On the Meir/Sharma/Hall/Meyer analysis of the spline interpolation error

Carl de Boor

The story

The basic idea of the Meir/Sharma/Hall/Meyer error analysis for cubic spline interpolation (see [SM66], [H68], [HM76]) has been extended to cover also certain deficient quartic and quintic spline interpolation schemes, the latter already by them, the former by Howell/Varma [HV89] and, perhaps, others. It is the purpose of this note to describe the most general situation to which this idea applies and thereby, perhaps, to obviate further papers on various special cases.

In full generality, the idea covers the following situation. Let

$$\tau = (\tau_1, \ldots, \tau_{k+1}) = (\underbrace{0, \ldots, 0}_{\rho \text{ times}} < \tau_{\rho+1} \le \cdots \le \tau_{\rho+r} < \underbrace{1, \ldots, 1}_{\rho \text{ times}}) \tag{1}$$

be any nondecreasing $(k+1)$-sequence in $[0\,..\,1]$ in which both 0 and 1 occur exactly $\rho > 0$ times, hence r is such that

$$k + 1 = 2\rho + r.$$

Denote by

$$H_\tau g$$

the Hermite interpolant to g at τ, i.e., the unique polynomial of degree $\le k$ which agrees with g at τ, repeated points in τ corresponding to the matching of derivative values in the standard way.

The corresponding Hermite spline interpolant, for a given break sequence

$$\xi = (a = \xi_1 < \cdots < \xi_{\ell+1} = b)$$

in the interval $[a\,..\,b]$, is, by definition, the function

$$Hg := H_{\tau,\xi} g := \sigma_i^{-1} H_\tau \sigma_i g \ \text{ on } \ [\xi_i\,..\,\xi_{i+1}], \quad i = 1, \ldots, \ell, \tag{2}$$

with σ_i the linear change of variables

$$\sigma_i f : t \mapsto f(\xi_i + t\,h_i), \qquad h_i := \xi_{i+1} - \xi_i, \quad i = 1, \ldots, \ell.$$

By the Schoenberg-Whitney Theorem, there exists, for given smooth g, exactly one element in the space

$$S := \Pi^{(\rho)}_{k,\xi}$$

of pp functions in $C^{(\rho)}[a\,..\,b]$ of degree k with break sequence ξ which matches g $\rho - 2$-fold at each breakpoint, except at a and b where the match is $\rho - 1$-fold, and matches also at the points

$$\xi_i + \tau_\kappa h_i, \quad \kappa = \rho + 1, \ldots, \rho + r,\ i = 1 \ldots, \ell.$$

We denote this element by

$$s = Ig = I_{\tau,\xi}g.$$

Special cases include:

(i) $k = 3$, $\rho = 2$, hence $r = 0$, leading to *complete cubic spline interpolation.*

(ii) $k = 3$, $\rho = 1$, hence $r = 2$, leading to *deficient cubic spline interpolation* studied by Dikshit/Powar [DP81] (and others).

(iii) $k = 2$, $\rho = 1$, hence $r = 1$, leading to *parabolic spline interpolation* (at midpoints, e.g.), first discussed by Subbotin [Su67].

(iv) $k = 4$, $\rho = 2$, hence $r = 1$, leading to C^2*-quartic spline interpolation*, as investigated by Howell/Varma [HV89].

(v) $k = 5$, $\rho = 3$, hence $r = 0$, leading to C^3*-quintic spline interpolation*, as investigated by Sharma/Meir [SM66], as well as by Hall [H68], and Hall/Meyer [HM76].

The *construction* of Ig proceeds as follows. Since each polynomial piece of Ig is completely determined once also the numbers $D^{\rho-1}Ig(\xi_i)$, $i = 2, \ldots, \ell$, are known, and thereby joins its neighboring piece(s) in $C^{(\rho-1)}$-fashion, one relies on the tridiagonal linear system

$$\mathrm{A}(D^{\rho-1}Ig) = \Lambda g \tag{3}$$

obtained from the requirement that the various polynomial pieces should join in $C^{(\rho)}$-fashion. Somewhat more explicitly,

