A NOTE ON THE LIMITED STABILITY OF SURFACE SPLINE INTERPOLATION

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Abstract. Given a finite subset $\Xi \subset \mathbb{R}^d$ and data $f_{|\Xi}$, the surface spline interpolant to the data $f_{|\Xi}$ is a function s which minimizes a certain seminorm subject to the interpolation conditions $s_{|\Xi} = f_{|\Xi}$. It is known that surface spline interpolation is stable on the Sobolev space W^m in the sense that $\|s\|_{L_\infty(\Omega)} \le const\, \|f\|_{W^m}$, where m is an integer parameter which specifies the surface spline. In this note we show that surface spline interpolation is not stable on W^γ whenever $\gamma < m - 1/2$.

1. Introduction

Let m, d be positive integers with $m > d/2$. The Beppo-Levi space H^m is defined to be the space of tempered distributions f for which $D^\alpha f \in L_2 := L_2(\mathbb{R}^d)$ for all $|\alpha| = m$, and the seminorm $|\cdot|_{H^m}$ is defined by

$$|f|_{H^m} := \left\| |\cdot|^m \widehat{f} \right\|_{L_2}, \quad f \in H^m,$$

where $\widehat{f}$ denotes the Fourier transform of f. Let Π_k denote the space of polynomials over $\mathbb{R}^d$ having total degree at most k, and let Ξ be an arbitrary nonempty subset of $\mathbb{R}^d$. Duchon [5] has shown that if Ξ is not contained in the zero-set of any nontrivial polynomial in Π_{m-1}, then for all $f \in H^m$ there exists a unique $s \in H^m$ which minimizes $|s|_{H^m}$ subject to the interpolation conditions

$$s(\xi) = f(\xi) \text{ for all } \xi \in \Xi. \tag{1.1}$$

The function s is called the *surface spline interpolant to f at Ξ*, and will be denoted $T_\Xi f$. In case Ξ is finite, Duchon has characterized $T_\Xi f$ as the unique function which satisfies (1.1) and has the form

$$T_\Xi f = q + \sum_{\xi \in \Xi} \lambda_\xi \phi(\cdot - \xi),$$

where

$$\phi(x) := \begin{cases} |x|^{2m-d} & \text{if } d \text{ is odd} \\ |x|^{2m-d} \log|x| & \text{if } d \text{ is even} \end{cases},$$

$q \in \Pi_{m-1}$, and λ satisfies the auxiliary conditions

$$\sum_{\xi \in \Xi} \lambda_\xi p(\xi) = 0 \text{ for all } p \in \Pi_{m-1}.$$

With the above formulation, the coefficients λ and the polynomial $q \in \Pi_{m-1}$ can be readily found by solving a system of linear equations, and this has made surface spline interpolation an attractive method for interpolating scattered data. The function $T_\Xi f$ is called a *radial basis function* because its essential part, $\sum_{\xi \in \Xi} \lambda_\xi \phi(\cdot - \xi)$, is a linear combination of translates of a single radially symmetric function. For a more general construction of radial basis function interpolants to scattered data, the reader is referred to the work of Light and Wayne [10].

Duchon has estimated the error in surface spline interpolation in terms of the *fill distance* from Ξ to Ω, given by

$$h := h(\Xi, \Omega) := \sup_{x \in \Omega} \inf_{\xi \in \Xi} |x - \xi|.$$

One formulation of Duchon's [6] error estimate is the following:

Theorem 1.2. *Let $\Omega \subset \mathbb{R}^d$ have the cone property. There exists $h_1 > 0$ (depending only on Ω and m) such that if Ξ satisfies $h := h(\Xi, \Omega) \le h_1$, then*

$$\|g\|_{L_p(\Omega)} \le const(\Omega, m) h^{m-d/2+d/p} |g|_{H^m}$$

for all $g \in H^m$ which vanish on Ξ and for all $2 \le p \le \infty$.

By choosing $g = f - T_\Xi f$ and noting that $|f - s|_{H^m} \le 2 |f|_{H^m}$ one immediately obtains the familiar error estimate:

$$\|f - T_\Xi f\|_{L_p(\Omega)} \le const(\Omega, m) h^{m-d/2+d/p} |f|_{H^m} .$$

We wish to draw the reader's attention to the fact that it follows from this error estimate that surface spline interpolation is stable on the Sobolev space $W^m(\mathbb{R}^d)$ in the sense that if $h \le h_1$, then

$$\|T_\Xi f\|_{L_\infty(\Omega)} \le const(\Omega, m) \|f\|_{W^m(\mathbb{R}^d)} ,$$

where $W^\gamma(\mathbb{R}^d)$ $(\gamma \ge 0)$ is defined to be the space of all $f \in L_2(\mathbb{R}^d)$ for which

$$\|f\|_{W^\gamma(\mathbb{R}^d)} := \left\| (1 + |\cdot|^2)^{\gamma/2} \widehat{f} \right\|_{L_2(\mathbb{R}^d)} < \infty.$$

