Wavelets and Their Associated Operators

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Abstract. This article is devoted to the study of wavelets based on the *theory of shift-invariant spaces*. It consists of two, essentially disjoint, parts. In the first part, the fiberization of the analysis operator of a shift-invariant system is discussed. That fiberization applies to wavelet systems via the notion of *quasi-wavelet* systems, and leads to the theory of *wavelet frames*. Highlights in this theory are the *unitary and mixed extension principles*, and the MRA construction of *framelets*. The second part of the article is devoted to the study of the cascade/transfer operators and the subdivision operator associated with a refinable function. The analysis there is primarily based on the interpretation of the cascade operator as a special quasi-interpolation scheme. This leads to a surprisingly simple analysis of certain properties of refinable functions, including their *smoothness* and the *convergence of the cascade and subdivision algorithms*. In particular, it follows that these latter algorithms, if handled properly, always converge.

1. Preface: Wavelets and Their Associated Operators

This article advocates the analysis of wavelet systems via the study of their associated operators. The goal is neither to survey the current state-of-the-art in this area, nor to provide the reader with in-depth comprehensive analysis of any of the issues addressed. Rather, my attempt is to provide a glimpse into various contemporary aspects of wavelets, in a way that may whet the reader's appetite for further reading. Based on this philosophy, I have chosen setups that simplify the discussion even in cases when the simplification is purely notational.

The notion of 'the operators associated with a wavelet system' is so broad that it allows me to discuss two essentially disjoint topics. The first topic concerns the intrinsic operators of the wavelet system: analysis and synthesis, with the main aim being to review the recent developments in the area of wavelet frames (cf. [66], [67], [68], [69], [70], [71], [72], [34], [22], [30]). The second topic is the analysis of the corresponding refinable/scaling function(s), a topic that is also pertinent to the area of *uniform subdivision algorithms* (cf. [56], [33], [15], [32]). The relevant operators in this discussion are the

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subdivision and the cascade. This second part is written as a short monograph, aiming to single out the few underlying principles, and to demonstrate the elegance and the simplicity of the resulting theory.

The *analysis and synthesis operators* are the two basic operators related to any setup where one represents functions in some function space with the aid of a basis or, more generally, a countable subset (referred to hereafter as a 'system') from that space. The study of the structure of the synthesis and analysis operators of a wavelet system is, probably, the most basic one. We show how the *analysis* operator of a wavelet system can be *fiberized*, *i.e.*, represented and thereby understood and analyzed with the aid of a collection of much simpler operators ('fibers'). It leads to a complete characterization of *wavelet frames* in terms of a certain collection of infinite-order non-negative definite matrices. The theory is based on an interplay between the wavelet system and a new (special) type of shift-invariant system: the quasi-wavelet system. As a result, simple *extension* techniques for constructing a wavelet system from a given MRA (multiresolution analysis, cf. [54], [55], [27], and [48]) become possible: the highlight of this part is that the refinable function need not satisfy any particular property, *i.e.*, its shifts need neither form a Riesz basis nor form a frame. The wavelet frames constructed by these principles are termed *framelets* and the example of B-spline framelets is discussed.

Another pair of operators associated with a wavelet system are the *cascade/transfer operator and the subdivision operator*. In contrast with the general synthesis/analysis operators, these operators are not associated directly with the wavelet system but rather with the underlying refinable function (known also as the scaling function, which may be either a scalar function or a vector-valued function). Thus, one assumes that the wavelets are constructed via MRA and wishes to understand first the properties of the generator(s) of the MRA. Of relevance here are the *stability and linear independence* of the shifts of the scaling function, its *smoothness*, and the *convergence of the corresponding subdivision and cascade algorithms*. The goal in this part is to demonstrate the relative simplicity of the theory that is obtained by using this approach: in fact, a single identity that involves the cascade operator, when combined with the existing knowledge concerning *quasi-interpolation schemes* provides a unified approach for the study of all these aspects!

I forgo discussing and treating the approximation orders of the scaling functions and wavelets. There are general treatments of approximation orders of shift-invariant spaces (cf. [4], [7]), and applying these theories to the case of a single scaling function is quite straightforward, hence does not require a special exposition. The technique for dealing with the vector case is primarily based on superfunction theory (cf. [5], [7], and [60]), a technique that does not invoke, at least not in an explicit way, any operator-based approach. The approximation orders of framelets are also derived from the general theory (cf. [30]), and while they do rely on the structure of the synthesis operator of a shift-invariant system, they seem to be a bit beyond the scope of this article. The relations between the approximation orders of the scaling function and its smoothness are discussed in [64].

2. The Analysis and Synthesis Operators

2.1. The Analysis and Synthesis Operators: General

Let X be a countable set in a Hilbert space $\mathcal{H}$. We can use X either in order to decompose or to reconstruct other elements in $\mathcal{H}$. Here, *reconstruction* means that we assemble functions from discrete data with the relevant operator, then the synthesis operator:

$$T_X : \ell_2(X) \to \mathcal{H} : c \mapsto \sum_{x \in X} c(x)x.$$

When this operator is well-defined and bounded we say that the system X is a Bessel system. The complementary use of X is for *decomposition*, *i.e.*, using X as a collection of linear functionals. The corresponding operator is then the analysis operator T_X^* which is the adjoint of T_X:

$$T_X^* : \mathcal{H} \to \ell_2(X) : f \mapsto T_X^* f := (\langle f, x \rangle)_{x \in X}.$$

Being the adjoint of T_X, the analysis operator is well-defined and bounded if and only if X is a Bessel system. There are situations when we restrict T_X^* to a closed subspace $H \subset \mathcal{H}$, and there is no need to assume then that $X \subset H$.

For most examples of interest, the Bessel property of the system X is easily verified. For example, if $\Phi \subset L_2(\mathbb{R}^d)$ is a finite set of functions, and if we let $E(\Phi)$ be the collection of shifts of Φ:

$$E(\Phi) := (E^\alpha \phi : \phi \in \Phi,\ \alpha \in \mathbb{Z}^d), \quad E^\alpha : f \mapsto f(\cdot - \alpha), \tag{2.1.1}$$

then $E(\Phi)$ is always Bessel, provided, say, that the bracket product

$$[\widehat{\phi}, \widehat{\phi}] := \sum_{j \in 2\pi\mathbb{Z}^d} E^j(|\widehat{\phi}|^2)$$

is continuous for every $\phi \in \Phi$. That continuity is implied by a mild decay condition on ϕ at ∞: since the Fourier coefficients of $[\widehat{\phi}, \widehat{\phi}]$ are given by the inner products $(\langle E^\alpha \phi, \phi \rangle)_{\alpha \in \mathbb{Z}^d}$, it suffices, for example, to require that these coefficients lie in $\ell_1(\mathbb{Z}^d)$.

However, one almost always would like to boundedly invert the analysis and synthesis operators, and these additional requirements turn out to be highly non-trivial. For example, even if we require the sequence Φ in the previous example to belong to the space of compactly supported C^∞ test functions, we cannot conclude that any of the two operators of interest is boundedly invertible.

Definition 2.1.2. *Let X be a Bessel system in Hilbert space $\mathcal{H}$, and let H be a closed subspace of $\mathcal{H}$. We say that:*

1) *X is a* stable system *in $\mathcal{H}$ (or that X forms a* Riesz basis *in $\mathcal{H}$) whenever the synthesis operator T_X is boundedly invertible;*

2) *X is a* frame *for H if the restriction of T_X^* to H is boundedly invertible.*

The frame bounds are the numbers $\|T_X^*\|^2$ (upper frame bound, may be referred to as the 'Bessel bound' if X is merely a Bessel system) and $\|T_X^{*\,-1}\|^{-2}$ (lower frame bound).

It is not hard to see that, given a stable basis X in $\mathcal{H}$ and a closed subspace $H \subset \mathcal{H}$, X is a frame for H if (and only if) T_X^* is injective on H.

Some of the basics concerning stable bases and frames are collected in the following proposition:

Proposition 2.1.3. *Let X be a Bessel system in $\mathcal{H}$. Then:*

1) *X is a stable basis in $\mathcal{H}$ if and only if there exists a map $\mathrm{R} : X \to \mathcal{H}$ such that $\mathrm{R}X$ is a* dual basis*: it is a Bessel system, and $T_{\mathrm{R}X}^* T_X = \mathrm{id}$, i.e.,*

$$\langle x', \mathrm{R}x \rangle = \delta_{x,x'}, \quad \forall x, x' \in X.$$

2) *X is a frame for $H \subset \mathcal{H}$ if and only if there exists a map $\mathrm{R} : X \to \mathcal{H}$ such that $\mathrm{R}X$ is a* dual system*: it is a Bessel system and $T_{\mathrm{R}X} T_X^* = \mathrm{id}$ on H, i.e.,*

$$\sum_{x \in X} \langle f, \mathrm{R}x \rangle\, x = f, \quad \forall f \in H.$$

As said, the Bessel property of a system X is usually easy to obtain and analyze. In contrast, the frame and Riesz basis properties are by far more demanding, and are also more challenging for mathematical analysis. There are two different possible approaches here: (a) an intrinsic analysis of the system X, and (b) an analysis of a pair $(X, \mathrm{R}X)$. The above proposition indicates that the second approach may be simpler: an intrinsic analysis of X requires one to check whether a certain operator is bounded below, while an analysis of the pair $(X, \mathrm{R}X)$ requires one to know whether a certain operator is the identity. On the other hand, the second approach requires one to augment first the given system X by a suitable system $\mathrm{R}X$, *i.e.*, a system which is a 'good candidate' for being dual to X, something that may not be simple at all. It is then important to emphasize the case a dual system is given for free:

Definition 2.1.4. *Let X be a system in a Hilbert space H. We say that X is a* tight frame *for H if the analysis operator T_X^* is unitary, or equivalently, if the condition*

$$T_X T_X^* = \mathrm{id}$$

holds (in H). □

The advantage of tight frames over other frames is obvious: the same system X may be used for reconstruction and for decomposition. That may be attractive for two different reasons: first, it eliminates the need to *find* a dual system. Second, even in the case when a dual system is easy to find, its properties may not as good as those of the original X. For example, if H is the subspace of $L_2(\mathbb{R})$ consisting of cardinal splines of order 2 (*i.e.*, continuous piecewise-linear functions with integer breakpoints), then, with B the hat function, $E(B)$ is a stable basis for H. However, the dual basis in H for $E(B)$ lacks the compact support.

In fact, our first example of a tight frame involves piecewise-linear functions as well.

Example 2.1.5. Let Φ be the set of the two piecewise-linear functions depicted in Figure 1 (the support of each is $[-1,1]$, and the max-norms are 1 for the function on the left and $\frac{\sqrt{2}}{2}$ for the function on the right.) Let

$$X := \cup_{k\in\mathbb{Z}} \mathcal{D}^k E(2^{k/2}\Phi),$$

with $\mathcal{D}$ the dyadic dilation operator:

$$\mathcal{D} : f \mapsto f(2\cdot).$$

Then X is a tight (wavelet) frame for $L_2(\mathbb{R})$. $\square$

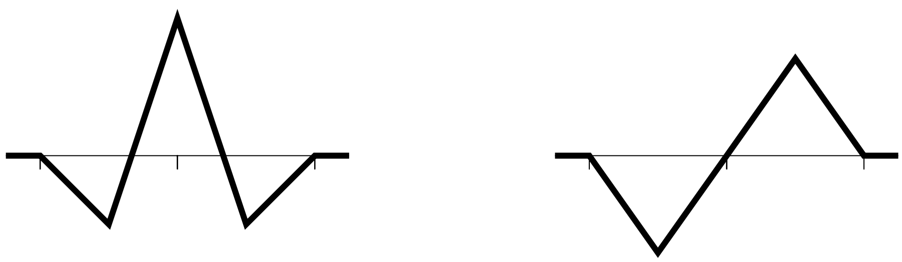

Figure 1. The generators of the piecewise-linear tight frame.