$$(\mathrm{A}f)(i) = \sum_j A(i,j)f(\xi_j), \qquad i = 2, \ldots, \ell$$

for a certain tridiagonal *matrix* A, with $A(i,j)$ expressible in terms of h_{i-1}, h_i, k, and ρ (see (18)). Also,

$$\Lambda g = (\lambda_i g : i = 2, \ldots, \ell),$$

with $\lambda_i g$ a certain linear combination of the data

$$g\big|_{\tau^{(i)}}$$

(see (8)) for g on the interval $[\xi_{i-1} \,.\,. \,\xi_{i+1}]$, all i. It is one of the results of the present note (see (19)) that, as has been known in all the special cases mentioned above, the matrix A of the linear system (3) is diagonally dominant, uniformly in ξ if scaled appropriately and if τ is symmetric, hence the linear system can be solved stably by Gauss elimination without pivoting.

The Meir/Sharma/Hall/Meyer *analysis of the error* $g - Ig$ is based on the split

$$e := g - Ig = (g - Hg) + (H - I)g,$$

as first used by Hall [H68] (for cubic and quintic spline interpolation).

The first part is dealt with locally, relying on known sharp bounds (such as those in Birkhoff and Priver [BP67] for the cubic and quintic case) for Hermite interpolation. Fortunately, Shadrin [Sh95] quite recently gave a complete treatment of such sharp error bounds, for arbitrary degree, thus providing a proof that

$$\|g - Hg\|_{\infty,(\xi_i..\xi_{i+1})} \le c_\tau h_i^{k+1} \|D^{k+1} g\|_{\infty,(\xi_i..\xi_{i+1})},$$

with

$$c_\tau := \|(1 - H_\tau)()^{k+1}\|_\infty/(k+1)!.$$

This implies that

$$\|g - Hg\|_\infty \le c_\tau (\max_i h_i)^{k+1} \|D^{k+1} g\|_\infty \tag{4}$$

with equality for any g for which $D^{k+1}g$ is absolutely constant and is constant in each break interval $(\xi_i \mathinner{..} \xi_{i+1})$.

The second part requires comparison of the (local) Hermite interpolant, Hg, with the (global) spline interpolant, $s = Ig$. On each break interval $[\xi_i \mathinner{..} \xi_{i+1}]$, both are polynomials of degree $\le k$ that match the same information about g *except* for the $(\rho - 1)$st derivative at the endpoints which is only matched by Hg, while Ig gets that information from the linear system (3). It follows that, on this break interval,

$$Hg - Ig = D^{\rho-1} e(\xi_i)\varphi_{0,i} + D^{\rho-1} e(\xi_{i+1})\varphi_{1,i},$$

with $\sigma_i \varphi_{j,i} := h_i^{\rho-1} \varphi_j$, $j = 0, 1$, and φ_0, φ_1 certain polynomials of degree $\le k$ with

$$(-1)^{k-\rho} \varphi_0 \varphi_1 \ge 0 \quad \text{on } [0 \mathinner{..} 1]. \tag{5}$$

Therefore,

$$\|Hg - Ig\|_{\infty,(\xi_i..\xi_{i+1})} \ \le\ \|D^{\rho-1} e|_\xi\|_\infty c_\varphi h_i^{\rho-1},$$

with

$$c_\varphi \ :=\ \|\varphi_0 + (-1)^{k-\rho} \varphi_1\|_\infty,$$

and with equality if $D^{\rho-1}e|_\xi$ is absolutely constant and $(-1)^{k-\rho} D^{\rho-1} e(\xi_i) D^{\rho-1} e(\xi_{i+1}) \ge 0$ for all i. Further, the sequence $(D^{\rho-1} e(\xi_j) : j = 2, \ldots, \ell)$ satisfies the linear system (3), but with an appropriately changed right side:

$$\mathrm{A}(D^{\rho-1} e) = \mathrm{A}(D^{\rho-1} g) - \Lambda g =: \mathrm{M}g. \tag{6}$$

