and $\mathcal{D}^j\gamma_k$ are piecewise in Z_{2+2j} and have Sobolev regularity $(2+2j, k)$. We can therefore define

$$\eta_{d,k} := \mathcal{D}^{(d-2)/2}\eta_k \text{ and } \gamma_{d,k} := \mathcal{D}^{(d-2)/2}\gamma_k, \text{ for } d \in \{2,4,6,\ldots\} \text{ and } k \in \mathbb{N} \text{ with } k \geq d/4,$$

and conclude that $\eta_{d,k}$ and $\gamma_{d,k}$ are piecewise in Z_d and have Sobolev regularity (d, k).

For $k = 2, 3, 4, \ldots$, define $\omega_k = \phi_{1,k-1}$, where $\phi_{1,k-1}$ is Wendland's function for $d = 1$. Then ω_k is piecewise in Z_1, belongs to $C^{2k-2}(0,\infty)$ and has Sobolev regularity $(1, k)$. It follows from Corollary 6.2 that $\mathcal{D}^j\omega_k$ is piecewise in Z_{1+2j} and has Sobolev regularity k for $j = 1, 2, \ldots, 2k-1$. We can therefore define,

$$\omega_{d,k} := \mathcal{D}^{(d-1)/2}\omega_k, \text{ for } d \in \{1,3,5,\ldots\} \text{ and } k \in \{2,3,4,\ldots\} \text{ with } k \geq (d+1)/4,$$

and conclude that $\omega_{d,k}$ is piecewise in Z_d and has Sobolev regularity (d, k). When $k \geq (d+1)/2$, the function $\omega_{d,k}$ corresponds to Wendland's function $\phi_{d,k-(d+1)/2}$, so the functions $\omega_{d,k}$ are only 'new' when $(d+1)/4 \leq k < (d+1)/2$.

7. Restriction to lower dimensions

Let $d, k \in \mathbb{N}$, with $d \geq 3$, $k \geq 2$, and suppose that $\Phi_d := \phi \circ \rho_d$ has regularity k. In this section, we show that if $\phi \in U_{d-2}$, then $\Phi_{d-2} := \phi \circ \rho_{d-2}$ has Sobolev regularity $k-1$ (see [7, section 4] for a relation between Φ_d and Φ_{d-2} in terms of unimodal distributions). Of course this result applies recursively to the effect that if $\ell \in \mathbb{N}$ satisfies $\ell \leq \min\{(d-1)/2, k-1\}$ and if $\phi \in U_{d-2\ell}$, then $\phi \circ \rho_{d-2\ell}$ has Sobolev regularity $k - \ell$. Our method of proof employs an extended notion of regularity, defined as follows.

Definition 7.1. Let $d, k \in \mathbb{N}$, $m \in \mathbb{N}_0$. We say that $\phi \in U_d$ has *regularity* (d, k, m) if there exist constants $B_j \geq A_j > 0$, $j = 0, 1, \ldots, m$, such that

$$A_j(1+r^2)^{-(k+j)} \leq (\mathcal{D}^j F_d\phi)(r) \leq B_j(1+r^2)^{-(k+j)}, \quad r > 0, \ j = 0, 1, \ldots, m.$$

Note that regularity $(d, k, 0)$ is the same as regularity (d, k), and regularity (d, k, m') implies regularity (d, k, m) if $m' \geq m$.

Lemma 7.2. *Let $f \in C^1(0,\infty)$ satisfy $\lim_{r\to\infty} f(r) = 0$. Let $j \in \mathbb{N}$ and suppose that there exist constants $B \geq A > 0$ such that*

$$A(1+r^2)^{-(j+1)} \leq (\mathcal{D}f)(r) \leq B(1+r^2)^{-(j+1)}, \quad r > 0.$$

Then

$$\frac{A}{2j}(1+r^2)^{-j} \le f(r) \le \frac{B}{2j}(1+r^2)^{-j}, \quad r > 0.$$

Proof. It follows from the hypothesis that $\mathcal{D}f \in U$ and hence, by Lemma 5.1, that $f = \mathcal{I}\mathcal{D}f$. Since $h \le g$ implies $\mathcal{I}h \le \mathcal{I}g$, it follows that $A\mathcal{I}g \le \mathcal{I}\mathcal{D}f \le B\mathcal{I}g$, where $g(r) = (1+r^2)^{-(j+1)}$. The desired conclusion now follows since $(\mathcal{I}g)(r) = \frac{1}{2j}(1+r^2)^{-j}$ and $\mathcal{I}\mathcal{D}f = f$. □

Our proof also employs the following identity, which appears (in much greater generality) in [16, section 4]. For the sake of completeness, we provide an elementary proof of the particular case of present interest.