I do not believe that it is straightforward to verify that the system in the above example is indeed a tight frame. It is my intent in the rest of this section to briefly review the theory that leads to this construction as well as to many other more involved ones: the theory of wavelet frames. That theory, which is detailed in [68], [69], is based on the *fiberization of shift-invariant systems*, [5], [66].

Fiberization. The idea behind fiberization is to analyze a complicated operator S with the aid of a collection $(S_\omega)_{\omega\in\Omega}$ of operators of simpler structure (each of which is a 'fiber'). For the fiberization to be useful, the fiber operators need, at least, help in determining whether S is bounded and/or invertible, and need also be of help for computing or estimating the norms of S and S^{-1}.

Example 2.1.6. The fiberization of the analysis and synthesis operators of a shift-invariant system, [66]. Let Φ be a countable subset of $L_2(\mathbb{R}^d)$, and let X be the collection $E(\Phi)$ of all the shifts of Φ. For (almost) every $\omega \in \mathbb{R}^d$, let J_ω be the pre-Gramian of X: the matrix whose rows are indexed by $2\pi\mathbb{Z}^d$, whose columns are indexed by Φ, and whose $(\alpha, \phi) \in 2\pi\mathbb{Z}^d \times \Phi$ entry is

$$J_\omega(\alpha, \phi) := \widehat{\phi}(\omega + \alpha).$$

The pre-Gramian J_ω is considered as a map from $\ell_2(2\pi\mathbb{Z}^d)$ into $\ell_2(\Phi)$.

The collection $(J_\omega)_{\omega\in\mathbb{R}^d}$ fiberize the synthesis operator T_X of X. The reference [66] contains detailed information as to the exact meaning and the possible value of such fiberization. In particular, we have that, with $X := E(\Phi)$, and $c \in \ell_2(X)$,

$$\widehat{T_X c}_{|_{\omega+2\pi\mathbb{Z}^d}} = J_\omega \widehat{c}(\omega),$$

where

$$\widehat{c}(\omega) := (\widehat{c_\phi}(\omega))_{\phi\in\Phi}, \quad \widehat{c_\phi}(\omega) := \sum_{j\in\mathbb{Z}^d} c(E^j\phi)e^{-ij\cdot\omega}.$$

If follows, [66], that

$$\|T_X\|^2 = \| \, \|J_\omega\| \, \|_{L_\infty(\mathbb{R}^d)}, \tag{2.1.7}$$

$$\|{T_X}^{-1}\|^2 = \| \, \|{J_\omega}^{-1}\| \, \|_{L_\infty(\mathbb{R}^d)}. \tag{2.1.8}$$

In a similar manner, the Gramian matrices $(J_\omega^* J_\omega)_\omega$ fiberize the self-adjoint operator $T_X^* T_X$. Note that $J_\omega^* J_\omega$ is an operator from $\ell_2(\Phi)$ into itself, and its $\phi \times \varphi \in \Phi \times \Phi$ entry is the bracket product $[\widehat{\varphi}, \widehat{\phi}](\omega)$:

$$[\widehat{\varphi}, \widehat{\phi}] := \sum_{j\in 2\pi\mathbb{Z}^d} \widehat{\varphi}(\cdot + j)\overline{\widehat{\phi}(\cdot + j)}.$$

Consequently, we can study the Bessel property and the Riesz basis property of a shift-invariant $X = E(\Phi)$ via the above Gramian fibers.

Fiberization of the analysis operator of $X = E(\Phi)$ is also possible, but is significantly more complicated (than that of the synthesis operator) unless we assume that X is fundamental in $L_2(\mathbb{R}^d)$ (*i.e.*, that the finite span of X is dense in that space). With the fundamentality assumption in hand, however, we get results as simple as in the synthesis case. Specifically, the matrices $(J_\omega^*)_\omega$ provide now fiberization for T_X^*, the dual Gramian matrices $(J_\omega J_\omega^*)_\omega$ fiberize $T_X T_X^*$, and results similar to (2.1.7) and (2.1.8) hold. Note that each dual Gramian fiber is a non-negative operator from $\ell_2(2\pi\mathbb{Z}^d)$ into itself, and its (α, β)-entry ($(\alpha, \beta) \in 2\pi\mathbb{Z}^d \times 2\pi\mathbb{Z}^d$) is

$$\sum_{\phi\in\Phi} \widehat{\phi}(\omega + \alpha)\overline{\widehat{\phi}}(\omega + \beta).$$

Fiberizations of the analysis operator are useful in the study of the Bessel property and the frame property of the original X. □

Here is an example that demonstrates the usefulness of the fiberization approach. Suppose that we would like to determine whether a system $X = E(\Phi)$ is a tight frame for $L_2(\mathbb{R}^d)$. X is a tight frame iff $T_X T_X^* = \mathrm{id}$ iff $J_\omega J_\omega^* = \mathrm{id}$ for almost every $\omega \in \mathbb{R}^d$. After suppressing some obvious repetitions (*e.g.*, different fibers that represent essentially the same operator), we obtain that X is a tight frame for $L_2(\mathbb{R}^d)$ if and only if, for every $j \in 2\pi\mathbb{Z}^d$,

$$\sum_{\phi\in\Phi} \widehat{\phi} \; \overline{E^j\widehat{\phi}} = \delta_j, \quad \text{a.e.} \tag{2.1.9}$$

Some readers may be able to find a simple direct proof for this result. Indeed, the tool of fiberization in the study of tight frames (as well as in the study of orthonormal bases and system-dual system setups) is not so essential: the fiber matrices in these cases are identity matrices, hence entrywise characterizations analogous to (2.1.9) are available. It is then plausible to claim that such characterizations can be obtained directly without assembling first the fiber matrices. Fiberization, however, is a powerful tool whenever the fiber matrices do not have an especially simple structure. □

Here is a striking example for the utility of the above fiberization. It is the simplest example of the so-called *duality principle of Weyl-Heisenberg systems*, [67].

Example 2.1.10. Self-adjoint Weyl-Heisenberg systems. Given $g \in L_2(\mathbb{R}^d)$, let

$$X := (e_{ij}E^k g : \ (j,k) \in 2\pi\mathbb{Z}^d \times \mathbb{Z}^d), \quad e_{ij} : \omega \mapsto e^{ij\cdot\omega}.$$

We have then that $X = E(\Phi)$, with $\Phi := (e_{ij}g)_{j\in 2\pi\mathbb{Z}^d}$. Indexing Φ by $2\pi\mathbb{Z}^d$, we obtain that the pre-Gramian fiber J_ω has the entries

$$J_\omega(k,j) = \widehat{g}(\omega + k + j).$$

This means that the pre-Gramians are self-adjoint (up to conjugation), hence that the condition that characterizes stability in terms of $(J_\omega)_\omega$ is identical to the condition that characterizes frames for L_2 in terms of $(J^*_\omega)_\omega$. This recovers the fact (cf. *e.g.*, [26], [1]) that the above X is a stable basis if and only if it is a fundamental frame. □

Wavelets. We want to focus now on the main theme of the discussion: wavelet systems. In order to simplify notation, we will mostly assume that we employ dyadic dilations (the general theory allows arbitrary dilations, for as long as the entries of the dilation matrix are integers, and the spectrum of the dilation matrix lies outside the closed unit disc). To recall, given a finite $\Psi \subset L_2(\mathbb{R}^d)$ of mother wavelets, the wavelet system $X(\Psi)$ generated by Ψ is the collection of all dyadic dilations of the shift-invariant $E(\Psi)$:

$$X(\Psi) := \cup_{k\in\mathbb{Z}} \mathcal{D}^k E(2^{kd/2}\Psi), \quad \mathcal{D} : f \mapsto f(2\cdot). \tag{2.1.11}$$

One observes that a wavelet system is not shift-invariant: the k-scale $\mathcal{D}^k E(\Psi)$ of $X(\Psi)$ is invariant under $2^{-k}\mathbb{Z}^d$-shifts, and these shifts become indefinitely sparse as $k \to -\infty$.

The attempt to apply the shift-invariant fiberization techniques to the almost shift-invariant wavelet system led in [68] to the introduction of a link between wavelet systems and shift-invariant systems in the form of quasi-wavelet systems.

Definition 2.1.12. Quasi-wavelet systems. *Given a collection of mother wavelets Ψ, the quasi-wavelet system generated by Ψ is*

$$X^q(\Psi) := \cup_{k=0}^{\infty} \mathcal{D}^k E(2^{kd/2}\Psi) \bigcup \cup_{k=-\infty}^{-1} E(2^{kd}\mathcal{D}^k\Psi).$$

As one observes, the quasi-wavelet system is obtained from the wavelet system by oversampling the negative scales of the latter. For instance, in the (-1)-scale, the even shifts of the function $2^{-d/2}\psi(\frac{\cdot}{2})$ are replaced by the integer shifts of the re-normalized function $2^{-d}\psi(\frac{\cdot}{2})$.

Theorem 2.1.13. [68], [69]. *Let $X := X(\Psi)$ be a wavelet system, X^q its quasi-wavelet system counterpart.*

1) *X is a Bessel system iff X^q is a Bessel system. The two systems have the same Bessel bound.*
2) *X is a frame for $L_2(\mathbb{R}^d)$ if and only if X^q is a frame for that space. The two systems have the same frame bounds.*
3) *Suppose X, X^q are frames for $L_2(\mathbb{R}^d)$, and let $\mathrm{R} : \Psi \to L_2(\mathbb{R}^d)$ be some map. Then $Y := X(\mathrm{R}\Psi)$ is a frame dual to X if and only if Y^q is a frame dual to X^q.*

We remark that a smoothness assumption (a mild one: it is satisfied, *e.g.*, by the univariate and multivariate Haar functions) is imposed on Ψ in [68], [69]. In [22] it is shown that the first two statements in the above result hold even without that assumption.

Since the quasi-wavelet system X^q is shift-invariant, it admits a fiberization. Thanks to the above theorem, the so-obtained fibers can be used to characterize the Bessel property and the frame property of the *original* wavelet system X.

The fiberization of wavelet systems. In order to describe the dual Gramian fibers of $X := X(\Psi)$ (more precisely: the dual Gramian fibers of the shift-invariant X^q), we introduce first the affine product:

$$\Psi[\omega, \omega'] := \sum_{k=\kappa(\omega-\omega')}^{\infty} \sum_{\psi\in\Psi} \widehat{\psi}(2^k\omega)\overline{\widehat{\psi}(2^k\omega')}, \tag{2.1.14}$$

where

$$\kappa(\omega) := \inf\{k \in \mathbb{Z} : 2^k\omega \in 2\pi\mathbb{Z}^d\}. \tag{2.1.15}$$

(Thus, for example, $\kappa = \infty$ off the 2π-dyadic numbers, $\kappa \le 0$ on $2\pi\mathbb{Z}^d$, and $\kappa(\omega) = -\infty$ iff $\omega = 0$.) Then, the $(\alpha, \beta) \in 2\pi\mathbb{Z}^d \times 2\pi\mathbb{Z}^d$-entry of the dual Gramian fiber $J_\omega J_\omega^*$ of X^q is [68],

$$\Psi[\omega + \alpha, \omega + \beta].$$

Thus we have the following fundamental result:

Theorem 2.1.16. [68]. *Let $X(\Psi)$ be a wavelet system. For each $\omega \in \mathbb{R}^d$, let S_ω be the operator from $\ell_2(2\pi\mathbb{Z}^d)$ to $\ell_2(2\pi\mathbb{Z}^d)$ defined by*

$$(S_\omega c)(\alpha) = \sum_{\beta \in 2\pi\mathbb{Z}^d} \Psi[\omega+\alpha, \omega+\beta]c(\beta).$$

Then:

(a) *$X(\Psi)$ is a Bessel system if and only if the function $\omega \mapsto \|S_\omega\|$ is essentially bounded. Furthermore, $\|T_X^*\|^2 = \| \|S_\omega\| \|_{L_\infty(\mathbb{R}^d)}$.*

(b) *Assume that $X(\Psi)$ is Bessel. Then $X(\Psi)$ is a frame for $L_2(\mathbb{R}^d)$ if and only if the map $\omega \mapsto \|S_\omega^{-1}\|$ is essentially bounded. Also, the lower frame bound is then $1/\| \|S_\omega^{-1}\| \|_{L_\infty(\mathbb{R}^d)}$.*

(c) *$X(\Psi)$ is a tight frame for $L_2(\mathbb{R}^d)$ if and only if almost all the fibers S_ω are the identity operators.*

We conclude this section with a variety of examples.