The coefficient matrix, A, for this linear system is tridiagonal, with positive diagonal entries. Hence, if

$$c_A \ :=\ \min_i (A(i,i) - |A(i,i-1)| - |A(i,i+1)|) \tag{20}$$

is positive, then

$$\|(D^{\rho-1} e)|_\xi\|_\infty \le \|\mathrm{M}g\|_\infty / c_A.$$

Further, if τ is symmetric, i.e., $\tau = 1 - \tau$, then (see (19)) the minimum in (20) is taken on for $i = 3, \ldots, \ell - 1$ and is independent of ξ. Finally,

$$\mathrm{M}g = (\mu_i g : i = 2, \ldots, \ell),$$

with each μ_i a certain linear combination of the linear functionals used for $\sigma_\nu^{-1} H_\tau \sigma_\nu$, $\nu = i-1, i$, i.e., $\mu_i g$ is a linear combination of values and derivatives of g at the entries of

$$\tau^{(i)} := (\xi_{i-1} + \tau h_{i-1}\ ,\ \xi_i + (\tau_\kappa : \kappa = \rho+1, \ldots, k+1) h_i\ ). \tag{8}$$

In the early papers, local Taylor expansion was appealed to to assert (sometimes incorrectly, e.g., in [H68] for the cubic case) that $\mu_i g = c_i D^{k+1} g(\eta_i)$ for some $\eta_i \in (\xi_{i-1} \mathinner{\ldotp\ldotp} \xi_{i+1})$. In any case, μ_i must vanish on Π_k, hence, as first used explicitly in Howell/Varma [HV89] (but see already Schultz [S73]), for smooth g,

$$\mu_i g = \int_{\xi_{i-1}}^{\xi_{i+1}} \widehat{\mu}_i D^{k+1} g, \quad i = 2, \ldots, \ell,$$

for a certain pp function $\widehat{\mu}_i$, with breaks only at the data points and with support in $[\xi_{i-1} \mathinner{\ldotp\ldotp} \xi_{i+1}]$. Therefore,

$$\|\mathrm{M}g\|_\infty \le \max_i \|\widehat{\mu}_i\|_1\ \|D^{k+1} g\|_\infty,$$

and a detailed analysis (see (22)) of the integrals

$$\|\widehat{\mu}_i\|_1 = \int_{\xi_{i-1}}^{\xi_{i+1}} |\widehat{\mu}_i(y)| \mathrm{d}y$$

provides the formula

$$\|\widehat{\mu}_i\|_1 \ = \ c_\mu \frac{h_i h_{i-1}^{k-\rho+2} + h_{i-1} h_i^{k-\rho+2}}{h_{i-1} + h_i} \tag{9}$$

with a certain ξ-independent constant c_μ, and so finishes the derivation of the error bound

$$\|g - Ig\|_\infty \le (c_\tau + (c_\mu / c_A) c_\varphi)(\max_i h_i)^{k+1} \|D^{k+1} g\|_\infty. \tag{10}$$

Curtis and Powell [CP67] seem to be the first to have proved sharpness of spline interpolation error bounds, for the special case of cubic spline interpolation on a uniform mesh. Their argument relies on a detailed analysis of the Peano kernel for the error.

In the present setting, Hall/Meyer [HM76] were the first to consider the sharpness of the bound (10) (and of corresponding bounds on the error in derivatives). To be precise, this sharpness is only asymptotic, as ℓ grows large, with the error bound almost exact for a uniform ξ. In all cases considered, i.e., Hall/Meyer in cubic and $C^{(3)}$-quintic spline interpolation, Howell/Varma in $C^{(2)}$-quartic spline interpolation, the sharpness has been established by exhibiting an extremizing function, in effect an Euler spline in the cases considered by Hall/Meyer, and the monomial $()^{k+1}$ in $C^{(2)}$-quartic spline interpolation. The argument for it relies on an often quite detailed analysis of the function $\widehat{\mu}_i$ to show its simple sign pattern (just one sign change, at ξ_i, in the cubic and quintic case, no sign change in the quartic case) and on a further, ad hoc, argument involving the matrix A.

