On Local Spline Approximation by Moments

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1. This note is intended to generalize the statements of [1]. Incidentally it should justify some of the steps taken in [1].

2. Let \( m \) be a positive integer, \( \pi : 0 = x_0 < x_1 < \cdots < x_n = 1 \) a partition of the unit interval, and denote by \( S = S_\pi \) the set of spline functions on \([0, 1]\) of degree \( 2m - 1 \) with (interior) joints \( x_1, \cdots, x_{n-1} \). We wish to investigate the behavior of

\[
\text{dist} (f, S) = \min_{s \in S} ||f - s||_\infty,
\]

for \( f \in C[0, 1] \), as the mesh of \( \pi \), \( |\pi| = \max_i |x_{i+1} - x_i| \), goes to zero. As is pointed out in [1],

\[
\text{dist} (f, S_\pi) = O(|\pi|^k)
\]

will not hold for \( k > 2m \), except for the trivial case that \( f \) is a polynomial of degree \( \leq 2m - 1 \). It is further stated there that if \( f \in C^{2m}[0, 1] \) and if the numbers

\[
M_\pi = \max_{i, i+1} (x_{i+1} - x_i)/(x_{i+1} - x_i)
\]

stay bounded, then there exists \( K \) independent of \( f \) or \( \pi \) and \( s_\pi \in S_\pi \) s.t.

\[
|f(x) - s_\pi(x)| \leq K |\pi|^{2m} ||f^{(2m)}||_\infty, \quad \forall x \in [x_m, x_{n-m}].
\]

It is one result of this note that in fact

\[
\text{dist} (f, S_\pi) = O(|\pi|^{2m}),
\]

for \( f \in C^{2m}[0, 1] \), and that (4) holds even without the assumption of bounded mesh ratios \( M_\pi \).

The argument in [1] relies on a linear approximation scheme, called local spline approximation by moments, which realizes the convergence rate \( O(|\pi|^{2m}) \). Briefly, the approximation \( P_\pi f \) to \( f \) is defined by

\[
(P_\pi f)(x) = p(x) + \sum_i G(x, x_i) \int_0^1 W_i(t) f^{(2m)}(t) \, dt.
\]

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Here, $p(x)$ is the polynomial of degree $2m - 1$ satisfying
\[ p^{(i)}(0) = f^{(i)}(0), \quad p^{(i)}(1) = f^{(i)}(1), \quad j = 0, \ldots, m - 1, \]
and $G(x, t)$ is Green's function for the boundary value problem $g^{(2m)}(t) = g(t),\ y^{(i)}(0) = y^{(i)}(1) = 0, \ j = 0, \ldots, m - 1$, so that
\[ f(x) = p(x) + \int_0^1 G(x, t)y^{(2m)}(t) \, dt. \]

"The weight functions $W_i(t)$ are distributed over the 2m mesh points $x_i$ nearest $t$ in such a way as to have sum one and k-th moment $\Sigma(x_i - t)^k W_i = 0$ for $k = 1, \ldots, 2m - 1."$ [1].

3. A little thought shows that, at least for "truly interior points" $t$, the $W_i(t)$ are the cardinal functions of an interpolation scheme which we will call, with [2], (2m)-point central (polynomial) interpolation. In this scheme, a function $g(x)$ is approximated on $[0, 1]$ by
\[ (Q, g)(x) = p_{h+m}(x), \quad x \in [x_{h-1}, x_h], \quad h = 1, \ldots, n, \]
where $p_n(x)$ is the polynomial of degree $\leq 2m - 1$ which interpolates $g(x)$ at the points $x_h, x_{h+1}, \ldots, x_{h+2m-1}$. This definition breaks down "near" $x = 0$ and $x = 1$. Since [1] gives no guidance in this matter, we pick one of the many supplemental definitions possible: Assume that $g(x)$ is defined in a neighborhood of $[0, 1]$ and that $x$ is supplemented by additional points satisfying
\[ x_{-m+1} < x_{-m+2} < \cdots < x_1 < x_{m+1} < \cdots < x_{n+1}. \]
Define
\[ W_i(t) = Q, \prod_{j=1}^i \frac{(t - x_j)}{(x_i - x_j)}, \quad i = -m + 1, \ldots, n + m - 1. \]
Then
\[ (Q, g)(x) = \sum_{i+1}^{n+m-1} g(x_i)W_i(x). \]

With this, we can use, more straightforwardly, Taylor's series with integral remainder,
\[ f(x) = \sum_{i=0}^{2m-1} f^{(i)}(0)x^i/i! + \frac{1}{(2m-1)!} \int_0^1 (x - t)^{2m-1} f^{(2m)}(t) \, dt, \]
and slightly redefine $P_*$ by
\[ (P_*, f)(x) = \sum_{i=0}^{2m-1} f^{(i)}(0)x^i/i! + \frac{1}{(2m-1)!} \sum_{i+1}^{n+m-1} (x - x_i)^{2m-1} \int_t^1 W_i(t) f^{(2m)}(t) \, dt. \]

It is clear that this definition differs from (5) only by a polynomial of degree $\leq 2m - 1$.