Example 2.1.17. Tight frames. In view of the above result, $X(\Psi)$ is a tight frame if and only if $\Psi[\omega+\alpha, \omega+\beta] = \delta_{\alpha,\beta}$ for every $\alpha, \beta \in 2\pi\mathbb{Z}^d$, and almost every $\omega \in \mathbb{R}^d$. Replacing ω by $\omega+\alpha$, we may assume $\alpha = 0$. Since the affine product is also dilation-invariant (*i.e.*, $\Psi[2\omega, 2\omega'] = \Psi[\omega, \omega']$), we may assume that, unless $\beta = 0$, $\kappa(\beta) = 0$, *i.e.*, that $\beta \in 2\pi(\mathbb{Z}^d \backslash 2\mathbb{Z}^d)$. We then obtain that $X(\Psi)$ is a tight frame if and only if, for a.e. ω,

$$\Psi[\omega,\omega] = 1, \quad \Psi[\omega, \omega+\beta] = 0, \quad \forall \beta \in 2\pi(\mathbb{Z}^d \backslash 2\mathbb{Z}^d).$$

This result was established independently by others (cf. [38] where the univariate dyadic case of the above is proved, and [35] where the general case is established.) □

Example 2.1.18. Univariate band-limited diagonal wavelets. Assume that $\operatorname{supp} \widehat{\psi} \subset [-2\pi, 2\pi] \backslash (-\pi, \pi)$, for every $\psi \in \Psi$. Then one easily confirms that $\Psi[\omega+\alpha, \omega+\beta] = 0$ (for $\alpha, \beta \in 2\pi\mathbb{Z}^d$) unless $\alpha = \beta$. This means that the fiber matrices S_ω are all diagonal with diagonal entries

$$\Psi[\omega, \omega] = \sum_{\psi \in \Psi, k \in \mathbb{Z}} |\widehat{\psi}(2^k \omega)|^2. \tag{2.1.19}$$

It follows that $X(\Psi)$ is a frame if and only if the function $P : \omega \mapsto \Psi[\omega, \omega]$ is essentially bounded together with its reciprocal. This recovers a well-known result.

Moreover, given an arbitrary wavelet system $X(\Psi)$, since the values of the above P comprise the diagonal entries of the fibers matrices, and since each fiber matrix is non-negative definite, the boundedness of P and $1/P$ is always a *necessary* condition for X to be a frame (since the norm of any non-negative operator is bounded below by the largest element on its diagonal, while the norm of its inverse is bounded below by the reciprocal of the smallest element on that diagonal). Again, this recovers a known result (cf. [26], [27], [19]). In fact, almost all the results that provide estimates on the frame bounds of a wavelet frame can be in retrospect understood as attempts to estimate norms and inverse norms of non-negative definite matrices in terms of their entries. □

Example 2.1.20. Univariate band-limited block diagonal systems with 2×2 blocks.
We assume that, for each $\psi\in\Psi$, $\operatorname{supp}\widehat{\psi}\subset[-8\pi/3,8\pi/3]\backslash(-2\pi/3,2\pi/3)$, and examine the affine product $\Psi[\omega+\alpha,\omega+\beta]$, $\beta\neq\alpha$. A direct computation shows that if $|\omega+\alpha|<2\pi/3$ all the summands in the affine product vanish; otherwise, due to symmetry considerations and to the invariance properties of $\Psi[\cdot,\cdot]$, we may assume that $2\pi/3\le\omega<4\pi/3$, that $\alpha=0$, and that $\beta\in 2\pi\mathbb{Z}$. This implies that $\widehat{\psi}(2^k\omega)=0$, unless $k=0,1$ and that, for $\beta\in 2\pi\mathbb{Z}\backslash 0$ and $k\in\{0,1\}$, $\widehat{\psi}(2^k(\omega+\beta))\neq 0$ only if $\beta=-2\pi$ (the 'magic' in this example is the fact that it is the same β for the two different values of k). As a result, it follows that S_ω is block diagonal, with all the blocks being (at most) 2×2, and with each block of the form

$$\begin{pmatrix} |a|^2+|b|^2 & \langle a,c\rangle+\langle b,d\rangle \\ \langle c,a\rangle+\langle d,b\rangle & |c|^2+|d|^2 \end{pmatrix}, \tag{2.1.21}$$

with $a:=(\widehat{\psi}(\omega))_{\psi\in\Psi}$, $b:=(\widehat{\psi}(2\omega))_{\psi\in\Psi}$, $c:=(\widehat{\psi}(\omega-2\pi))_{\psi\in\Psi}$, $d:=(\widehat{\psi}(2(\omega-2\pi)))_{\psi\in\Psi}$, and with $|\cdot|$ being the ℓ_2-norm, and $\omega\in[2\pi/3,4\pi/3)$. Thus, such a wavelet system is a frame for $L_2(\mathbb{R}^d)$ if and only if each matrix of the form (2.1.21) is invertible, and, in addition, the norms as well as inverse norms of these matrices are bounded independently of ω. Of particular interest is the case when Ψ is a singleton $\{\psi\}$. In this case, each matrix in (2.1.21) is of the form BB^*, with B the *square* matrix

$$B=\begin{pmatrix} \widehat{\psi}(\omega) & \widehat{\psi}(2\omega) \\ \widehat{\psi}(\omega-2\pi) & \widehat{\psi}(2\omega-4\pi) \end{pmatrix}.$$

Obviously, the result can now be stated directly in terms of the norms and inverse norms of the matrices of the form B. This case was studied in [75], where it was observed that the frame property is equivalent here to the (seemingly stronger) Riesz basis property. (From the point of view of the current discussion, that can be attributed to the fact that each B is square, hence the norms and inverse norms of BB^* are identical to those of B^*B). □

Example 2.1.22. Oversampling. In order to explore this case we need also to consider systems that are shift-invariant with respect to a superlattice of $\mathbb{Z}^d$. For simplicity, we consider only lattices that are scalar scales of an integer lattice (see [68] for the general case). Thus, we let

$$E_n(\Phi):=(E^\alpha\phi:\phi\in\Phi,\ \alpha\in\mathbb{Z}^d/n),$$

with n some integer, and define the wavelet system $X_n(\Psi)$ similarly to (2.1.11), with $E(\Psi)$ there replaced by $E_n(\Psi)$ here. Following [18], we consider the new system as an oversampling of $X(\Psi)$. The study of relations between properties of $X(\Psi)$ and properties of $X_n(\Psi)$ was originated in the work of Chui and Shi (cf. [18], [20], [21]).

In order to analyze the oversampling via the fiberization theory, we need to know the fibers of the oversampling system. These fibers are simple (and straightforward) variants of the fibers in the integer case: this time, each fiber $S_{n,\omega}$ is indexed by $2\pi n\mathbb{Z}^d \times 2\pi n\mathbb{Z}^d$, and its (α,β)-entry is

$$n^d \Psi_n[\omega+\alpha, \omega+\beta],$$

with the only difference between $\Psi_n[,]$ and $\Psi[,]$ in the definition of the valuation κ: here κ is replaced by

$$\kappa_n(\omega) := \inf\{k \in \mathbb{Z} : 2^k\omega \in 2\pi n\mathbb{Z}^d\}.$$

One then observes that $\kappa_n = \kappa$ on $2\pi n\mathbb{Z}^d$ whenever n is odd. This means that for an odd n, $n^{-d}S_{n,\omega}$ is a submatrix of S_ω. Since S_ω is non-negative definite, it follows that

$$\|S_{n,\omega}\| \le n^d \|S_\omega\|.$$

Hence, the fiberization theory yields that

$$\|T^*_{X_n(\Psi)}\| \le n^{d/2} \|T^*_{X(\Psi)}\|,$$

and a similar argument provides an analogous estimate on the other frame bound. It follows then that $X_n(\Psi)$ is tight if $X(\Psi)$ is.

The particular oversampling result above is due to Chui and Shi (and was proved originally by different means). We note that the situation is different if the oversampling rate is even (cf. [18] and [68]). □

2.2. Wavelets: Extension Principles, Framelets, B-Spline Framelets

The fiberization of wavelet systems, when combined with the vehicle of *multiresolution analysis* (MRA), leads to new ways for constructing wavelet frames.

To recall, a function $\phi \in L_2(\mathbb{R}^d)$ is (dyadically) refinable if there exists a 2π-periodic mask function τ_0 such that

$$\widehat{\phi}(2\cdot) = \tau_0 \widehat{\phi}. \qquad (2.2.1)$$

We assume that $\widehat{\phi}(0) = 1$, but do not impose any other standard assumption; in particular, the shifts of ϕ need not form a Riesz basis. For notational convenience, we set $\psi_0 := \phi$.

Let V_0 be the PSI space generated by ϕ, *i.e.*, the smallest closed subspace of $L_2(\mathbb{R}^d)$ that contains $E(\phi)$. The refinability assumption (2.2.1) is equivalent to the condition that $V_1 := \mathcal{D}V_0$ is a superspace of V_0 (cf. [6]). We then attempt to construct a wavelet frame that is generated by finitely many mother wavelets from V_1:

$$\Psi := (\psi_1, \ldots, \psi_n) \subset V_1.$$

This means that each ψ_i, $i = 1, \ldots, n$ satisfies a relation of the form

$$\widehat{\psi_i}(2\cdot) = \tau_i \widehat{\psi_i},$$

for some 2π-periodic wavelet mask τ_i. Since the wavelet system $X(\Psi)$ is completely determined by choice of the wavelet masks, we attempt to construct the wavelet frame by an appropriate selection of the wavelet masks. As one encounters in the Riesz basis case (cf. [55], [48]), the construction is based on *matrix extensions*. However, in stark contrast with Riesz bases constructions, no *a priori* assumption need be imposed on the refinable ϕ.

We start with the discussion of tight wavelet frames. [68] contains a complete characterization of all tight wavelet frames that can be constructed by using the above MRA approach. That characterization easily leads to the following *unitary extension principle*.