It is one purpose of this note to point out that the error bound (10) is asymptotically sharp in the general case, and that this can be seen quite easily. In particular, $\widehat{\mu}_i$ is necessarily a linear combination of the $k+1-\rho$ B-splines of order $k+1$ associated with the knot sequence $\tau^{(i)}$ (see (8)) and has a zero of order $k-\rho$ at ξ_i, and this already determines it uniquely, up to sign. In particular, $\widehat{\mu}_i$ is of one sign on each of the two breakpoint intervals in its support. Furthermore, the $\widehat{\mu}_i$ are of one sign (resp., change sign) exactly when the inverse of A is of one sign (resp., checkerboard), making it possible to exhibit a function for which the error bound is (asymptotically) sharp. In particular, the error bound (10) can thereby be seen to be exact in a very simple case, in which the error can be computed directly, thus making the separate calculation of the four constants, c_τ, c_μ, c_A, and c_φ unnecessary.

Here is the formal statement.

Theorem. *Let $\tau = (\tau_i : i = 1, \ldots, k+1)$ be a nondecreasing sequence in $[0\,..\,1]$ in which both 0 and 1 occur exactly $\rho > 0$ times, and set $r := k+1-2\rho$ (hence $0 < \tau_{\rho+1} \le \cdots \le \tau_{\rho+r} < 1$). Let $\xi = (a = \xi_1 < \cdots < \xi_{\ell+1} = b)$ and let $S = \Pi^{(\rho)}_{k,\xi}$ be the space of pp functions in $C^{(\rho)}$ of degree k with breaks ξ. Then, for every $g \in C^{(\rho-1)}[a\,..\,b]$, there exists exactly one $s = Ig$ in S which interpolates to g in the sense that*

$$s \;=\; g \;\text{ at } \xi_j + (\tau_2, \ldots, \tau_k)\Delta\xi_j, \quad j = 1, \ldots, \ell$$

(with repetitions indicating the matching of derivatives in the usual way), and also

$$D^{\rho-1}s \;=\; D^{\rho-1}g \;\text{ at } a, b. \tag{11}$$

Further, if τ is symmetric, i.e., $\tau = 1-\tau$, then

$$\|g - Ig\|_\infty \;\le\; c\,(\max_i \Delta\xi_i)^{k+1}\|D^{k+1}g\|_\infty, \tag{12}$$

with $c := \|g_0 - I_0 g_0\|_\infty$ the maximum error in the special case that $\xi = (-1, 0, 1)$, g_0 is 2-periodic with $D^{k+1}g_0(x)$ equal to -1 for $-1 < x < 0$ and equal to $(-1)^{k-\rho-1}$ for $0 < x < 1$, and with (11) replaced by the periodic *end-conditions*

$$D^{\rho-j}s(b) = D^{\rho-j}s(a), \quad j = 0, 1. \tag{13}$$

In particular (12) is asymptotically sharp, for a uniform ξ, as $\ell \to \infty$.

It should be noted that the symmetry assumption for τ is a convenience. However, diagonal dominance cannot be had without *some* assumption on τ.

Also, it should be pointed out that sharp *pointwise* error bounds are available for interpolation by splines with simple (interior) knots to data at simple data sites which satisfy the Schoenberg-Whitney conditions; see Sections 5.2 and 5.3 of Korneichuk's book [K91]. Those results can easily be extended to the present situation (which involves non-simple knots and data sites). This note's virtue (if any) lies in pointing out a simple way to derive the error bound (12) and compute the exact constant c in it for the particular spline interpolation schemes considered.

The remainder of this note proves the various assertions made, thus providing all missing details for the proof of the theorem.

Some facts concerning H_τ

In this preparatory section, we derive various facts concerning the Hermite interpolant, $H_\tau g$, in particular its dependence on the data $D^{\rho-1}g(\nu)$, $\nu = 0, 1$, of use later, in the analysis of the matrix of the linear system (3), and of the norms $\|\widehat{\mu}_i\|_1$.

Since $H_\tau g$ matches g at τ, we may write it in the convenient form

$$H_\tau g = D^{\rho-1}g(0)\varphi_0 + D^{\rho-1}g(1)\varphi_1 + Q_\tau g, \tag{14}$$

with

$$\varphi_\nu = \alpha_\nu\,(\cdot + \nu - 1)\psi, \quad \nu = 0, 1,$$

where

$$\psi(t) := \prod_{\kappa=2}^{k}(t - \tau_\kappa),$$

and the constant α_ν is such that $D^{\rho-1}\varphi_\nu(\nu) = 1$, $\nu = 0, 1$, hence $D^{\rho-1}Q_\tau g(\nu) = 0$, $\nu = 0, 1$, and $Q_\tau g$ depends only on the data $g|_{(\tau_\kappa:\kappa=2,\ldots,k)}$. In other words, (14) gives $H_\tau g$ in 'Lagrange form', but with only two of the data, namely $D^{\rho-1}g$ at 0 and at 1, mentioned explicitly, and the rest of the information collected in the term $Q_\tau g$.