To see this, we recall that by the Sobolev imbedding theorem [1, p.97] (as $m > d/2$),

$$\|f\|_{L_\infty(\mathbb{R}^d)} \le const(d, m) \|f\|_{W^m(\mathbb{R}^d)} , \quad f \in W^m(\mathbb{R}^d).$$

Hence,

$$\begin{aligned} \|T_\Xi f\|_{L_\infty(\Omega)} &\le \|T_\Xi f - f\|_{L_\infty(\Omega)} + \|f\|_{L_\infty(\mathbb{R}^d)} \\ &\le const(\Omega, m) h^{m-d/2} |f|_{H^m} + const(d, m) \|f\|_{W^m(\mathbb{R}^d)} \le const(\Omega, m) \|f\|_{W^m(\mathbb{R}^d)} , \end{aligned}$$

where we have used the imbedding $|f|_{H^m} \le \|f\|_{W^m(\mathbb{R}^d)}$ in the last inequality.

The purpose of this note is to show that surface spline interpolation is not stable on $W^\gamma(\mathbb{R}^d)$ whenever $d/2 < \gamma < m - 1/2$. Specifically we prove the following:

Theorem 1.3. *Let Ω be the closed unit ball $\{x \in \mathbb{R}^d : |x| \leq 1\}$, and assume $d/2 < \gamma < m - 1/2$. For every $h_0 > 0$ there exists $f \in W^\gamma(\mathbb{R}^d)$ and a sequence of finite pointsets $\Xi_n \subset \Omega$, with $h(\Xi_n, \Omega) \leq h_0$, such that $\|T_{\Xi_n} f\|_{L_1(\Omega)} \to \infty$ as $n \to \infty$.*

Note that this theorem leaves open the interesting possibility that surface spline interpolation remains stable on $W^\gamma(\mathbb{R}^d)$ when $\gamma \geq m - \frac{1}{2}$.

We mention that the question of whether univariate spline interpolation is stable on $C(\mathbb{R})$ has been addressed by de Boor [3, pp. 194–197]. In contrast to Theorem 1.3, Brownlee and Light [4] (see also [11] and [12]) show that if the interpolation points are quasi-uniformly scattered, then in addition to being stable on $W^\gamma(\mathbb{R}^d)$ $(\gamma \in \mathbb{Z} \cap (\frac{d}{2}, m-1])$ surface spline interpolation actually acheives the expected order of approximation.

2. Construction of a function in $W^\gamma(\mathbb{R}^d)$

Our first task is to construct the function f which will be used in the proof of Theorem 1.3. Let $\sigma \in C_c^\infty([-4,4])$ be such that $\sigma = 1$ on $[-3,3]$. Here $C_c^\infty(A)$ denotes the space of compactly supported C^∞ functions whose support is contained in A. For $\alpha > 0$ and $\beta \in \mathbb{R}$, we define $f_{\alpha,\beta} \in C(\mathbb{R})$ by

$$f_{\alpha,\beta}(x) := \sigma(x-\beta)\,|x-\beta|^\alpha\,.$$

Lemma 2.1. *If $\alpha > 0$ and $0 \leq \gamma < \alpha + 1/2$, then $f_{\alpha,\beta} \in W^\gamma(\mathbb{R})$ for all $\beta \in \mathbb{R}$.*

Proof. Assume $\alpha > 0$ and $0 \leq \gamma < \alpha + 1/2$. If α is an even integer, then $f_{\alpha,\beta} \in C_c^\infty(\mathbb{R}) \subset W^\gamma(\mathbb{R})$, so assume α is not an even integer. Since $f_{\alpha,\beta}$ is simply a translate of $f_{\alpha,0}$ and $W^\gamma(\mathbb{R})$ is translation invariant, we may assume without loss of generality that $\beta = 0$. Define $\psi \in C(\mathbb{R})$ by $\psi(x) := |x|^\alpha$ so that we can write $f_{\alpha,0} = \sigma\psi$. Note that since ψ has at most polynomial growth, ψ is a tempered distribution. It is known [7] that $\widehat{\psi}$ can be identified on $\mathbb{R}\backslash\{0\}$ with $c\,|\cdot|^{-\alpha-1}$ for some constant c. Writing