Theorem 2.2.2. The unitary extension principle [68]. *Let $X(\Psi)$ be a wavelet system constructed by the above MRA recipe, and assume further that all the masks involved are bounded. Then $X(\Psi)$ is a tight frame for $L_2(\mathbb{R}^d)$ if the following condition holds: for almost every $\omega \in \mathbb{R}^d$, and for every $\gamma \in \{0,\pi\}^d$,*

$$\sum_{i=0}^{n} \tau_i(\omega)\overline{\tau_i(\omega+\gamma)} = \begin{cases} 1, & \gamma = 0, \\ 0, & \text{otherwise.} \end{cases}$$

Discussion. One may interpret the condition in Theorem 2.2.2 as a matrix extension. Let v_i be the row vector $v_i = (\tau_i(\cdot+\gamma))_{\gamma\in\{0,\pi\}^d}$, and let V be the matrix whose rows are $v_0,\ldots,v_n$. The refinable ϕ determines the vector v_0, and selecting the wavelets $(\psi_1,\ldots,\psi_n)$ is tantamount to selecting the additional rows $v_1,\ldots,v_n$. The above extension principle asserts that $X(\Psi)$ is a tight frame once the columns of V are orthonormal for almost every ω. Since the number of columns is fixed (viz. 2^d), then by selecting in an appropriate manner a large number of wavelets, it is plausible that we can find a suitable unitary extension *without imposing any further condition on the refinement mask* τ_0.

B-spline framelets. We refer to a wavelet frame that is constructed from MRA by a matrix extension principle (either the above unitary one, or the mixed extension principle detailed in the sequel) as framelet. The simplest construction of framelets are those derived from the B-spline MRA. For notational convenience, we discuss here the construction of B-spline framelets of even order; the odd case is treated similarly.

The mask of a centered B-spline of order n, n even, is

$$\tau_0(\omega) = \cos^n(\omega/2).$$

Thus, τ_0^2 is the first term in the binomial expansion of

$$1 = (\cos^2(\omega/2) + \sin^2(\omega/2))^n. \tag{2.2.3}$$

Let τ_j, $j = 1,2,\ldots,n$, be the squareroots of the other terms in this expansion, *i.e.*,

$$\tau_j(\omega) = \sqrt{\binom{n}{j}} \cos^j(\omega/2)\sin^{n-j}(\omega/2).$$

Setting $\tau := (\tau_j)_{j=0}^n$, (2.2.3) ensures that $\tau(\omega)$ is a unit vector for every ω. The only other required condition is that, for every ω, $\tau(\omega)$ be orthogonal to $\tau(\omega+\pi)$: since $\tau_j(\omega)\tau_j(\omega+\pi) = (\cos(\omega/2)\sin(\omega/2))^n\binom{n}{j}(-)^j$, that additional requirement follows from the fact that $(1-1)^n = 0$.

One can now verify that the mother wavelet set in Figure 1 corresponds to the case $n = 2$. In Figure 2 we show the case $n = 4$. □

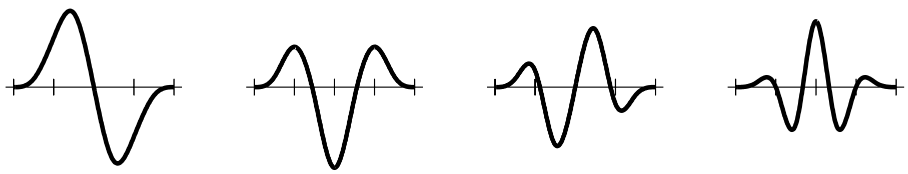

Figure 2. The generators of the cubic B-spline tight framelet.

We refer the reader to [70], [34], and [30] for further construction methods that are based on the unitary extension principle. It is worth mentioning that in more than one variable, dyadically refinable functions may not be the prime candidates for framelet constructions. The refinement mask on the one hand has a relatively large spectrum, and the relevant extension matrix has, on the other hand, 2^d columns, forcing one to use many mother wavelets in the construction. Thus, dilation matrices with small determinants may be preferred. For example, the Powell-Zwart element ϕ (which is a bivariate C^1 piecewise-quadratic spline supported in an octagon that lies in $[0,3]^2$, cf. [9]) is refinable with respect to the dilation matrix

$$\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}.$$

The mask has only four terms (and is identical to the dyadic mask of the support function of $[0,1]^2$), and there are only two columns in the extension matrix. In [70], two different wavelet systems are derived from that function: one with three mother wavelets and the other with two mother wavelets. It is worth mentioning that the shifts of this ϕ do not form a Riesz basis, and this explains the essential lack of prior constructions based on this function (the only exception I know is the 4-direction frames that were constructed in [17] by oversampling. However, the dual systems of those frames do not have compact support). □

An added flexibility to the construction of framelets is obtained when one uses MRA to construct bi-frames *i.e.*, a wavelet frame together with a dual wavelet frame. The setup and development is similar to the 'tight case.' One starts with *two* refinable functions ϕ and φ with refinement masks τ_0 and t_0, and attempts to extend each τ_0 and t_0 to an n-column vector (τ and t, respectively). Two wavelet systems are then constructed. The first is $X(\psi_1, \ldots, \psi_n)$ with

$$\widehat{\psi}_i(2\cdot) := \tau_i\widehat{\phi},\ i = 1, \ldots, n,$$

and the second is $X(\mathrm{R}\psi_1, \ldots, \mathrm{R}\psi_n)$, where

$$\widehat{\mathrm{R}\psi_i}(2\cdot) := \widehat{t_i}\widehat{\varphi}.$$

The mixed extension principle: Constructing bi-framelets. Suppose that we construct two wavelet systems $X(\Psi)$ and $X(\mathrm{R}\Psi)$ via the extensions of τ_0 and t_0 as above. Assuming that both $X(\Psi)$ and $X(\mathrm{R}\Psi)$ are Bessel systems (and imposing a mild smoothness condition on ϕ and φ), it is proved in [69] that $(X(\Psi), X(\mathrm{R}\Psi))$ forms a bi-framelet (*i.e.*, a framelet together with a dual framelet) if the following bi-orthogonality relation holds: for every $\gamma \in \{0, \pi\}^d$

$$\sum_{i=0}^{n} \tau_i \, \overline{t_i(\cdot + \gamma)} = \begin{cases} 1, & \gamma = 0, \\ 0, & \text{otherwise.} \end{cases}$$

See [69], [71], and [30] for more details as well as specific examples of bi-framelets. □

Quasi-wavelet systems may not be a mere theoretical tool. These systems were introduced in [68] for the sole purpose of the eventual fiberization of the wavelet operators. However, experiments that were done (independently) with 'translation-invariant' wavelet systems [14] revealed superior results compared to the standard wavelet systems. Since those latter systems are (essentially) dilations of quasi-wavelet systems, there might be intrinsic promise in quasi-wavelet systems. At present, we lack a theory that explains the results of [14]. □

Framelets and their extension principles possess great potential in several areas of applications: feature detection, noise removal, image compression, and possibly for solving PDEs. All these applications require the implementation of the system with the aid of a *fast transform*. Most of these applications require the system to have good *approximation order*. These aspects of framelets are dealt with in [30].

It is worthwhile to note that a different type of wavelet fiberization technique appears in [75], [76]. The results there appear helpful in analyzing oversampling systems, when the oversampling ratio is even.

3. The Cascade/Transfer and the Subdivision Operators

This part is devoted to the study of refinable functions via the exploitation of two relevant operators: the cascade and the subdivision (with the transfer operator being the 'Hermitian form' of the cascade operator, hence suitable for efficient treatment of L_2-problems; cf. Section 3.4 for the precise meanings of that). There is already a very rich literature devoted to this approach, and it is beyond the scope of this section to review that literature to any extent. Instead, my goal here is to highlight a specific approach: a treatment that is based on the existing tools from and the acquired knowledge in the theory of shift-invariant spaces: first and foremost, quasi-interpolation basics (cf. [77], [10], [2], [3], [11], [46], [9]). I do not claim any of the results in this part to be novel, although assume that some of them *are* new. The proofs given (whenever given) are not borrowed from elsewhere, but, again, similar arguments might already exist somewhere in the literature.

As stated before, we prefer to carry out the analysis under simplifying assumptions:

1) *The dilation is dyadic.* The extensions of the results to other dilations are almost entirely notational. If a general dilation matrix s is employed, one should use dilation by s on the original domain, and by s^* on the Fourier domain. The group , $= \{0, 2\pi\}^d = 2\pi(\mathbb{Z}^d/2\mathbb{Z}^d)$ should be replaced by $2\pi(\mathbb{Z}^d/s^*\mathbb{Z}^d)$, and the dyadic lattices

$$\mathcal{Z}_k := \mathbb{Z}^d/2^k \tag{3.0.1}$$

should be replaced by the s-adic lattices $s^{-k}\mathbb{Z}^d$. The only place I am aware of where the results do not carry over to general dilations is in the context of *smoothness*: there, one needs to assume that the dilation is isotropic, *i.e.*, that all the eigenvalues of s have the same modulus. Without this assumption, one can only get upper and lower bounds on the smoothness (cf. [16]).

2) *The refinable ϕ is scalar-valued* (rather than vector-valued). This assumption simplifies substantially the *notations*; however, it should be stressed that many of the arguments here can be easily carried over to that setup (specifically, the treatment of convergence and the treatment of smoothness).

3) *The refinable ϕ is compactly supported.* Some of the results are valid without this assumption; however, one then loses a major component of the analysis, viz. that the underlying spaces of interest are *finite dimensional*. So, this assumption should be regarded as essential.

Unless explicitly stated, we do *not* assume that the mask τ (cf. (3.1.1)) is a trigonometric polynomial. With rare exceptions, such an assumption leads neither to improved results nor to simpler arguments. It is worth noting that, since we assume ϕ to have compact support, the mask τ is necessarily a rational trigonometric polynomial (cf. [63]). Throughout the entire analysis we do assume (without further mentioning) that τ is *bounded*, and that $\widehat{\phi}(0) = \tau(0) = 1$.

3.1. The Transfer Operator: Stability and Related Properties

Let ϕ be a compactly supported refinable distribution, *i.e.*,

$$\widehat{\phi}(2\cdot) = \tau\widehat{\phi}, \tag{3.1.1}$$

for some 2π-periodic τ. Set

$$m := |\tau|^2.$$

The (Fourier transform version of the) transfer operator $\mathcal{T} := \mathcal{T}_m$ is defined as

$$\mathcal{T} : f \mapsto \sum_{\gamma\in\Gamma} E^{\gamma}\mathcal{D}^{-1}(mf),$$

with

$$, := \{0, 2\pi\}^d, \quad E^t : f \mapsto f(\cdot - t), \quad \mathcal{D}^{-1} f = f(\cdot/2).$$

Thus, in one variable, for instance,

$$\mathcal{T} f = (mf)(\frac{\cdot}{2}) + (mf)(\frac{\cdot}{2} + \pi).$$

Before we begin the discussion, we remind the reader the following basic consequence of Poisson's summation formula:

Lemma 3.1.2. *Let f be a function in the Wiener algebra $A(\mathbb{R}^d)$, i.e., $\widehat{f} \in L_1(\mathbb{R}^d)$. Then the 2π-periodization $\sum_{j \in 2\pi \mathbb{Z}^d} E^j \widehat{f}$ of $\widehat{f}$ lies in the span of $\{e_{i\alpha} : \ f(\alpha) \neq 0\}$. Here,*

$$e_\theta : \omega \mapsto e^{\theta \cdot \omega}. \tag{3.1.3}$$

Proof: The Fourier coefficients of the 2π-periodization of $\widehat{f}$ are, up to a multiplicative constant, the values of f at the integers. □

The most useful property of the transfer operator in wavelet analysis is the following lemma. It is somewhat awkward to state its most general case, hence I put instead two separate statements (that together suffice for the subsequent applications):

Lemma 3.1.4. *Let $\mathcal{T}$ be the transfer operator of a refinable compactly supported distribution ϕ.*

1) *Let ν be a compactly supported distribution, and assume that $\widehat{\nu}|\widehat{\phi}|^2 \in L_1(\mathbb{R}^d)$. Let $\Phi_\nu \in L_1(\mathbb{T}^d)$ be the 2π-periodization of $\widehat{\nu}|\widehat{\phi}|^2$*

$$\Phi_\nu := \sum_{j \in 2\pi \mathbb{Z}^d} E^j (\widehat{\nu} |\widehat{\phi}|^2). \tag{3.1.5}$$

   Then, for every k, $\mathcal{T}^k(\Phi_\nu)$ is the 2π-periodization of $\widehat{\nu}(\frac{\cdot}{2^k})|\widehat{\phi}|^2$.