Set
\[ e(x) = f(x) - (P_*, f)(x). \]

Then, for $k = 0, 1, \ldots, 2m - 1$,
\[ e^{(k)}(x) = \int_0^1 E_{k}(x, t)f^{(2m)}(t) \, dt, \]
with
\[ E_k(x, t) = \left( \frac{\partial}{\partial x} \right)^{k} \frac{1}{(2m-1)!} \left[ (x - t)^{2m-1} - \sum_{i} (x - x_i)^{2m-1} W_i(t) \right] \]
\[ = (1 - Q_*)^{(k)} \frac{1}{(2m-1-k)!} (x - t)^{2m-1-k}. \]

4. The investigation of the behavior of $e^{(k)}(x)$ reduces, therefore, to the study of the error in applying central interpolation, $Q_*$, to
\[ g(t) = (x - t)^{2m-1}/(2m - 1 - k)!. \]

Write $s = 2m - 1 - k$, for short. Let $a, b \in [0, 1], x_{i-1} \leq a \leq x_i, x_{i-1} \leq b \leq x_i, \ldots, x_{i-1} \leq y \leq x_i, a \leq h \leq n$. It follows from the definition of $Q_*$ that if $g(t)$ is a polynomial of degree $\leq 2m - 1$ on $[x_{-h}, x_{h+1}]$, then
\[ (Q, g)(t) = g(t), \quad \text{all } t \in [x_{i-1}, x_i]. \]

Hence
\[ E_k(x, t) = 0 \quad \text{for } |j - h| \geq m. \]

Further, since also
\[ g(t) = (-x + t)^{s} + (x - t)^{s}/s!, \]
we can assume without loss that $a \leq t$ and $0 \leq h - j < m$. Let $p_{s, j}(t)$ denote the polynomial of degree $\leq r$ which interpolates $g(t)$ at $x_i, x_{i+1}, \ldots, x_{i+r}$. Then
\[ (1 - Q_*) g(t) = p_{s, j-m+s-1}(t). \]

To estimate $p_{s, j-m+s-1}(t)$, we express it in terms of certain of the $p_{i, i}(t)$.

**Lemma 1.** Let $t_0 < t_1 < \cdots < t_r$, and, for given $g(t)$, let $p_{s, j}$ denote the $s$-th degree polynomial which interpolates $g(t)$ at the points $t_1, t_{i+1}, \ldots, t_{i+s}, t$, $s \geq 0$, $i + s \leq r$. Then
\[ p_{s, j}(t) = \sum_{i=0}^{s-1} p_{i, i}(t)L_{s-i}(t), \]
\[ p_{s, j}(t) = \sum_{i=0}^{s-1} p_{i, i}(t)L_{s-i}(t), \]

\[ (1 - Q_*) g(t) = p_{s, j-m+s-1}(t). \]

To estimate $p_{s, j-m+s-1}(t)$, we express it in terms of certain of the $p_{i, i}(t)$.

**Lemma 1.** Let $t_0 < t_1 < \cdots < t_r$, and, for given $g(t)$, let $p_{s, j}$ denote the $s$-th degree polynomial which interpolates $g(t)$ at the points $t_1, t_{i+1}, \ldots, t_{i+s}, t$, $s \geq 0$, $i + s \leq r$. Then
\[ p_{s, j}(t) = \sum_{i=0}^{s-1} p_{i, i}(t)L_{s-i}(t), \]

\[ (1 - Q_*) g(t) = p_{s, j-m+s-1}(t). \]
where the $L_i(\ell)$ depend on the points $t$, but not on $g(\ell)$. Specifically,

(i) $L_i(\ell) = \alpha_{i, s} (t - t_s) \cdots (t - t_{i-1})(t - t_{i+s}) \cdots (t - t_r)$;

(ii) $1 \geq L_i(\ell) \geq 0$ for all $\ell \in [t_{i-1}, t_{i+s}]$.