$$\widehat{\psi} = \sigma\widehat{\psi} + (1-\sigma)c\,|\cdot|^{-\alpha-1} =: \widehat{g_1} + \widehat{g_2},$$

we see that $\widehat{g_2}$ admits the estimate $|\widehat{g_2}(w)| \le const(\alpha,\sigma)(1+|w|)^{-\alpha-1}$ whence it readily follows that $g_2 \in W^\gamma(\mathbb{R})$; hence, $\sigma g_2 \in W^\gamma(\mathbb{R})$. As for g_1, we note that $g_1 \in C^\infty(\mathbb{R})$ since $\widehat{g_1}$ is compactly supported; hence, $\sigma g_1 \in C_c^\infty(\mathbb{R}) \subset W^\gamma(\mathbb{R})$. Therefore, $f_{\alpha,0} = \sigma g_1 + \sigma g_2 \in W^\gamma(\mathbb{R})$. □

Let $\nu \in C_c^\infty([1/4,4])$ satisfy $0 \le \nu \le 1$ and $\nu = 1$ on $[1/2,2]$. Let $\rho : \mathbb{R}^d \to [0,\infty)$ denote the modulus mapping

$$\rho(x) := |x|\,.$$

We define the linear operator $M : C_c(\mathbb{R}) \to C_c(\mathbb{R}^d)$ by $Mf := (\nu f)\circ\rho$; in other words

$$Mf(x) := \nu(|x|)f(|x|), \quad x \in \mathbb{R}^d.$$

Having defined M on $C_c(\mathbb{R})$, we note that by changing to spherical coordinates one sees that the $L_2(\mathbb{R}^d)$-norm of Mf is dominated by a constant multiple of the $L_2(\mathbb{R})$-norm of f:

$$\|Mf\|_{L_2}^2 = const(d)\int_0^\infty t^{d-1}\,|\nu(t)f(t)|^2\,dt \le const(d)\int_{1/4}^4 |f(t)|^2\,dt \le const(d)\,\|f\|_{L_2(\mathbb{R})}^2\,.$$

Since M is linear and $C_c(\mathbb{R})$ is dense in $L_2(\mathbb{R})$ it follows that M can be uniquely extended to a continuous linear operator from $L_2(\mathbb{R})$ into $L_2(\mathbb{R}^d)$. For brevity, let us denote this extension also by M.

Proposition 2.2. *For all $\gamma \ge 0$ and $f \in W^\gamma(\mathbb{R})$,*

$$\|Mf\|_{W^\gamma(\mathbb{R}^d)} \le const(d,\nu,\gamma)\,\|f\|_{W^\gamma(\mathbb{R})}\,.$$

Proof. Let us first consider the case $\gamma = 2n$ for an integer $n \geq 0$. We will employ the following equivalent norms for $\|\cdot\|_{W^{2n}(\mathbb{R})}$ and $\|\cdot\|_{W^{2n}(\mathbb{R}^d)}$, respectively:

$$\|g\|_{W^{2n}(\mathbb{R})} \sim \sum_{k=0}^{2n} \left\|g^{(k)}\right\|_{L_2(\mathbb{R})}, \quad g \in W^{2n}(\mathbb{R});$$
$$\|g\|_{W^{2n}(\mathbb{R}^d)} \sim \|g\|_{L_2(\mathbb{R}^d)} + \|\Delta^n g\|_{L_2(\mathbb{R}^d)}, \quad g \in W^{2n}(\mathbb{R}^d),$$

where $\Delta := \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_d^2}$ denotes the Laplacian operator. Since $C_c^\infty(\mathbb{R})$ is dense in $W^{2n}(\mathbb{R})$, it suffices to consider the case $f \in C_c^\infty(\mathbb{R})$. Put $g := \nu f$. It is shown in the proof of [8, Lem. 2] that there exist functions $p_0, p_1, \ldots, p_{2m-1} \in C^\infty(0, \infty)$ such that

$$\Delta^n(g \circ \rho) = g^{(2n)}(\rho) + p_{2n-1}(\rho) g^{(2n-1)}(\rho) + \cdots + p_0(\rho) g(\rho).$$

Hence,

$$\|\Delta^n Mf\|_{L_2(\mathbb{R}^d)}^2 = const(d) \int_0^\infty t^{d-1} \left| g^{(2n)}(t) + p_{2n-1}(t) g^{(2n-1)}(t) + \cdots + p_0(t) g(t) \right|^2 dt$$
$$\leq const(d,n) \int_{1/4}^4 \sum_{k=0}^{2n} \left| g^{(k)}(t) \right|^2 dt \leq const(d,n) \|g\|_{W^{2n}(\mathbb{R})}^2 \leq const(d,n,\nu) \|f\|_{W^{2n}(\mathbb{R})}^2.$$