2) *Assume $\phi \in L_2(\mathbb{R}^d)$. Let $t \in L_\infty(\mathbb{R}^d)$. Then, with F_t the 2π-periodization of $t|\widehat{\phi}|^2$, the function $\mathcal{T}^k(F_t)$ is the 2π-periodization of $t(\frac{\cdot}{2^k})|\widehat{\phi}|^2$.*
3) *Assume $\phi \in L_2(\mathbb{R}^d)$. Let Φ be the 2π-periodization of $|\widehat{\phi}|^2$. Then $(1, \Phi)$ is an eigenpair of $\mathcal{T}$.*

We omit the simple proof of the first part; the second part is merely another variant of the first (with the same proof). Of course, the third part is the special case of the first one corresponding to the choice $\nu := \delta$.

Here are some illustrations of the power of the above lemma:

Corollary 3.1.6. [73]. *Let $\mathcal{T}$ be the transfer operator of the compactly supported refinable distribution ϕ. Let $Z_\phi := \mathbb{Z}^d \cap \operatorname{supp} \varphi$, with φ the autocorrelation of ϕ, i.e., the compactly supported distribution whose Fourier transform is $|\widehat{\phi}|^2$. Set (cf. (3.1.3))*

$$H := \operatorname{span}\{e_{i\alpha} : \ \alpha \in Z_\phi\}. \tag{3.1.7}$$

Then, given a function of the form Φ_ν *(as defined in* (3.1.5)*) there exists* k_0 *(that depends only on* $\operatorname{diam}\operatorname{supp}\nu$*) such that* $\mathcal{T}^k(\Phi_\nu) \in H$ *for every* $k \ge k_0$*.*

Proof: By Lemma 3.1.4, $\mathcal{T}^k(\Phi_\nu)$ is the 2π-periodization of $\widehat{\nu}(\frac{\cdot}{2^k})|\widehat{\phi}|^2$. Note that the inverse transform of that latter function is (up to a constant) the convolution $\mathcal{D}^k\nu * \varphi$. By choosing a sufficiently large k, we can ensure that the support of $\mathcal{D}^k\nu$ lies in a sufficiently small neighborhood of the origin. The result then follows from Lemma 3.1.2. □

We will use in the next result, as well as in some subsequent results, the following definition:

Definition 3.1.8. The E-condition and the weak E-condition. *We say that a linear endomorphism* S *on a finite dimensional space satisfies the* weak E-condition *if the spectral radius of* S *is* 1*, and all its eigenvalues on the unit circle are non-defective. We further say that* S *satisfies the* E-condition *if, in addition,* 1 *is the unique eigenvalue on the unit circle and is simple.* □

Under the polynomiality assumption on the mask, the next result can be found in [52].

Theorem 3.1.9. *Let* $\mathcal{T}$ *be the transfer operator of a compactly supported refinable distribution* ϕ*. Let*

$$H_\phi \tag{3.1.10}$$

be the largest $\mathcal{T}$*-invariant subspace of* H*, and let*

$$\mathcal{T}_\phi$$

be the restriction of $\mathcal{T}$ *to* H_ϕ*. If* $\mathcal{T}_\phi$ *satisfies the weak E-condition then* $\phi \in L_2(\mathbb{R}^d)$*.*

Remarks. 1) Note that the result does not assume the mask to be a (trigonometric) polynomial. 2) The non-defectivity assumption on the dominant eigenvalues is necessary: The first derivative ϕ' of Daubechies' first scaling function ϕ [25], [27] is a suitable example ($\phi \notin W_2^1(\mathbb{R})$, while the spectral radius of $\mathcal{T}_{\phi'}$ is 1. The 'culprit' is the eigenvalue 1 which is defective). 3) We also note that the converse of this statement is false. [73] shows that the spectral radius of the transfer operator $\mathcal{T}_\phi$ of a compactly supported refinable $\phi \in L_2(\mathbb{R})$ can be as large as one wishes (the corresponding masks in the examples there are polynomial; in the case where the mask is polynomial and the dilation is dyadic, we have that $H = H_\phi$). □

Proof: Note that the weak E-condition guarantees that, given any $f \in H_\phi$, the sequence $(\mathcal{T}^k f)_k$ is bounded (in any norm, since H_ϕ is finite dimensional).

Let ν be a compactly supported function such that (i) $\widehat{\nu}|\widehat{\phi}|^2 \in L_1(\mathbb{R}^d)$, (ii) $\widehat{\nu}(0) = 1$, (iii) $\widehat{\nu} \ge 0$ everywhere, (iv) $\widehat{\nu} \in L_\infty$. Then, as $k \to \infty$,

$$\|\widehat{\nu}(\frac{\cdot}{2^k})|\widehat{\phi}|^2\|_{L_1(\mathbb{R}^d)} \to \|\widehat{\phi}\|^2_{L_2(\mathbb{R}^d)}.$$

However, Lemma 3.1.4 together with the non-negativity of $\widehat{\nu}$ implies that

$$\|\widehat{\nu}(\frac{\cdot}{2^k})|\widehat{\phi}|^2\|_{L_1(\mathbb{R}^d)} = \|\mathcal{T}^k(\Phi_\nu)\|_{L_1(\mathbb{T}^d)},$$

while Corollary 3.1.6 guarantees $\mathcal{T}^k(\Phi_\nu)$ to lie, for all sufficiently large k, in the domain H_ϕ of $\mathcal{T}_\phi$. Hence, by the weak E-condition, $(\mathcal{T}^k\Phi_\nu)_k$ is bounded in H_ϕ (say, in the $L_1(\mathbb{T}^d)$-norm). Thus $\|\widehat{\phi}\|_{L_2(\mathbb{R}^d)} < \infty$. □

The next result was proved first in [51], under the assumption that the mask is polynomial; see also [16]. We remind the reader that, for a compactly supported $\phi \in L_2(\mathbb{R}^d)$, the stability of $E(\phi)$ is characterized by the positivity *everywhere* of the function Φ from Lemma 3.1.4 (cf. [77], [24], [47], [48], [5]).

Theorem 3.1.11. [65]. *Let ϕ be a compactly supported refinable distribution. Then the following conditions are equivalent:*

(i) *$\phi \in L_2(\mathbb{R}^d)$ and the shifts $E(\phi)$ of ϕ are stable.*

(ii) *$\mathcal{T}_\phi$ satisfies the E-condition, and there is an eigenvector of the eigenvalue 1 which is positive everywhere.*

(iii) *$\mathcal{T}_\phi$ satisfies the weak E-condition, 1 is a simple eigenvalue of $\mathcal{T}_\phi$, and an eigenvector of it is positive everywhere.*

Proof: We prove that (i)$\Longrightarrow$(ii)$\Longrightarrow$(iii)$\Longrightarrow$(i).

The implication (ii)$\Longrightarrow$(iii) is trivial. Also, assuming (iii), we conclude from Theorem 3.1.9 that $\phi \in L_2(\mathbb{R}^d)$; hence (Lemma 3.1.4) that $(1, \Phi)$ is an eigenvector of $\mathcal{T}$. In view of the assumption in (iii), this implies that $c\Phi > 0$ for some constant c, which must be positive since $\Phi \geq 0$.

Now, assume (i) and let f be any trigonometric polynomial such that $f(0) = 0$. Since $\Phi > 0$ everywhere, we can write $f = t\Phi$, with $t \in C(\mathbb{T}^d)$ and $t(0) = 0$. By Lemma 3.1.4, $\mathcal{T}^k f$ is the 2π-periodization of $t(\frac{\cdot}{2^k})|\widehat{\phi}|^2$, the latter converges to 0 pointwise (since t is continuous at the origin and vanishes there). The dominated convergence theorem then implies that $\|t(\frac{\cdot}{2^k})|\widehat{\phi}|^2\|_{L_1(\mathbb{R}^d)} \to 0$, hence that $\|\mathcal{T}^k f\|_{L_1(\mathbb{T}^d)} \to 0$. Since the subspace of all trigonometric polynomials that vanish at 0 has co-dimension 1 in the space of all trigonometric polynomials, this implies that at most one eigenvalue of $\mathcal{T}_\phi$ lies outside the open unit disc, and that this eigenvalue, if it exists, must be algebraically simple. The E-condition then follows from the fact that, Lemma 3.1.4, $(1, \Phi)$ is an eigenpair of $\mathcal{T}_\phi$. The positivity of the eigenvector follows directly from the stability assumption. □

3.2. The Cascade and Subdivision Algorithms Always Converge

Let

$$\mathcal{Q}_k$$

be the space of all complex-valued sequences defined on $\mathcal{Z}_k := \mathbb{Z}^d/2^k$ (*i.e.*, $\mathcal{Q}_k = \mathbb{C}^{\mathcal{Z}_k}$). Also set

$$\mathcal{Q} := \mathcal{Q}_0.$$

Given a sequence λ defined (at least) on the lattice $\mathcal{Z}_k$, and a compactly supported distribution f, we define the k-scale semi-discrete convolution as

$$f *_k' \lambda := \sum_{j\in\mathbb{Z}^d} \lambda(2^{-k}j) f(2^k \cdot -j).$$

Note that $f *_k' \lambda$ is actually a linear combination of the $\mathcal{Z}_k$-shifts of $\mathcal{D}^k f$, with coefficients λ. Also,

$$f *' \lambda := f *_0' \lambda.$$

We have preferred in the previous section to carry out the analysis on the Fourier domain. Since the results of this section target L_p-norms for $p \neq 2$, we must switch to the time/space domain. Thus, the refinability assumption now reads as

$$\phi = \phi *_1' a, \tag{3.2.1}$$

with a a sequence defined on the half-integers $\mathcal{Z}_1$, still referred to the mask (and which essentially comprises the Fourier coefficients of the previous mask τ).

The transfer operator is a 'folded version' of another operator (the connection is made explicit in Section 3.4). Since the iterations of the latter operator form the *cascade algorithm* (see below), we have chosen to name the operator the *cascade operator*.

The cascade algorithm. The cascade algorithm aims at computing the refinable ϕ much in the same way the power method computes an eigenvector of a matrix. Starting with some initial compactly supported function f_0, it generates a sequence of functions $(f_k)_k$ by applying the fixed point iteration

$$f_k := C^k f_0 = C f_{k-1} := f_{k-1} *_1' a,$$

with a the mask of ϕ. The cascade operator is then the map

$$C : f \mapsto f *_1' a.$$

Given $1 \le p \le \infty$ and $\alpha > 0$, we say that the cascade algorithm converges in the p-norm on the function set F at a rate α if, for every $f \in F$,

$$\|C^k f - \phi\|_{L_p(\mathbb{R}^d)} \le \text{const}_f\, 2^{-\alpha k}. \quad \square$$

In order to analyze the cascade algorithm, I find it convenient to assume that the initial function f above is of the form $f := g *' u$, with u some finitely supported element of $\mathcal{Q}$ and g some fixed function. For example, g can be taken to be the support function of $[0,1]^d$, or a tensor product spline, a box spline, etc. The cascade algorithm is intimately tied to the issue of *quasi-interpolation from shift-invariant spaces* (cf. [11], [9]). It is not hard to see that

$$C^k(g *' u) = g *_k' \mathcal{C}^k u, \tag{3.2.2}$$

with

$$\mathcal{C} : u \mapsto a * \mathcal{D}u,$$

i.e., $\mathcal{C}u(j/2) = \sum_{n\in\mathbb{Z}^d} a(n/2)u(j-n)$ and $\mathcal{C}(\mathcal{Q}_k) \subset \mathcal{Q}_{k+1}$. Approximation theory basics then tell us that, in order for (3.2.2) to approximate ϕ at a rate α, three conditions should be satisfied:

(i) The shifts $E(g)$ of g should provide high approximation order. For *convergence* only, approximation order 1 (*i.e.*, partition of unity) suffices. If we are interested in an α-rate of convergence, we should choose g to provide approximation order $\geq \alpha$.