Here are some details concerning the function φ_1 which, together with the analogous information about φ_0, will be needed in the discussion of the diagonal dominance of A and the sign pattern of its inverse. Since

$$D^q(\cdot - \beta)\psi = qD^{q-1}\psi + (\cdot - \beta)D^q\psi, \quad q = 0, 1, 2, \ldots,$$

we have

$$1 = D^{\rho-1}\varphi_1(1) = \alpha_1(\rho - 1)D^{\rho-2}\psi(1) + \alpha_1 D^{\rho-1}\psi(1).$$

Since

$$D^{\rho-2}\psi(1) \;=\; 0 \;\neq\; D^{\rho-1}\psi(1),$$

this implies that $\alpha_1 = 1/D^{\rho-1}\psi(1)$, hence

$$D^\rho\varphi_1 = \left(\rho D^{\rho-1}\psi + ()^1 D^\rho\psi\right)/D^{\rho-1}\psi(1).$$

In particular,

$$D^\rho\varphi_1(0) = \rho D^{\rho-1}\psi(0)/D^{\rho-1}\psi(1),$$

while

$$D^\rho \varphi_1(1) = \rho + D^\rho \psi(1)/D^{\rho-1}\psi(1) \ > \ \rho \tag{15}$$

since, by Rolle's theorem, $D^{\rho-1}\psi$ has all its zeros in the open interval $(0\,..\,1)$.

Since $D^{\rho-1}\psi$ has exactly $\rho - 1 + r = k - \rho$ zeros in $(0\,..\,1)$, we have

$$(-1)^{k-\rho} D^{\rho-1}\psi(0) D^{\rho-1}\psi(1) \ > \ 0,$$

hence

$$(-1)^{k-\rho} D^\rho \varphi_1(0) \ > \ 0\,. \tag{16}$$

This implies that $(-1)^{k-\rho} D^{2\rho-1}(\varphi_0\varphi_1)(0) > 0$, hence, since $\varphi_0\varphi_1$ vanishes to exact order $2\rho - 1$ at 0 and has only even zeros in $(0\,..\,1)$, (5) follows. More explicitly than (16),

$$D^\rho \varphi_1(0) = (-1)^{k-\rho}\rho, \quad \text{if } \tau \text{ is symmetric.}$$

It seems most efficient to deduce the corresponding statements for φ_0 from the fact that

$$\varphi_0 = \varphi_{0,\tau} = (-1)^{\rho-1}\varphi_{1,1-\tau}(1 - \cdot).$$

This implies that

$$D^\rho \varphi_0 = (-\rho\, D^{\rho-1}\psi + (1 - \cdot) D^\rho \psi)/D^{\rho-1}\psi(0)$$

and, in particular,

$$D^\rho \varphi_0(0) = -\rho + D^\rho \psi(0)/D^{\rho-1}\psi(0) \ < \ -\rho$$

(since $D^{\rho-1}\psi$ has all its zeros in the open interval $(0\,..\,1)$). Therefore,

$$D^\rho \varphi_0(0) \ < 0 \ < D^\rho \varphi_1(1). \tag{17}$$

The tridiagonal linear system

Since the Schoenberg-Whitney theorem guarantees existence and uniqueness of the interpolant, we know that the coefficient matrix A in (3) is invertible. We now show that, for a symmetric τ, A is diagonally dominant, independently of ξ, with positive diagonal entries and with the next-to-diagonal entries negative (positive) exactly when $k - \rho$ is even (odd), hence the inverse of A is positive (checkerboard).