Theorem 7.3. *Let $d \in \mathbb{N}$ and let $f \in U_d \cap U_{d+2}$. Then $F_d f \in C^1(0,\infty)$ and $\mathcal{D}F_d f = F_{d+2}f$.*

Proof. Since $F_{d+2}f$ is continuous, it suffices to show that $\lim_{r\to r_0} \frac{(F_d f)(r)-(F_d f)(r_0)}{r-r_0} = -r_0(F_{d+2}f)(r_0)$ for all $r_0 > 0$. Fix $r_0 > 0$ and define $G(r,t) := r^{1-d/2}J_{d/2-1}(rt)$, $r,t > 0$. Then, with (2.2) in view, $G_r(r,t) = \frac{\partial}{\partial r}G(r,t) = -tr^{1-d/2}J_{d/2}(rt)$ and

$$\frac{(F_d f)(r)-(F_d f)(r_0)}{r-r_0} = \int_0^\infty f(t)t^{d/2}\frac{G(r,t)-G(r_0,t)}{r-r_0}\,dt. \tag{7.1}$$

Note that the integrand on the right side of (7.1) converges pointwise to $f(t)t^{d/2}G_r(r_0,t)$ as $r \to r_0$. In preparation for Lebesgue's Dominated Convergence Theorem, we first recall that there exists a constant $C_{d/2}$ such that $\left|J_{d/2}(t)\right| \le C_{d/2}t^{d/2}$, $t > 0$. It follows that if $r \in [\frac{1}{2}r_0, 2r_0]$, then $|G_r(r,t)| \le \widetilde{C}t^{1+d/2}$ for all $t > 0$, where $\widetilde{C}$ is a constant depending only on d and r_0. By the Mean Value Theorem, for each $t > 0$ and $r \in [\frac{1}{2}r_0, 2r_0]\backslash\{r_0\}$, there exists r_t between r_0 and r such that $\frac{G(r,t)-G(r_0,t)}{r-r_0} = G_r(r_t,t)$. Hence, the integrand on the right side of (7.1) is dominated by $g(t) := |f(t)|\, t^{d/2}\widetilde{C}t^{1+d/2} = \widetilde{C}\,|f(t)|\, t^{(d+2)-1}$. Since $f \in U_{d+2}$, g is integrable and therefore by Lebesgue's Dominated Convergence Theorem, $\lim_{r\to r_0}\frac{(F_d f)(r)-(F_d f)(r_0)}{r-r_0} = \int_0^\infty f(t)t^{d/2}G_r(r_0,t)\,dt = -r_0(F_{d+2}f)(r_0)$. □

Theorem 7.4. *Let $d,k \in \mathbb{N}$, with $d \ge 3$, $k \ge 2$, and suppose that $\phi \in U_d$ has regularity (d,k,m) for some $m \in \mathbb{N}_0$. Let $\ell \in \mathbb{N}$ satisfy $\ell \le \min\{(d-1)/2, k-1\}$. If $\phi \in U_{d-2\ell}$, then ϕ has regularity $(d-2\ell, k-\ell, m+\ell)$.*

Proof. Let $\ell = 1$ and assume $\phi \in U_{d-2}$. It follows from Theorem 7.3 that $F_{d-2}\phi \in C^1(0,\infty)$ and $\mathcal{D}F_{d-2}\phi = F_d\phi$. Replacing $F_d\phi$ with $\mathcal{D}F_{d-2}\phi$, in Definition 7.1 yields

$$A_j(1+r^2)^{-(k+j)} \le (\mathcal{D}^{j+1}F_{d-2}\phi)(r) \le B_j(1+r^2)^{-(k+j)}, \quad r > 0,\ j = 0,1,\dots,m.$$

With the case $j = 0$ of the above in view, we apply Lemma 7.2 to obtain

$$\frac{A_0}{2(k-1)}(1+r^2)^{k-1} \le (F_{d-2}\phi)(r) \le \frac{B_0}{2(k-1)}(1+r^2)^{k-1}, \quad r > 0,$$

and we conclude that ϕ has regularity $(d-2, k-1, m+1)$. The proof is then completed by induction, where the induction step is very similar to the case $\ell = 1$, provided one notes that $U_d \cap U_{d-2\ell} \subset U_{d-2(\ell-1)}$. $\square$

As a quick illustration, consider the function η_3 which is given at the beginning of section 3. The function $\eta_{4,3} = \mathcal{D}\eta_3$ has regularity $(4, 3, 0)$ and the first piece of $\eta_{4,3}$ equals $\frac{1}{5}(-b_{12} + (5 - 2b_{14})t^2 + 20t^2 \log t)$. It follows that $\eta_{4,3} \in U_2$ and therefore, by Theorem 7.4, $\eta_{4,3}$ has regularity $(2, 2, 1)$.