(ii) We can get approximation rate α in the p-norm only if $\phi \in W_p^\alpha(\mathbb{R}^d)$. We discuss the smoothness of refinable functions in the next subsection. (Warning: for general anisotropic dilations, the rate of convergence may only be a fraction of the smoothness parameter).

(iii) The coefficient sequence u_k in the approximation $\phi \approx g *_k' u_k$ should be selected according to a *quasi-interpolation rule*, [11], [9]. One selects a compactly supported ν such that $1 - \widehat{\nu}\widehat{g}$ has a zero at the origin of order $\geq \alpha$ and that ν is sufficiently smooth (so that $\mathcal{D}^k\nu * \phi \in A(\mathbb{R}^d)$), and defines u_k as the restriction to $\mathcal{Z}_k$ of $2^{kd}\mathcal{D}^k\nu * \phi$.

The first condition is entirely benign: after all, the selection of g is within our control. The second condition belongs to another topic, viz., the smoothness of refinable functions; it establishes the actual upper bound on any attempt to get fast convergence with the cascade algorithm. The interesting condition is the last one, and one may initially look at this requirement with utmost despair. After all, the sequence u_k is determined by the cascade algorithm as $\mathcal{C}^k u$, and we can only control the initial u. Fortunately, the counterpart of Lemma 3.1.4 changes despair into joy: the cascade iterations respect the rules of quasi-interpolation.

Lemma 3.2.3. *Let ϕ be a refinable distribution with mask a. Let ν be a compactly supported distribution such that $\nu * \phi \in A(\mathbb{R}^d)$, and set $u := (\nu * \phi)_{|\mathbb{Z}^d}$. Then $\mathcal{C}^k u$ is the restriction to $\mathcal{Z}_k$ of $2^{kd}\mathcal{D}^k\nu * \phi$.*

Thus we have the following result:

Theorem 3.2.4. [59]. The cascade algorithm always converges fast. *Let ϕ be a refinable function that lies in $W_p^\alpha(\mathbb{R}^d)$ for some $1 \leq p \leq \infty$ and some $\alpha > 0$. Let ν be some compactly supported distribution so that $\widehat{\nu}\widehat{\phi} \in L_1(\mathbb{R}^d)$, and set $u := (\nu * \phi)_{|\mathbb{Z}^d}$. Let g be a compactly supported bounded function whose shifts provide approximation order $n \geq \alpha$, and let β be the order of the zero $1 - \widehat{\nu}\widehat{g}$ has at the origin. Then, choosing the initial seed to be $g *' u$, the cascade algorithm converges in the p-norm to ϕ at a rate $\min\{\alpha, \beta\}$. Moreover, given any $m < \alpha$ and assuming $g \in W_\infty^m(\mathbb{R}^d)$, the cascade algorithm converges on $g *' u$ to ϕ in the $W_p^m(\mathbb{R}^d)$-norm at a rate $\min\{\alpha - m, \beta\}$.*

The usefulness of the above result depends on the ability to compute a good initial sequence u. If, for example, ϕ is continuous, we can choose ν to be supported on $\mathbb{Z}^d$, in a way that $1-\widehat{g}\widehat{\nu}$ has a zero of order n at the origin (and with n, say, any number $\geq \alpha$). Then, in order to implement the above theorem we need to find $u_0 := \phi_{|_{\mathbb{Z}^d}}$. Note the $(1, u_0)$ is an eigenpair of the operator

$$\mathcal{C}_0 : u \mapsto (\mathcal{C}u)_{|_{\mathbb{Z}^d}};$$

(this eigenpair is analogous to the eigenpair $(1, \Phi)$ in Lemma 3.1.4).

Discussion 3.2.5. The idea that the convergence of the cascade iterations can be accelerated by a 'smart' choice of the initial function is not entirely new. First, in [58], it is shown how, for a box spline ϕ, the convolution of $\mathcal{C}^k\delta$ with $(\mathcal{D}^k\phi)_{|_{\mathbb{Z}^d}}$ removes undesired artifacts from the surface obtained (the actual discussion there is in terms of the subdivision operator). Second, in [28], [29], it was shown that if an initial seed f for the cascade algorithm is chosen in a way that $f-\phi$ vanishes to order n at the integers, then the cascade algorithm converges to ϕ at a rate n (provided that f and ϕ are smooth, and that $E(f)$ provide approximation order n). That technique is more restrictive. As Theorem 3.2.4 asserts, given any g whose shifts provide high enough approximation order, one can accelerate the convergence of the cascade iterations by replacing g by a suitable element in the span of $E(g)$. In contrast, only for exceptional examples of (necessarily smooth) g, one can find in the span of $E(g)$ a function f that interpolates ϕ at the integers to a high order. □

We close the discussion here with the following results in which we use

$$K_\phi := \ker(\phi *') := \{\lambda \in \mathcal{Q} : \ \phi *' \lambda = 0\}. \tag{3.2.6}$$

Proposition 3.2.7. *Let ϕ be a compactly supported distribution (not necessarily refinable) and let $u \in \mathcal{Q}$ be finitely supported. Then the following conditions are equivalent:*

(i) $u * K_\phi = 0$.

(ii) *There exists a smooth compactly supported function ν that $(\nu * \phi)_{|_{\mathbb{Z}^d}} = u$.*

Theorem 3.2.8. *Let $1 \leq p \leq \infty$, and let $\phi \in W_p^\alpha(\mathbb{R}^d)$. Let g be a bounded compactly supported function for which $\widehat{\phi} - \widehat{g} = O(|\cdot|^n)$ near the origin. Let $u \in \mathcal{Q}$ be a finitely supported sequence such that $u * K_\phi = 0$ and $1 - \widehat{u} = O(|\cdot|^\alpha)$. If the shifts of g provide approximation order $\geq \alpha$, then the cascade algorithm converges at rate $\min\{\alpha, n\}$ on $g *' u$. Moreover, if $g \in W_\infty^m(\mathbb{R}^d)$ for some positive m, then $(C^k(g *' u))_k$ converges in $W_p^m(\mathbb{R}^d)$ to ϕ at a rate $\min\{\alpha - m, n\}$.*

Note that, if $K_\phi = 0$ (*e.g.*, if the shifts of ϕ are orthonormal, or form a Riesz basis with a compactly supported dual basis), then we may choose $u := \delta$ in the above theorem.

The subdivision algorithm. Given the mask a of a refinable function ϕ and a sequence $\lambda \in \mathcal{Q}$, the main aim of the subdivision algorithm is to produce fast a good approximation for $\phi *' \lambda$ without computing ϕ first.

The subdivision algorithm involves the iterations of the *subdivision operator*. The subdivision literature usually assumes that the subdivision operator is an endomorphism on $\mathcal{Q}$, with the only advantage in this description that the same operator S is employed in all the iterations. I find it more convenient to align the subdivision iterations with the cascade iterations, and to assume that after k iterations the sequence obtained lives on $\mathcal{Z}_k = \mathbb{Z}^d/2^k$. This entails that the kth order subdivision operator is not exactly the kth power of the 1st order one.

The kth order subdivision operator S_k. That operator maps $\mathcal{Q}$ into $\mathcal{Q}_k$ and is defined inductively as follows: S_0 is the identity, and

$$S_k\lambda := \mathcal{D}^{k-1}a * S_{k-1}\lambda, \tag{3.2.9}$$

i.e., $S_k\lambda(j) := \sum_{n\in\mathcal{Z}_{k-1}} a(2^{k-1}(j-n))\, S_{k-1}\lambda(n)$.

The standard definition for 'convergence of subdivision' [32] is also somewhat inconvenient for analysis. I prefer the following (equivalent, at least for $p=\infty$) definition, in which we use

$$G_\alpha$$

for the collection of all compactly supported functions g whose shifts provide approximation order $\geq \alpha$, and whose Fourier transform is α-flat at the origin: $1-\widehat{g}$ has a zero at the origin of order $\geq \alpha$. Examples of functions in G_α include functions whose shifts are orthonormal, and cardinal interpolants, but there are many others. In fact, given any compactly supported f whose shifts provide approximation order $\geq \alpha$, there exists $g \in G_\alpha$ which is finitely spanned by $E(f)$. Given $g \in G_\alpha$, it is well-known that, if $\phi \in W_p^\beta(\mathbb{R}^d)\cap C(\mathbb{R}^d)$ and if $\beta \leq \alpha$, then $\|\phi - g *'_k \phi\|_{L_p(\mathbb{R}^d)} \leq \text{const}\, 2^{-\beta k}$.

Definition 3.2.10. Convergence of subdivision. *Let ϕ be refinable with mask a, and let $1 \leq p \leq \infty$ and $\alpha > 0$. Given a subset Q of $\mathcal{Q}$, we say that* the subdivision converges on Q in the p-norm at α rate *if for any $g \in G_\alpha$, for every $q \in Q$, and for every compact set K,*

$$\|(\phi *' q) - (g *'_k S_k q)\|_{L_p(K)} \leq \text{const}_{q,K}\, 2^{-\alpha k}. \quad \square$$

Note that S_k commutes with (integer) shifts, and hence $S_k q = S_k\delta *' q$. Thus, the subdivision converges on the entire $\mathcal{Q}$ (for some fixed p and α) iff it converges on δ. As to the convergence of the subdivision on δ, we have the following simple observation:

Proposition 3.2.11. *Let ϕ be refinable with mask a. Then, for every k, $\mathcal{C}^k\delta = S_k\delta$, and hence the subdivision converges in the p-norm at α-rate on the entire $\mathcal{Q}$ if and only if the cascade converges at that norm and at that rate on δ (i.e., on G_α).*

The above proposition has one immediate consequence: since we can force the cascade algorithm to converge, and even at fast rates, we can do the same with the subdivision:

Theorem 3.2.12. [59]. The subdivision always converges, and converges fast. *Let ϕ be a refinable function that lies in $W_p^\alpha(\mathbb{R}^d)$ for some $1 \le p \le \infty$ and some $\alpha > 0$. Let ν be some compactly supported distribution so that $\widehat{\nu}\widehat{\phi} \in L_1(\mathbb{R}^d)$. In addition, let $u := (\nu * \phi)_{|_{\mathbb{Z}^d}}$ and β be the order of the zero that $1 - \widehat{\nu}$ has at the origin. Then, $(\mathcal{D}^k u * S_k \delta)_k$ converges in the p-norm at a rate* $\min\{\alpha, \beta\}$.

The theorem implies, in particular, that for $p = \infty$,

$$\|(\phi *' q)_{|_{\mathcal{Z}_k}} - \mathcal{D}^k u * S_k q\|_{\ell_\infty(\mathcal{Z}_k)} \le \text{const}\, 2^{-\alpha k},$$

provided that $\phi \in W_\infty^\alpha(\mathbb{R}^d)$, and that u is selected as in the theorem (and that $\beta \ge \alpha$). A highlight here is that the sequence u depends only on ϕ and not on the initial sequence q of the subdivision.