Hence

(iii) $\sum_i |L_i(\ell)| = 1$, all $\ell \in [t_{i-1}, t_{i+s}]$;

(iv) $\sum_i |L_i(\ell)| \leq C_r(M)$, all $\ell \in [t_s, t_r]$,

with $C_r(M)$ an increasing function of $M$ and $M = \max_{i=i-1} (t_{i-1} - t_i)/(t_{i+s} - t_i)$.

**Proof.** Obviously, the $L_i(\ell)$ are generalizations of the Lagrange polynomials which they reduce for $s = 0$. The $L_i(\ell)$ may be found recursively. Using Neville's formula,

$$p_{i,s+1}(\ell) = \frac{\ell - t_{i+s+1}}{t_{i+s+1} - t_i} p_{i,s}(\ell) + \frac{t_i - \ell}{t_{i+s+1} - t_i} p_{i+1,s}(\ell),$$

one gets

$$L_i(\ell) = \frac{t - t_{i-1}}{t_{i+s} - t_i} L_{i-1}(\ell) + \frac{t_i - \ell}{t_{i+s} - t_i} L_{i+1}(\ell),$$

with the convention that $L_i(\ell) = 0$ for $i < 0$ or $i + s > r$. With $L_r(\ell) = 1$, all statements of the lemma are clearly true for $s = r$. Using induction and the identity (20), (18) and (i), (ii), (iv) follow for all $s < r$. In particular, (ii) follows from the observation that the two "weights" in (20) are non-negative for $t_{i-1} \leq \ell \leq t_{i+s}$, and that, by (18),

$$\sum_i L_i(\ell) = 1.$$ 

This, together with (ii), also establishes (iii).

Applying this lemma to (17), we get

$$\|\phi - (P_s g)(\ell)\| \leq \max_i |p_i(\ell)| \sum_i |L_i(\ell)|,$$

where $i$ runs from $h - m$ to $h + m - 1 - s$. The term

$$\max_i |p_i(\ell)|$$

is easily estimated. With $g(x_1, \ldots, x_{i+s})$ the $q$-th divided difference of $g(\ell)$ at the points $x_1, \ldots, x_{i+s}$, and $\phi^{(q)}(\ell) = (\ell - t_i)^q/(q - g)$, Newton's interpolation formula gives

$$|p_i(\ell)| \leq \sum_{q=0}^i |g(x_1, \ldots, x_{i+s})| \prod_{s=0}^{i-1} (\ell - x_{i+s})$$

$$\leq \sum_{q=0}^i \max_{t \in [x_i, x_{i+s}]} |g^{(q)}(\ell)| \frac{1}{q!} \alpha |\ell|^q \leq C_r |\ell|^q,$$

where $C_r$ is independent of $r$ or $g$. Hence

$$\|\phi - (P_s g)(\ell)\| \leq C_r |\ell|^q \sum_i |L_i(\ell)|.$$ 

Hence, with (iii) of the lemma,

$$\|\phi - (P_s g)(\ell)\| \leq C_r |\ell|^q$$

for $s \geq m - 1$,

with $C_r(\alpha)$ some increasing function of $\alpha$.

5. It is now straightforward to prove the following

**Theorem 1.** For $f \in C^{2m-1}[0, 1]$, and $\pi : x_{-m+1} < x_{-m+2} < \cdots < x_{n+m-1}$, with $x_0 = 0$, $x_n = 1$, and $P_s f$ as given by (10), we have

$$\|f^{(2k)}(\ell) - (P_s f)^{(2k)}(\ell)\| \leq N_s |\pi|^{2m-2} \|f^{(2m)}(\ell)\|, \quad k = 0, \ldots, 2m - 1,$$

with $N_s$ independent of $f$. If $k \leq m$, $N_s$ is also independent of $\pi$, while for $k > m$, $N_s$ can be bounded in terms of $M_r$.

**Proof.** With (12), (13) and (15),

$$\|f^{(2k)}(\ell) - (P_s f)^{(2k)}(\ell)\| = \left| \int_{[\ell - \ell; \ell + \ell]} E_s(\ell, \ell') f^{(2m)}(\ell') \, d\ell' \right| \leq 2m |\pi| \|E_s(\ell, \ell')\| \|f^{(2m)}(\ell')\|.$$

For $s = 2m - 1 - k \geq m - 1$, i.e., for $k \leq m$, (24) gives

$$\|E_s(\ell, \ell')\| \leq C_s |\pi|^{2m-1-k}$$

with $C_s$ independent of $\pi$; hence (26) follows with $N_s = 2mC_s$. If, else, $k > m$, (26) follows similarly from (25).

Q.E.D.