The ith equation in (3) expresses the requirement that the interpolant, $s = Ig$, have a continuous ρth derivative at ξ_i:

$$D^\rho s(\xi_i-) = D^\rho s(\xi_i+).$$

Written out in more detail, this reads, after reordering so as to put the unknown terms on the left and the given information on the right (except that, for $i = 2$ and $i = \ell$, we leave the known endpoint derivative on the left side),

$$\begin{gathered}(D^{\rho-1}s(\xi_{i-1})D^{\rho}\varphi_0(1) \;+\; D^{\rho-1}s(\xi_i)D^{\rho}\varphi_1(1))/h_{i-1}\\ -\\ (D^{\rho-1}s(\xi_{i+1})D^{\rho}\varphi_1(0) \;+\; D^{\rho-1}s(\xi_i)D^{\rho}\varphi_0(0))/h_i\\ = D^{\rho-1}(\sigma_{i-1}^{-1}Q_\tau\sigma_{i-1}g)(\xi_i-) \;-\; D^{\rho-1}(\sigma_i^{-1}Q_\tau\sigma_i g)(\xi_i+).\end{gathered}$$

After multiplying both sides by $(h_{i-1}h_i)/(h_{i-1}+h_i)$, we obtain the ith equation of (3), with

$$A(i,j) = \frac{1}{h_{i-1}+h_i}\begin{cases} D^{\rho}\varphi_0(1)h_i, & j = i-1;\\ D^{\rho}\varphi_1(1)h_i - D^{\rho}\varphi_0(0)h_{i-1}, & j = i;\\ -D^{\rho}\varphi_1(0)h_{i-1}, & j = i+1.\end{cases} \tag{18}$$

By (17), $A(i,i) > 0$ while, by (16), $A(i,i\pm 1)$ is negative (positive) exactly when $k-\rho$ is even (odd).

If now τ is symmetric, i.e., $\tau = 1-\tau$, then

$$A(i,j) = \frac{1}{h_i+h_{i-1}}\begin{cases} -\rho(-1)^{k-\rho}h_i, & j = i-1;\\ (\rho + D^{\rho}\psi(1)/D^{\rho-1}\psi(1))(h_i+h_{i-1}), & j = i;\\ -\rho(-1)^{k-\rho}h_{i-1}, & j = i+1.\end{cases} \tag{19}$$

Therefore,

$$A(i,i) - \sum_{j\neq i}|A(i,j)| \geq D^{\rho}\psi(1)/D^{\rho-1}\psi(1) =: c_A, \tag{20}$$

with c_A positive (by (15)) and with equality for all i except for the first and last, unless we switch to the periodic end conditions (13), in which case there is equality here for all i.

While it is trivial that, therefore, A is also diagonally dominant for all 'nearby' τ, A is not diagonally dominant for all choices of τ. E.g., for the simplest possible case $\rho = 1 = r$, hence $k = 2$, we have $\phi_0 = (\cdot - \tau_2)(\cdot - 1)/\tau_2$, and therefore

$$A(i,i)-|A(i,i-1)|-|A(i,i+1)| = (h_i(\frac{2-\tau_2}{1-\tau_2}-\frac{1-\tau_2}{\tau_2})+h_{i-1}(\frac{\tau_2+1}{\tau_2}-\frac{\tau_2}{1-\tau_2}))/(h_i+h_{i-1}),$$

which, for some ξ, becomes negative when $\tau_2 \notin [2-\sqrt{3}\,..\,1/\sqrt{2}]$. (However, this matrix is *column* diagonally dominant for all $\tau_2 \in (0\,..\,1)$).

A representer for μ_i

Since the error, $e = g - Ig$, is zero for any $g \in S$ and the matrix A is invertible, it follows that, for each i, the functional μ_i necessarily vanishes on S. This implies that

$$\mu_i g = \int_{\xi_{i-1}}^{\xi_{i+1}} \widehat{\mu}_i D^{k+1} g, \quad i = 2, \ldots, \ell,$$

for a certain pp function $\widehat{\mu}_i$, with breaks only at the data sites and with support in $[\xi_{i-1} \mathinner{..} \xi_{i+1}]$. Precisely, any smooth g can be written

$$g(t) = \sum_{j \le k} D^i g(a)(t-a)^i/j! + \int_a^b (t-y)_+^k/k!\; D^{k+1} g(y)\mathrm{d}y,$$