Note that if the subdivision algorithm converges on δ (for some p and α), it converges on each $\lambda \in \mathcal{Q}$ (in that norm and rate); in particular, it converges on each $\lambda \in K_\phi$ to $\phi *' \lambda = 0$. One of the main result of [59] provides a converse for this result:

Theorem 3.2.13. *Let ϕ be a compactly supported refinable function with finitely supported mask a. Let $1 \le p \le \infty$, and let $0 < \alpha \le 1$. Assume that $\phi \in W_p^\alpha(\mathbb{R}^d)$. Then the following conditions are equivalent:*

(a) *The cascade algorithm converges in the p-norm at an α-rate on any compactly supported initial seed f whose shifts provides approximation order* 1.

(b) *The subdivision algorithm converges in the p-norm at an α-rate on all sequences in $\mathcal{Q}$.*

(c) *The subdivision converges in the p-norm at an α-rate to zero on K_ϕ.*

The theorem leads to several important consequences. For instance, if $\phi \in W_\infty^\alpha(\mathbb{R})$ for some $\alpha > 0$, and if the subdivision converges (on all sequences) in some norm at some rate, it must converge in all other p-norms (albeit at possibly different rates). We refer to [59] and [13] for further discussions.

3.3. Smoothness of Refinable Functions

Lemma 3.1.4 and Lemma 3.2.3 are key tools in almost any analysis of refinable functions that is based on the transfer operator. Here is a very brief discussion of one of the major implications of the latter lemma: smoothness of refinable functions.

Definition 3.3.1. *Let ϕ be a compactly supported distribution. Given $1 \le p \le \infty$, we define the p-smoothness parameter of ϕ, $\alpha_p(\phi)$, as follows:*

$$\alpha_p(\phi) := \sup\{\alpha \in \mathbb{R} : \phi \in W_p^\alpha(\mathbb{R}^d)\}.$$

Note that $\alpha_p(\phi)$ may be negative.

Many articles in the literature exploit the cascade/transfer operator for the analysis of smoothness. The early, univariate, results were based on the idea of *factorization* (cf. *e.g.*, [39], [78]). Riemenschneider and Shen were probably the first to provide lower bounds on the $L_2(\mathbb{R}^d)$-smoothness without factorization, and Jia [43] used that approach to characterize the L_2-smoothness of a single scaling function ϕ in several variables, under the assumptions that $E(\phi)$ are stable, that the dilation is isotropic, and that the mask is a trigonometric polynomial. The treatment of L_2-smoothness for single $\phi \in L_2(\mathbb{R}^d)$ and for non-isotropic dilations (still under the stability and polynomiality assumptions) is contained in [16]: as said, one cannot compute exactly the smoothness parameter in that general setup. That Cohen et al. article stimulated [73] where the L_2-smoothness parameter was completely characterized (several variables, isotropic dilations, positive/negative smoothness, a vector ϕ, no stability and/or polynomiality assumptions). An algorithm (and software) that implements the results of [73] is contained in [74]. There are fewer treatments of L_p-smoothness, $p \neq 2$, and, in particular, I could find only one reference that characterizes the $L_p(\mathbb{R}^d)$-smoothness, $d > 1$, [53]. The reader is referred to [73], [49], and [45] for further references and discussions.

In this section, I merely wish to explain in the simplest possible terms the natural connection between the cascade iterations and the smoothness of the refinable ϕ. The treatment here is purely theoretical, and one must keep in mind, [78], [74], [80], that implementing results of this type in practical situations is non-trivial.

The connection between the cascade iterations and smoothness is clear once we compare the basic Lemma 3.2.3 with the definition of smoothness in terms of difference operators (cf. *e.g.*, [31]). Specifically, the following is one of the possible equivalent definitions of smoothness:

Proposition 3.3.2. *Let ϕ be a compactly supported distribution. In addition, let $1 \le p \le \infty$ and β be some positive number. Then the following conditions are equivalent:*

(a) *The smoothness exponent $\alpha_p(\phi)$ is $\ge \beta$.*

(b) *Let n be any integer $\ge \beta$. Then, for any sufficiently smooth compactly supported ν, if $\widehat{\nu}$ has a zero at the origin of order n, then, for every $\alpha < \beta$, $2^{kd}\|\mathcal{D}^k \nu * \phi\|_{L_p(\mathbb{R}^d)} = O(2^{-k\alpha})$.*

The smoothness problem can now be efficiently attacked by combining the above proposition with Lemma 3.2.3 and Proposition 3.2.7. As said, the result is also valid in the FSI (vector) case, as well as for general isotropic dilations (the technique handles also negative smoothness parameters). It is convenient here to normalize the ℓ_p-norm of sequences defined on $\mathcal{Z}_k$, so that the total mass of the points in the unit cube always equals 1:

$$\|c\|^p_{\ell_p(\mathcal{Z}_k)} := 2^{-kd} \sum_{j \in \mathcal{Z}_k} |c(j)|^p.$$

We recall [11], [12], [4], [41], [50] that if ϕ is compactly supported and its shifts provide approximation order n, then it satisfies the Strang-Fix conditions of order n, *i.e.*, $\widehat{\phi}$ has a zero of order n at each $j \in 2\pi\mathbb{Z}^d\backslash 0$. It follows then from Poisson's summation formula that, if ν is a smooth compactly supported function, and if $\widehat{\nu}$ has a zero of order n' at the origin, then, with $u := (\nu * \phi)_{|_{\mathbb{Z}^d}}$, $\widehat{u}$ has a zero at the origin of order $\geq \min\{n, n'\}$. Moreover, if n' is the exact order of the zero of $\widehat{\nu}$ at the origin, and if $n' \leq n$, then n' is also the exact order of the zero $\widehat{u}$ has at the origin (provided $\widehat{\phi}(0) \neq 0$).

Theorem 3.3.3. *Let ϕ be a compactly supported refinable distribution with mask a and cascade operator $\mathcal{C}$. Assume $\widehat{\phi}(0) = 1$. Let $1 \leq p \leq \infty$ be given, and let N be the approximation order provided by $E(\phi)$. Let K be a compact set that contains* $\operatorname{supp}\phi$, *and let $U \subset \mathcal{Q}$ be the space of all sequences u that satisfy the following conditions:*
(i) $\operatorname{supp} u \subset K$.
(ii) *$\widehat{u}$ has an N-fold zero at the origin.*
(iii) $u * K_\phi = 0$.
Let $\widetilde{\alpha}$ be the supremum of all α that satisfy the following condition: '$\|\mathcal{C}^k\| = O(2^{-k\alpha})$ as a map from U to $\ell_p(\mathcal{Z}_k)$'. Then $\widetilde{\alpha} = \alpha_p(\phi)$.

Proof: We prove first the lower bound inequality ($\alpha_p(\phi) \geq \widetilde{\alpha}$). Let n be any number larger than $\max\{N, \alpha_p(\phi)\}$. Also, let ν be a smooth compactly supported function of small support, whose Fourier transform has a zero of order n at the origin. By Proposition 3.2.7 and the discussion preceding the current theorem, the sequence $u_t := (E^t\nu * \phi)_{|_{\mathbb{Z}^d}}$ lies in U (provided that t lies in small neighborhood of the origin). Now, suppose that $\|\mathcal{C}^k\| = O(2^{-k\alpha})$ as a map from U to $\ell_p(\mathcal{Z}_k)$ (note that U is finite dimensional hence the choice of its norm is immaterial here). Since $\nu * \phi$ is continuous, the sequences $(u_t)_t$ lie in some bounded subset of U, hence $\|\mathcal{C}^k u_t\|_{\ell_p(\mathcal{Z}_k)} = O(2^{-k\alpha})$, uniformly in $t \in [0,1]^d$. But Lemma 3.2.3 asserts that $\mathcal{C}^k u_t = 2^{kd}((\mathcal{D}^k E^t \nu) * \phi)_{|_{\mathcal{Z}_k}}$, and thus $\mathcal{C}^k u_t$ is the restriction to $t/2^k + \mathcal{Z}_k$ of $2^{kd}\mathcal{D}^k\nu * \phi$. So we can integrate our estimate over $[0,1]^d/2^k$ to conclude that $2^{kd}\|\mathcal{D}^k\nu * \phi\|_{L_p(\mathbb{R}^d)} = O(2^{-k\alpha})$. Thus, by Proposition 3.3.2, $\alpha_p(\phi) \geq \alpha$.

For the converse, assume that $\phi \in W_p^\alpha(\mathbb{R}^d)$ for some $\alpha > 0$. Since 'smoothness implies approximation orders' (cf. [15], [64], [44]), we know that $\alpha < N$. We need to show that $\alpha \leq \widetilde{\alpha}$, too. Since U is finite dimensional, it suffices to prove that, for each $u \in U$,

$$\|\mathcal{C}^k u\|_{\ell_p(\mathcal{Z}_k)} = O(2^{-k\alpha}).$$

Fix such u, and let ν be a smooth compactly supported function for which $u = (\nu * \phi)_{|_{\mathbb{Z}^d}}$ (Proposition 3.2.7). By the remarks preceding this theorem, since $u \in U$, $\widehat{\nu}$ must have an N-fold zero at the origin. Let B be a compactly supported function whose shifts provide approximation order N, and let μ be a compactly supported smooth function such that $1 - \widehat{B}\widehat{\mu}$ has an N-fold zero at the origin. Then, quasi-interpolation basics tell us that the approximation scheme $f \approx B *_k' (2^{kd}\mathcal{D}^k\mu * f)$ provides approximation order N (cf. [9]), hence, by standard interpolation arguments (and since $N \geq \alpha$), we have that

$$\|\phi - B *_k' (2^{kd}\mathcal{D}^k\mu * \phi)\|_{L_p(\mathbb{R}^d)} = O(2^{-k\alpha}).$$

The above also holds when μ is replaced by $\mu + \nu$, hence we obtain that

$$\|B *_k' (2^{kd}\mathcal{D}^k\nu * \phi)\|_{L_p(\mathbb{R}^d)} = O(2^{-k\alpha}).$$

However, by Lemma 3.2.3, $B *_k' (2^{kd}\mathcal{D}^k\nu * \phi) = B *_k' \mathcal{C}^k u$. Assuming, without loss, that the shifts of B are stable in the p-norm (cf. [47]), we conclude that $\|\mathcal{C}^k u\|_{\ell_p(\mathcal{Z}_k)} = O(2^{-k\alpha})$. □

As a nice application, we obtain the following result. The case $n = 1$ in this result is essentially very well known (cf. [33], [42], [57], [36]).

Corollary 3.3.4. *Let ϕ be compactly supported and refinable and assume that $\widehat{\phi}(0) = 1$. Let n be a positive integer such that* (i) *$E(\phi)$ provide approximation order $\geq n$, and* (ii) *$1 - \widehat{\phi}$ has a zero of order $\geq n$ at the origin. Let K be a compact set that contains* $\operatorname{supp}\phi$, *and let $U_0 \subset \mathcal{Q}$ be the space of all sequences u that satisfy the following conditions:*
(i) $\operatorname{supp} u \subset K$.
(ii) *$\widehat{u}$ has a zero of order $\geq n$ at the origin.*
Let $0 < \alpha \leq n$ and $1 \leq p \leq \infty$ be given. Then the subdivision converges in the p-norm at any rate $< \alpha$ on the entire sequence space $\mathcal{Q}$ whenever the following condition holds: "With $\mathcal{C}$ the cascade operator associated with ϕ, we have that $\|\mathcal{C}^k\| = O(2^{-k\alpha})$, when considering $\mathcal{C}^k$ as a map from U_0 to $\ell_p(\mathcal{Z}_k)$."