Since we have already established

$$\|Mf\|_{L_2(\mathbb{R}^d)} \leq const(d) \|f\|_{L_2(\mathbb{R})}, \tag{2.3}$$

we obtain the desired estimate

$$\|Mf\|_{W^{2n}(\mathbb{R}^d)} \leq const(d,\nu,n) \|f\|_{W^{2n}(\mathbb{R})}. \tag{2.4}$$

We consider now the general case $\gamma \geq 0$. Let n be the smallest integer satisfying $\gamma \leq 2n$. Since $M : L_2(\mathbb{R}) \to L_2(\mathbb{R}^d)$ is a linear operator satisfying (2.3) and (2.4), we can interpolate between these inequalities (see [2, p. 301,302] and [13, p. 39,40]) to obtain

$$\|Mf\|_{W^\gamma(\mathbb{R}^d)} \leq const(d,\nu,\gamma) \|f\|_{W^\gamma(\mathbb{R})}. \quad \square$$

Combining this proposition with Lemma 2.1 yields

Corollary 2.5. *If $\alpha > 0$ and $0 \leq \gamma < \alpha + 1/2$, then $Mf_{\alpha,\beta} \in W^\gamma(\mathbb{R}^d)$ for all $\beta \in \mathbb{R}$.*

3. Proof of Theorem 1.3

Let us invoke the hypothesis of Theorem 1.3; namely that $d/2 < \gamma < m - 1/2$ and $h_0 > 0$. We will assume, without loss of generality, that $h_0 < \min\{1, h_1\}$, where h_1 is as described in Theorem 1.2. Let α be a non-integer satisfying $\gamma < \alpha + 1/2 < m - 1/2$. Put $\beta_0 := 1 - h_0/3$, and let us assume that $\beta \in (1 - h_0/2, \beta_0)$. Note that by Corollary 2.5,

$$f := Mf_{\alpha,\beta_0} \in W^\gamma(\mathbb{R}^d).$$

The construction of Mf_{α,β_0} ensures that f is C^∞ on the compliment of the sphere $S_{\beta_0} := \{x \in \mathbb{R}^d : |x| = \beta_0\}$ and that

$$f(x) = ||x| - \beta_0|^\alpha \text{ for } 1/2 < |x| < 2.$$

Let $B := \{x \in \mathbb{R}^d : |x| < 1\}$ denote the open unit ball. Although we cannot claim that f belongs to H^m, we note that $T_{\beta B} f$ is well-defined since there exists $g \in H^m$ which satisfies $g_{|\beta B} = f_{|\beta B}$ (eg. $g = \psi f \in C_c^\infty(\beta_0 B)$, with $\psi \in C_c^\infty(\beta_0 B)$ satisfying $\psi = 1$ on βB).

Lemma 3.1.

$$\|T_{\beta B} f\|_{L_1(B)} \to \infty \text{ as } \beta \uparrow \beta_0.$$

Proof. Since the seminorm $|\cdot|_{H^m}$ is rotationally invariant and f is radially symmetric, it follows that $T_{\beta B} f$ is radially symmetric. It is shown in [9, Th. 4.1] that there exists a polynomial $q \in \Pi_{m-1}$ and a distribution μ, with $supp\, \mu \subset \beta\overline{B}$ such that $T_{\beta B} f = q + \phi * \mu$. Since ϕ is C^∞ on $\mathbb{R}^d \backslash \{0\}$, it follows that $T_{\beta B} f$ is C^∞ on $\mathbb{R}^d \backslash \beta\overline{B}$. In [9, Lem. 5.9], it is shown that $\Delta^m(\phi * \mu) = c\mu$ for some constant c. In particular, $\Delta^m(T_{\beta B} f) = 0$ on $\mathbb{R}^d \backslash \beta\overline{B}$. To reiterate, we have shown that $T_{\beta B} f$ is radially symmetric, C^∞ on $\mathbb{R}^d \backslash \beta\overline{B}$, and satisfies