In order to give a complete explanation of how Theorem 7.4 can be applied to the families $\{\eta_{d,k}\}$, $\{\gamma_{d,k}\}$ and $\{\omega_{d,k}\}$, we need to pay closer attention to the first piece in these piecewise functions. Let us extend the definition of Z_d (currently defined for $d \in \mathbb{N}$) to integers $d \leq 0$ as follows.

$$Z_{-1} = span\{1; t^2, t^3, t^4, \dots\}, \qquad Z_0 = span\{1; t^2, t^2 \log t, t^4, t^4 \log t, \dots\},$$
$$Z_{-3} = span\{1, t^2; t^4, t^5, t^6, \dots\}, \qquad Z_{-2} = span\{1, t^2; t^4, t^4 \log t, t^6, t^6 \log t, \dots\},$$
$$Z_{-5} = span\{1, t^2, t^4; t^6, t^7, t^8, \dots\}, \qquad Z_{-4} = span\{1, t^2, t^4; t^6, t^6 \log t, t^8, t^8 \log t, \dots\},$$

and in general, $Z_{d-2} = span\{1\} + t^2 Z_d$, $d \leq 2$. We note that the properties mentioned in Remark 2.3 remain valid for all $d \in \mathbb{Z}$. With the hypothesis of Theorem 7.4 in mind, consider the case when ϕ is a piecewise function in Z_d having bounded support and regularity (d, k, m). Then the condition $\phi \in U_{d-2\ell}$ holds if and only if the first piece of ϕ belongs to $Z_{d-2\ell}$. Regarding the family $\{\eta_k\}$, we recall that the first piece of η_k belongs to Z_{4-2k} and consequently the first piece of $\eta_{d,k} = \mathcal{D}^{(d-2)/2}\eta_k$ belongs to Z_{d-2k+2}. Now suppose $\ell \in \mathbb{N}$ satisfies $\ell \leq \min\{(d-1)/2, k-1\}$. Since $\ell \leq k-1$, we have $Z_{d-2k+2} \subset Z_{d-2\ell}$ and it follows that $\eta_{d,k} \in U_{d-2\ell}$. We can now apply Theorem 7.4 to conclude that $\eta_{d,k}$ has regularity $(d-2\ell, k-\ell, \ell)$. Combining the restrictions on d and k in the definition of $\{\eta_{d,k}\}$ with the above restriction on ℓ leads to the following.

Corollary 7.5. *Let $d \in 2\mathbb{N}$, $k \in \mathbb{N}$ and $m \in \mathbb{N}_0$, with $m \geq \frac{d}{2} - 2k$. Then*

$$\eta_{d,k,m} := \eta_{d+2m,k+m} = \mathcal{D}^{(d-2)/2+m}\eta_{k+m}$$

has regularity (d, k, m).

Regarding the family $\{\gamma_{d,k}\}$, we recall, for $j \in \mathbb{N}$, that Γ_{2j} is piecewise in Z_{2j+2} and Γ_{2j+1} is piecewise in Z_{2j+4}. From this it follows that γ_{2j} is piecewise in $Z_{2j+2-2(3j-1)} = Z_{-4j+4}$ and γ_{2j+1} is piecewise in $Z_{2j+4-2(3j+1)} = Z_{-4j+2}$, and so in either case, γ_k is piecewise in Z_{4-2k}. Following exactly the same line of reasoning as above, we conclude that $\gamma_{d,k,m} := \gamma_{d+2m,k+m} = \mathcal{D}^{(d-2)/2+m}\gamma_{k+m}$ has regularity (d, k, m), where d, k, m are as specified in the above corollary.

Wendland's family $\{\omega_{d,k}\}$ can be treated in a similar fashion. In brief, the first piece of ω_k belongs to Z_{3-2k}, and consequently the first piece of $\omega_{d,k} = \mathcal{D}^{(d-1)/2}\omega_k$ belongs to $Z_{d-2(k-1)}$. Applying Theorem 7.4 then yields the following.

Corollary 7.6. *Let $d \in 2\mathbb{N}_0 + 1$, $k \in \mathbb{N}$ and $m \in \mathbb{N}_0$, with $m \geq \frac{d+1}{2} - 2k$ and $k + m \geq 2$. Then*

$$\omega_{d,k,m} := \omega_{d+2m,k+m} = \mathcal{D}^{(d-1)/2+m}\omega_{k+m}$$

has regularity (d, k, m).