Proof: Let $\alpha' < \alpha$. We need to show that for each $g \in G_\alpha$, $(g *_k' \mathcal{C}^k\delta)_k$ converges in the p-norm at a rate α' to ϕ.

First, since the space U_0 here is a superspace of the space U of Theorem 3.3.3, we can invoke the latter to conclude that, under the current assumptions, $\phi \in W_p^{\alpha'}(\mathbb{R}^d)$. Now, let u be a sequence supported on K such that $1 - \widehat{u}$ has a zero of order n at the origin, and such that $u * K_\phi = 0$. (We are tacitly assuming that K contains such a sequence; a suitable K is *e.g.*, $\operatorname{supp}\phi + [0, n-1]^d$.) By Theorem 3.2.8, the cascade iterations converge on $g *' u$ to ϕ at a rate α' in the p-norm. On the other hand, $\delta - u \in U_0$, hence, by our assumption here $\|\mathcal{C}^k(\delta - u)\|_{\ell_p(\mathcal{Z}_k)} = O(2^{-k\alpha})$, hence the cascade iterations converge to 0 on $g *' (\delta - u)$ at a rate α. Thus, those iterations converge to ϕ on $g = g *' \delta$ at a rate α'. □

The cascade operator can be represented as the composition of 2^d basic operators: with $j \in \mathcal{Z}_1 \cap [0, 1)^d$, the jth component $\mathcal{C}_j$ of $\mathcal{C}$ is

$$\mathcal{C}_j c := (E^{-j}\mathcal{C}c)_{|_{\mathbb{Z}^d}}.$$

Using this approach, one may interpret the condition

$$\|\mathcal{C}^k\| = O(2^{-k\alpha})$$

(with $\mathcal{C}^k$ viewed as an operator from U to $\ell_p(\mathcal{Z}_k)$) that appears in Theorem 3.3.3 as an equivalent statement on the joint spectral radius of these 2^d operators (acting on U: that joint spectral radius should be in the p-norm $< 2^{-\alpha}$). I forgo providing further details in this regard, since, at the time this article is written, I am not convinced that the formulation of the previous theorem in that equivalent language provides a more efficient venue compared to the straightforward attempt of estimating $\|\mathcal{C}^k u\|_{\ell_p(\mathcal{Z}_k)}$ for $k = 1, 2, 3, \ldots$, and with u varies over a basis for U.

Remark. In the above analysis, we took into account the fact that each point in $\mathcal{Z}_k$ represents a shift of the dilated cube $[0,1]^k/2^k$, and that the L_p-norm of that dilated cube is $2^{-k/p}$. We thus attached mass $2^{-k/p}$ to each point in $\mathcal{Z}_k$ when defining the $\ell_p(\mathcal{Z}_k)$-norm. For $p < \infty$, this removes the unnecessary artifact in the original definition of the p-joint spectral radius ([79], [42]). For example, in these terms, the subdivision algorithm converges on the entire $\mathcal{Q}$ space in the L_p-norm if and only if the p-joint spectral radius on U_0 is < 1, [42], [36].

Remark. The formulation of Theorem 3.3.3 in terms of the kernel is convenient, especially since we do not assume ϕ to be continuous, hence cannot restrict it or a translate of it to the integers. However, it might be hard, in general, to compute K_ϕ; the alternative is to invoke Proposition 3.2.7 and to compute instead all sequences of the form $(\nu * \phi)_{|_{\mathbb{Z}^d}}$. In case ϕ is continuous, that may not be hard (cf. [40] and the discussion in the next section). Since, in general, we do not know in advance whether ϕ is continuous, we may instead try the following idea, which is an adaptation of the approach used in [53] and [45], and which was suggested to me by D. X. Zhou, [80]. If we choose ν to be a well-understood smooth refinable function (e.g., a box spline), and if we guarantee that $\nu * \phi$ *is* continuous, we should be able to compute $u_\nu := (\nu * \phi)_{|_{\mathbb{Z}^d}}$ (cf. the discussion after Theorem 3.2.4). By varying the above ν (and applying suitable difference operators to each so-obtained u_ν), one may hope to get a spanning set for the space U in Theorem 3.3.3. A rigorous treatment using this approach has yet to be found.

3.4. Miscellaneous Results

The connection between the transfer operator and the cascade operator. We started the discussion of the second part of this article with L_2-analysis via the transfer operator. We then presented the L_p-approach via the cascade operator. As said, the two operators are intimately related. Indeed, it is straightforward to prove the following (where we define $\widehat{a}(\omega) := \sum_{j\in\mathcal{Z}_1} a(j)e^{ij\cdot 2\omega}$):

Proposition 3.4.1. *Let ϕ be refinable with mask a and set $m := 2^{-d}|\widehat{a}|^2$. Let $\mathcal{C}$ be the cascade operator associated with a and let $\mathcal{T}$ be the transfer operator associated with m. Given any $c \in \ell_2(\mathbb{Z}^d)$, we have*

$$\|\mathcal{C}^k c\|^2_{\ell_2(\mathcal{Z}_k)} = \|\mathcal{T}^k(|\widehat{c}|^2)\|_{L_1(\mathbb{T}^d)}.$$

In view of the fact that $\mathcal{T}$ is an endomorphism on the space H_ϕ (cf. Section 3.1), while $\mathcal{C}$ is not an endomorphism of any non-trivial space, it is preferable to use the transfer operator for studies in the L_2-norm. For example, the contractivity assumption on the space U in Corollary 3.3.4 is equivalent, when $p=2$, to the condition

$$\|(\mathcal{T}_{|H_0})^k\| = O(2^{-2k\alpha}),$$

with $H_0 := \{f \in H : \ f(0)=0\}$ (with H as in (3.1.7)). This eventually leads to the characterization of the convergence of the cascade/subdivision algorithms in terms of the E-condition on $\mathcal{T}$ (cf. [52]). Similar remarks can be made with respect to the smoothness problems. Usually it is easier to get the L_2-results directly from the transfer operator, compared to the alternative of converting the cascade operator results via Proposition 3.4.1.

Linear independence. The recent papers [37] and [40] suggest an interesting way for analyzing the *local linear independence* of the shifts of a refinable function ϕ. To recall, given a compactly supported (not necessarily refinable) ϕ, and an open set $\Omega \subset \mathbb{R}^d$, we say that the shifts of ϕ are independent on Ω if, whenever

$$\phi *' q = 0, \qquad \text{on } \Omega$$

for some $q \in \mathcal{Q}$, we have that $q(j)\phi(\cdot - j) = 0$ on Ω, for every $j \in \mathbb{Z}^d$. We assume, for simplicity, that $\phi \in C(\mathbb{R}^d)$ (this allows us to choose $\Omega := [0,1]^d$, though that set is not open).

Let us look closer at this problem. Let

$$Z_\Omega := \{j \in \mathbb{Z}^d : \Omega \cap (j + \operatorname{supp}\phi) \neq \emptyset\}.$$

Note that $q(j)\phi(\cdot - j) = 0$ on Ω unless $j \in Z_\Omega$. Now, if $\phi *' q = 0$ on Ω, then for every $x \in \Omega$,

$$\sum_{j \in Z_\Omega} q(j)\phi(x-j) = 0.$$

So, with

$$\phi_x : Z_\Omega \to \mathbb{C} : j \mapsto \phi(x-j),$$

the question is whether or not $(\phi_x)_{x\in\Omega}$ span

$$U_\Omega := \mathbb{C}^{Z_\Omega}.$$

In [37], [40], the following idea was devised for finding the local dependence relations of ϕ on Ω. If we find first the vector $u := \phi_{|\mathbb{Z}^d}$ (cf. the discussion after Theorem 3.2.4), then, by Lemma 3.2.3, $\mathcal{C}^k u = \phi_{|\mathcal{Z}_k}$. The iterations thus provide us eventually with the sequences ϕ_x, x dyadic, which suffice here since ϕ is continuous. The only remaining practical problems are: (i) to compute the initial u, something that usually is not hard when ϕ is continuous, and (ii) determining a stopping criterion for the iterations: the tree structure of the cascade operator (which we have largely ignored) entails that, for $\Omega := [0,1]^d$, we stop exactly when

$$\operatorname{span}\{\phi_x : x \in \mathcal{Z}_k \cap \Omega\} = \operatorname{span}\{\phi_x : x \in \mathcal{Z}_{k+1} \cap \Omega\}.$$

One can also devise a stopping criterion when, *e.g.*, Ω is a box whose corners lie in some $\mathcal{Z}_k$. I do no know of a strategy for choosing a stopping criterion for a general Ω.

The local linear independence is nicely connected with the problem of *global* linear independence: this is the case when $K_\phi = 0$ (cf. (3.2.6)). We recall, [23], [62], that the shifts of a compactly supported distribution ϕ are globally linearly dependent if there exists an exponential $\xi \in K_\phi$. Here,

$$\xi : j \mapsto \xi^j, \quad j \in \mathbb{Z}^d,$$

and $\xi \in (\mathbb{C}\backslash 0)^d$. The following connection between global independence and local independence is a consequence of that characterization:

Lemma 3.4.2. *Let ϕ be a compactly supported continuous function. Let $\Omega := [0,1]^d$, and let Λ be all the local dependence relations of $E(\phi)$ on Ω:*

$$\Lambda := \{q \in \mathcal{Q} : \operatorname{supp} q \subset \Omega + \operatorname{supp} \phi,\ (\phi *' q)_{|\Omega} = 0\}.$$

Then the shifts of ϕ are globally linearly independent if and only if Λ contains a sequence that coincides on $\mathbb{Z}^d \cap (\Omega + \operatorname{supp} \phi)$ with an exponential ξ.

Proof: If q and ξ coincide on $\mathbb{Z}^d \cap (\Omega + \operatorname{supp} \phi)$, and if $q \in \Lambda$, then, on Ω, $\phi *' \xi = \phi *' q = 0$. But since ξ is an exponential, it is obvious that $\phi *' \xi = 0$ on any integer translate of Ω. Those integer translates cover $\mathbb{R}^d$, hence $\phi *' \xi = 0$ everywhere. The converse is trivial. $\square$

This leads to the following result, [40]:

Corollary 3.4.3. *Let ϕ be a compactly supported continuous function, and let V be any spanning set for* $\operatorname{span}(\phi_x)_{x\in[0,1]^d}$. *Then $E(\phi)$ are linearly dependent if and only if there exists an exponential ξ such that $v \perp \xi$, for every $v \in V$, i.e., such that $\sum_{j\in\mathbb{Z}^d} v(j)\xi^j = 0$, for every $v \in V$.*

4. A Conjecture

I want to close this article with the following conjecture concerning the convergence of the cascade algorithm. At the time this article is written, I strongly believe it to be true, but cannot say whether it is easy or hard to solve it. The conjecture is proved in [59] under various additional assumptions (for example, under the assumption that $K_\phi = 0$, and under the weaker assumption that $f = g *' u$, where $u * K_\phi = 0$. These results readily imply that the conjecture is true in one dimension.)

Conjecture. *Let ϕ be a compactly supported refinable function in $W_p^\alpha(\mathbb{R}^d)$, $d \geq 2$, and assume $\widehat{\phi}(0) \neq 0$ (the mask need not be finite). Let g be a compactly supported bounded function that satisfies the following three conditions:*

(a) *The shifts of g provide approximation order $\geq \alpha$ (and $\widehat{g}(0) = 1$).*
(b) *$\widehat{\phi} - \widehat{g} = O(|\cdot|^\alpha)$ near the origin.*
(c) *$K_\phi \subset K_g$.*

Then the cascade algorithm converges on g to ϕ in the p-norm at rate α.

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