$\Delta^m(T_{\beta B}f) = 0$ on $\mathbb{R}^d \backslash \beta\overline{B}$. Consequently, we can invoke [8, Lem. 2] to conclude that $T_{\beta B}f$ can be identified on $\mathbb{R}^d \backslash \beta\overline{B}$ with an element v_f of the finite dimensional space

$$
\begin{aligned}
V := span\{1, \rho^2, \dots, \rho^{2m-2}\}& \\
+ \begin{cases} span\{\rho^{2-d}, \rho^{4-d}, \dots, \rho^{2m-d}\} & \text{if } d \text{ is odd,} \\ span\{\rho^{2-d}, \rho^{4-d}, \dots, \rho^{-2}, \log\rho, \rho^2\log\rho, \dots, \rho^{2m-d}\log\rho\} & \text{if } d \text{ is even.} \end{cases}&
\end{aligned}
$$

Since the functions in V are radially symmetric and C^∞ on $\mathbb{R}^d \backslash \{0\}$, for each $v \in V$, there exists a unique $\widetilde{v} \in C^\infty(0, \infty)$ such that $v = \widetilde{v} \circ \rho$. Similarly, we can write $T_{\beta B}f = \widetilde{f} \circ \rho$ for some $\widetilde{f} \in C([0, \infty))$. Note that $\widetilde{f} = (\beta_0 - \cdot)^\alpha$ on $[1/2, \beta]$ and $\widetilde{f} = \widetilde{v_f}$ on (β, ∞). Since $\widetilde{f} \circ \rho \in H^m$, it follows from [1, Th. 7.55] that

$$\widetilde{v_f}^{(j)}(\beta) = \frac{d^j}{dt^j}(\beta_0 - t)^\alpha_{\big|t=\beta} \quad \text{for } j = 0, 1, \dots, m-1.$$

In particular, $\widetilde{v_f}^{(m-1)}(\beta) = \alpha(\alpha-1)\cdots(\alpha-m+2)(\beta_0-\beta)^{\alpha-m+1}$, and since α is not an integer and $\alpha - m + 1 < 0$, it follows that

$$\left|\widetilde{v_f}^{(m-1)}(\beta)\right| \to \infty \text{ as } \beta \uparrow \beta_0. \tag{3.2}$$

For $t \in [1/2, 1]$, let λ_t denote the continuous linear functional on V defined by

$$\langle v, \lambda_t \rangle := \widetilde{v}^{(m-1)}(t),$$

and note that the family $\{\lambda_t\}_{1/2 \le t \le 1}$ is equicontinuous; hence, there exists a constant c such that

$$|\langle v, \lambda_t \rangle| \le c \, \|v\|_{L_1(B \backslash \beta_0 \overline{B})} \text{ for all } 1/2 \le t \le 1, v \in V,$$

where we have used the fact that $\|\cdot\|_{L_1(B \backslash \beta_0 \overline{B})}$ is a norm on V. It thus follows from (3.2) that $\|v_f\|_{L_1(B \backslash \beta_0 \overline{B})} \to \infty$ as $\beta \uparrow \beta_0$. Since $T_{\beta B}f = v_f$ on $B \backslash \beta_0 \overline{B}$, we obtain the desired conclusion that $\|T_{\beta B}f\|_{L_1(B)} \to \infty$ as $\beta \uparrow \beta_0$. □

Proof of Theorem 1.3. It suffices to show that for each $N > 0$ (large), there exists a finite subset $\Xi \subset B$, with $h(\Xi, B) \le h_0$, such that $\|T_\Xi f\|_{L_1(B)} \ge N$.

Let $N > 0$. By Lemma 3.1, there exists $\beta \in (1 - h_0/2, \beta_0)$ such that $\|T_{\beta B} f\|_{L_1(B)} > N$. Let Ξ_n be an increasing sequence of finite subsets of βB which satisfy $h(\Xi_n, \beta B) \le h_0/2n$, $n \in \mathbb{N}$, and note that $h(\Xi_n, B) \le h_0$ for all n. We recall that there exists $g \in H^m$ such that $g_{|\beta B} = f_{|\beta B}$, and hence $T_{\beta B} f = T_{\beta B} g$ and $T_{\Xi_n} f = T_{\Xi_n} g$. Duchon [6] has shown that $|T_{\beta B} f - T_{\Xi_n} f|_{H^m} \to 0$ as $n \to \infty$ (see also [9, Th. 1.5]). We invoke Duchon's inequality, Theorem 1.2, to write

$$\|T_{\beta B} f - T_{\Xi_n} f\|_{L_\infty(B)} \le const(d,m) h_0^{m-d/2} \, |T_{\beta B} f - T_{\Xi_n} f|_{H^m} \to 0 \text{ as } n \to \infty.$$

It follows that $\|T_{\Xi_n} f\|_{L_1(B)} \to \|T_{\beta B} f\|_{L_1(B)}$ and hence $\|T_{\Xi_n} f\|_{L_1(B)} > N$ for sufficiently large n. □

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