Remark 7.7. Although there does not appear to be any direct relationship between the parameter m in Definition 7.1 and the smoothness of ϕ on $(0, \infty)$, it is striking that there is such a relationship in the families $\{\eta_{d,k,m}\}$, $\{\gamma_{d,k,m}\}$, $\{\omega_{d,k,m}\}$ (and also $\{\psi_{d,k,m}\}$ appearing in the next section). For any function ϕ, in one of these families, having regularity (d, k, m), we also have $\phi \in C^s(0, \infty)$, where $s = 2k + m - 1 - d/2$ if d is even, and $s = 2k + m - 1 - (d + 1)/2$ if d is odd.

8. The regularity of $\mathcal{D}^j \psi_k$

For $k \in \mathbb{N}$, let ψ_k be the restriction to $(0, \infty)$ of the B-spline (see [3]) having knots $0, 0, \pm 1, \pm 2, \dots, \pm k$. It is shown in [2] that ψ_k has regularity $(1, k)$, which is regularity $(1, k, 0)$ in the language of the previous section. In this section, we first prove that ψ_k has regularity $(1, k, 1)$, and then we define and discuss the regularity of the families $\{\psi_{d,k}\}$ and $\{\psi_{d,k,m}\}$.

It is shown in [2] that $F_1\psi_k$ can be written in the form

$$(F_1\psi_k)(r) = \frac{d_k}{r^{2k+1}} \int_0^r (1 - \cos t)^k \, dt, \quad r > 0,$$

where $d_k > 0$ is a constant. Differentiating the above yields

$$(\mathcal{D}F_1\psi_k)(r) = \frac{d_k}{r^{2k+2}} \left(\frac{2k+1}{r} \int_0^r (1 - \cos t)^k \, dt - (1 - \cos r)^k \right). \tag{8.1}$$

Lemma 8.1. *For $k \in \mathbb{N}$, the following hold.*
(i) $\int_0^\pi (1 - \cos t)^k \, dt = \frac{1}{2}\frac{3}{4} \cdots \frac{2k-1}{2k} \pi 2^k$. *(ii)* $(2k + 1) \int_0^\pi (1 - \cos t)^k \, dt \ge \frac{3}{2}\pi 2^k$.

Proof. Item (i) holds for $k = 1$ since $\int_0^\pi (1 - \cos t) \, dt = \pi$. Proceeding by induction, assume that (i) holds for k and consider $k + 1$. Employing the identity $2 \sin^2 \frac{t}{2} = 1 - \cos t$, and making a change of variable yields $\int_0^\pi (1 - \cos t)^k \, dt = 2^k \int_0^\pi \sin^{2k} \frac{t}{2} \, dt = 2^{k+1} \int_0^{\pi/2} \sin^{2k} t \, dt$. Applying the well-known reduction formula for $\int \sin^n t \, dt$ (and noting that $\cos \frac{\pi}{2} = \sin 0 = 0$), we have

$$2^{k+2} \int_0^{\pi/2} \sin^{2k+2} t \, dt = 2^{k+2} \frac{2k+1}{2k+2} \int_0^{\pi/2} \sin^{2k} t \, dt = 2 \frac{2k+1}{2k+2} \left(\frac{1}{2}\frac{3}{4} \cdots \frac{2k-1}{2k} \pi 2^k \right),$$

which proves (i) for $k + 1$ and completes the induction. Now it follows from (i) that $(2k + 1) \int_0^\pi (1 - \cos t)^k \, dt = \frac{3}{2}\frac{4}{3} \cdots \frac{2k+1}{2k} \pi 2^k \ge \frac{3}{2}\pi 2^k$, hence (ii). □

Lemma 8.2. *Let $k \in \mathbb{N}$ and define*

$$G(r) = \frac{2k+1}{r} \int_0^r (1 - \cos t)^k \, dt - (1 - \cos r)^k.$$

Then $G(r)>0$ *for* $r\in(0,2\pi]$ *and* $2^{k-2}\le G(r)\le(2k+1)2^k$ *for* $r>2\pi$.