and, since $\Pi_k|_{[a..b]} \subset S|_{[a..b]}$, this implies that

$$\widehat{\mu}_i(y) = \mu_i(\cdot - y)_+^k/k! \;=\; -\mu_i(\cdot - y)_-^k/k!,$$

the second equality since $(\cdot - y)_+^k + (\cdot - y)_-^k \in \Pi_k$. Consequently, $\widehat{\mu}_i$ is an element of the space $S_{k+1,\tau^{(i)}}$ of splines of order $k+1$ with knot sequence $\tau^{(i)}$ (see (8)). Any spline space has dimension equal to (number of knots) – (order), hence,

$$\dim S_{k+1,\tau^{(i)}} = \rho + r = k - \rho + 1. \tag{21}$$

Since also $(\cdot - \xi_i)_+^q$, $q = \rho+1, \ldots, k$, is in S, hence is annihilated by μ_i, it follows that $\widehat{\mu}_i$ vanishes $k-\rho$-fold at ξ_i. It follows from (21) (and, e.g., the Schoenberg-Whitney Theorem) that this condition alone determines $\widehat{\mu}_i$ uniquely, up to a scalar factor. Further, since $\widehat{\mu}_i \neq 0$, it follows that $\widehat{\mu}_i$ vanishes in $(\xi_{i-1} \mathinner{..} \xi_{i+1})$ only at ξ_i, hence is of one sign, or changes sign only at ξ_i, depending on whether $k-\rho$ is even or odd.

This simple observation suffices for the derivation of an explicit formula for $\|\mu_i\| = \|\widehat{\mu}_i\|_1$, as follows. Consider the functions

$$\varphi_\pm(t) := (t - \beta_\pm)\varphi_0\Big(\frac{t-\xi_i}{h_\pm}\Big)\,(h_\pm)^k\,\gamma_\pm\, D^{\rho-1}\psi(0)/(k+1)!,$$

with $\gamma_- := -1$, $\gamma_+ := -(-1)^{k-\rho}$, $h_\pm := \xi_{i\pm1} - \xi_i$, with φ_0 and ψ as in the detailed discussion of H_τ, and with $\beta_\pm$ to be determined in such a way that

$$D^{\rho-j}\varphi_-(\xi_i) \;=\; D^{\rho-j}\varphi_+(\xi_i), \quad j = 0, 1.$$

It turns out that $\beta_\pm$ can so be determined. The resulting function

$$g(t) := \begin{cases} \varphi_-(t) & t \le \xi_i; \\ \varphi_+(t) & t \ge \xi_i, \end{cases}$$

is seen to be piecewise polynomial of degree $k+1$, with $D^{k+1}g$ equal to γ_- on $(\xi_{i-1} \mathinner{\ldotp\ldotp} \xi_i)$, and equal to γ_+ on $(\xi_i \mathinner{\ldotp\ldotp} \xi_{i+1})$, hence

$$\int_{\xi_{i-1}}^{\xi_{i+1}} \widehat{\mu}_i D^{k+1} g \;=\; \pm\|\widehat{\mu}_i\|_1.$$

On the other hand, $g|_{\tau^{(i)}} = 0$, and even $D^{\rho-1}g(\xi_{i\pm1}) = 0$. This implies that the spline interpolant to g for the break sequence $(\xi_{i-1}, \xi_i, \xi_{i+1})$ is zero, therefore $A(i,i)D^{\rho-1}g(\xi_i) = \int_{\xi_{i-1}}^{\xi_{i+1}} \widehat{\mu}_i D^{k+1} g$.

This gives (9) in the more explicit form

$$\|\widehat{\mu}_i\|_1 \;=\; \frac{\rho|D^{\rho-1}\psi(0)|}{(k+1)!}\,\frac{h_i(h_{i-1})^{k-\rho+2} + h_{i-1}(h_i)^{k-\rho+2}}{h_i + h_{i-1}}, \tag{22}$$

and even shows that $\widehat{\mu}_i(t) \le 0$ for $t \le \xi_i$, while $(-1)^{k-\rho}\widehat{\mu}_i(t) \le 0$ for $t \ge \xi_i$ (by virtue of the fact that $A(i,i)D^{\rho-1}g(\xi_i)$ is positive). To be sure, the g constructed is only in $C^{(\rho)}$. However, it is the error in the spline interpolant to any $(k+1)$st primitive of the function $D^{k+1}g$, and that is all that really matters.