Proof. We first establish the inequality

$$\frac{t}{2}\sin t<1-\cos t,\qquad 0<t<2\pi. \tag{8.2}$$

That (8.2) holds for $\pi\le t<2\pi$ is clear since then $\frac{t}{2}\sin t\le 0<1-\cos t$; so assume $0<t<\pi$, and put $\theta=t/2$. Employing the well known inequality $\theta<\tan\theta$, we obtain $\frac{t}{2}\sin t=\theta\sin 2\theta<\tan\theta\sin 2\theta=2\sin^2\theta=1-\cos t$, which proves (8.2). We next prove that $G(r)>0$ for $0<r<2\pi$. Applying integration by parts and (8.2), we have $\int_0^r(1-\cos t)^k\,dt=r(1-\cos r)^k-k\int_0^r t\sin t(1-\cos t)^{k-1}\,dt>r(1-\cos r)^k-2k\int_0^r(1-\cos t)^k\,dt$, whence $G(r)>0$ readily follows. Note also that $G(2\pi)>0$, by inspection. Thus we have established $G(r)>0$ for $0<r\le 2\pi$. Now let $r>2\pi$, say $r=2\pi\ell+r'$ where $\ell\in\mathbb{N}$ and $r'\in(0,2\pi]$. That $G(r)\le(2k+1)2^k$ is a simple consequence of the inequality $0\le 1-\cos t\le 2$. Note that $(2k+1)\int_0^{2\pi\ell}(1-\cos t)^k\,dt=2\ell(2k+1)\int_0^{\pi}(1-\cos t)^k\,dt\ge 3\ell\pi 2^k$, by Lemma 8.1 (ii). Hence, $\frac{2k+1}{r}\int_0^r(1-\cos t)^k\,dt\ge\frac{1}{r}\left(3\ell\pi 2^k+(2k+1)\int_0^{r'}(1-\cos t)^k\,dt\right)$. Since $G(r')>0$, it follows that $(2k+1)\int_0^{r'}(1-\cos t)^k\,dt>r'(1-\cos r')^k=r'(1-\cos r)^k$, and writing $3\ell\pi 2^k=\ell\pi 2^k+2\pi\ell 2^k\ge\ell\pi 2^k+2\pi\ell(1-\cos r)^k$, we obtain

$$\frac{2k+1}{r}\int_0^r(1-\cos t)^k\,dt\ge\frac{1}{r}\left(\ell\pi 2^k+2\pi\ell(1-\cos r)^k+r'(1-\cos r)^k\right)=\frac{\ell\pi 2^k}{r}+(1-\cos r)^k.$$

Hence, $G(r)\ge\frac{\ell\pi 2^k}{r}=\frac{2\pi\ell}{2\pi\ell+r'}2^{k-1}\ge 2^{k-2}$. □

Theorem 8.3. *For* $k\in\mathbb{N}$, ψ_k *has regularity* $(1,k,1)$.

Proof. Since ψ_k has regularity $(1,k)$, it suffices to show that there exist constants $B_1\ge A_1>0$ such that

$$A_1(1+r^2)^{-(k+1)}\le(\mathcal{D}F_1\psi_k)(r)\le B_1(1+r^2)^{-(k+1)},\quad r>0. \tag{8.3}$$

Since ψ_k is positive on $(0,k)$ and 0 elsewhere, and with Theorem 7.3 in view, it follows that $(\mathcal{D}F_1\psi_k)(0^+)=(F_3\psi_k)(0^+)=(2\pi)^{-3/2}\int_{\mathbb{R}^3}\psi_k(\|x\|)\,dx>0$. Consequently, (8.3) follows from (8.1) and Lemma 8.2. □

For $k\in\mathbb{N}$, the function ψ_k is piecewise in Z_1 and belongs to $C^{2k-1}(0,\infty)$ (see [2]). It follows from Corollary 6.2 that for $j=1,2,\dots,2k$, $\mathcal{D}^j\psi_k$ is piecewise in Z_{1+2j} and has regularity $(1+2j,k,1)$. We can therefore define

$$\psi_{d,k}:=\mathcal{D}^{(d-1)/2}\psi_k,\ \text{ for } d\in\{1,3,5,\dots\} \text{ and } k\in\mathbb{N} \text{ with } k\ge(d-1)/4,$$

and conclude that $\psi_{d,k}$ is piecewise in Z_d and has regularity $(d,k,1)$. As with ω_k, the first piece of ψ_k belongs to Z_{3-2k} and consequently the first piece of $\psi_{d,k}$ belongs to $Z_{d-2(k-1)}$. Applying Theorem 7.4 then yields the following.

Corollary 8.4. *Let $d \in 2\mathbb{N}_0 + 1$, $k \in \mathbb{N}$ and $m \in \mathbb{N}_0$, with $m \geq \frac{d-1}{2} - 2k$. Then*

$$\psi_{d,k,m+1} := \psi_{d+2m,k+m} = \mathcal{D}^{(d-1)/2+m}\psi_{k+m}$$

has regularity $(d, k, m+1)$.

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