A similar construction for the simple break sequence $(-1, 0, 1)$ can be used to provide the 2-periodic function g_0, with $g_0(x) = ((x(x-1) + \rho D^{\rho-1}\psi(0)/D^\rho\psi(0))\psi(x)/(k+1)!$ on $(0 \mathinner{\ldotp\ldotp} 1)$ and odd (even) when $\rho - 1$ is odd (even), for which $I_0 g_0 = 0$ while $\|g_0\|_\infty = c_\tau + (c_\mu/c_A)c_\varphi$.

It also follows that, if $k-\rho$ is even, then all the μ_i take on their norm on the function $()^{k+1}$, while, if $k-\rho$ is odd, they all take on their norm on the function whose $k+1$st derivative is absolutely constant and changes sign across each interior ξ_i. In the former case, the corresponding right side in (6) is of one sign, while, in the latter case, it is maximally alternating in sign. In the latter case, and for a symmetric choice of τ and for a uniform ξ, it follows that $D^{\rho-1}e$ vanishes at all the breaks ξ_i when $g = ()^{k+1}$, hence the error in $D^{\rho-1}e$ at the breaks is at least one order higher than expected. This was first observed for cubic spline interpolation, in [BB65].

Now recall that, correspondingly, the inverse of the matrix A in (6) is of one sign when $k-\rho$ is even, and is checkerboard when $k-\rho$ is odd. Therefore, (12) is asymptotically sharp for a uniform ξ (using the essentially local character of spline interpolation in that case).

References

[BB65] G. Birkhoff, C. R. de Boor (1965): *Piecewise polynomial interpolation and approximation*. In: Approximation of Functions (H. L. Garabedian, ed). New York: Elsevier, pp. 164–190.

[BP67] G. Birkhoff, A. S. Priver (1967): *Hermite interpolation errors for derivatives*. J. Math. Phys., **46**: 440–447.

[CP67] A. R. Curtis, M. J. Powell (1967): *Error analysis for equal-interval interpolation by cubic splines*. Theoret. Phys. Div. A.E.R.E. - R.5600

[DP81] H. P. Dikshit, P. L. Powar (1981): *On deficient cubic spline interpolation*. J. Approx. Theory, **31**: 99–106.

[H68] C. A. Hall (1968): *On error bounds for spline interpolation*. J. Approx. Theory, **1**: 209–218.

[H69] C. A. Hall (1969): *Error bounds for periodic quintic splines*. CACM, **12**: 450–452.

[H73] C. A. Hall (1973): *Uniform convergence of cubic spline interpolants*. J. Approx. Theory, **7**: 71–75.

[HM76] C. A. Hall, W. W. Meyer (1976): *Optimal error bounds for cubic spline interpolation*. J. Approx. Theory, **16**: 105–122.

[HV89] Gary Howell, A. K. Varma (1989): *Best error bounds for quartic spline interpolation*. J. Approx. Theory, **58**: 58–67.

[K91] N. Korneichuk (1991): Exact Constants in Approximation Theory. Cambridge, England: Encyclopedia of Mathematics and its Applications 38, Cambridge University Press.

[S73] M. H. Schultz (1973): Spline Analysis. Englewood Cliffs, NJ: Prentice–Hall.

[Sh95] A. Yu. Shadrin (1995): *Error bounds for Lagrange interpolation*. J. Approx. Theory, **80**: 25–49.

[SM66] A. Sharma, A. Meir (1966): *Degree of approximation of spline interpolation*. J. Math. Mech., **15(5)**: 759–767.

[Su67] Y. N. Subbotin (1967): *On piecewise-polynomial (spline) interpolation*. Mat. Zametki, **1**: 63–70.

[RD] S. S. Rana, Y. P. Dubey (1996): *Local behaviour of the deficient discrete cubic spline interpolator*. J. Approx. Theory, **86(1)**: 120–127.