Certain generalizations are possible. For one, it is sufficient in the above to assume merely that $f \in C^{2m-1}[0, 1]$ with $f^{(2m-1)}$ of bounded variation. Also, it is possible to let some of the joints coalesce. Precisely, we have the

**Corollary.** Let $\pi : 0 = x_0 < x_1 < \cdots < x_n = 1$, and let $S_r$ denote the set of piecewise polynomial functions of degree $2m - 1$ which have continuous derivatives up to and including the $2m - d_r$-th at $x_i$, $d_r$ a positive integer not exceeding $m$, $i = 1, \ldots, n - 1$. If $f \in C^{2m-1}[0, 1]$, then there exists $e \in S_r$ s.t.
(28) \[ \|s^{(k)} - s^{(k)}\|_\infty \leq N_k \|f\|^{2m-k}_{2m} \|f^{(2m)}\|_\infty, \quad k = 0, \ldots, m, \]
with \(N_k\) independent of \(f\) or \(\pi\).

6. But, more important, the arguments leading up to Theorem 1 can be used to establish the analogous results for even-degree splines. Specifically, let \(x_{-m} < x_{-m+1} < \cdots < x_m\), with \(x_0 = x_m = 1\), and let \(S_{\pi}\) denote the set of spline functions of degree \(2m\) on \([0, 1]\) with (interior) joints at \(x_1, \ldots, x_{m-1}\). For \(f \in C^{(2m+1)}[0, 1]\), define \(P_{\pi}f \in S_{\pi}\) by

\[ (P_{\pi}f)(x) = \sum_{k=0}^m f^{(k)}(0) x^k / k! + \frac{1}{(2m)!} \int_0^1 Q_{\pi}(x-t, f^{(2m+1)}(t)) dt, \]

with \(Q_{\pi}\) denoting \((2m + 1)\)-point central (polynomial) interpolation. Specifically,

\[ (Q_{\pi}g)(x) = p_{j-m}(x), \quad x \in (x_{j-1/2}, x_{j+1/2}), \quad j = 0, \ldots, n, \]

where \(p_{j-m}(x)\) is the polynomial of degree \(\leq 2m\) interpolating \(g(x)\) at \(x_{j-m}, \ldots, x_{j+1/2}\), and \(x_{j+1/2} = \frac{1}{2}(x_j + x_{j+1})\). To follow [2], the definition is completed by

\[ (Q_{\pi}g)(x) = \frac{1}{2}(Q_{\pi}g(x+1) + Q_{\pi}g(x-1)), \quad \text{all} \quad x \in [0, 1], \]

although it would do just as well to define \(Q_{\pi}g\) to be left-continuous or right-continuous everywhere.

With \(e(x) = f(x) - (P_{\pi}f)(x)\), one gets, for \(k = 0, \ldots, 2m,

\[ e^{(k)}(x) = \int_0^1 E_k(x, t) f^{(2m+1)}(t) dt, \]

with

\[ E_k(x, t) = (1 - Q_{\pi}(x-t)) (2m-k)! / (2m-k)! \]

Proceeding now just as in §4, set \(s = 2m - k\) and consider

\[ g(t) = (x-t)^{2m-k} / (2m-k)! \]

Let \(\tilde{x}, \tilde{t} \in [0, 1], x_{i-1} \leq \tilde{x} \leq \tilde{t} \leq x_{i+1/2}\), say. Then

\[ E_k(\tilde{x}, \tilde{t}) = 0 \quad \text{for} \quad |j - h| \geq m. \]

Assume, without loss, that \(\tilde{x} \leq \tilde{t}\) and \(0 \leq h - j < m\). As before,

\[ |E_k(\tilde{x}, \tilde{t})| \leq C_\pi \|\pi\| \sum_i |E_i(\tilde{t})|, \]

with \(C_\pi\) independent of \(\pi\). Here \(i\) runs from \(h - m\) to \(h + m - s\).

Using once again Lemma 1, this gives

**Theorem 2.** For \(f \in C^{(2m+1)}[0, 1]\), and \(\pi : x_{-m} < \cdots < x_{m}\), with \(x_0 = 0, x_m = 1\), and \(P_{\pi}f\) given by (29), we have

\[ \|f^{(k)} - (P_{\pi}f)^{(k)}\|_\infty \leq N_k \|\pi\|^m \|f^{(2m+1)}\|_\infty, \quad k = 0, \ldots, 2m, \]

with \(N_k\) independent of \(f\). If \(k \leq m\), \(N_k\) is also independent of \(\pi\), while for \(k > m\), \(N_k\) can be bounded in terms of \(M_{\pi}\).

**Corollary.** Let \(\pi : 0 = x_0 < x_1 < \cdots < x_n = 1\), and let \(S_{\pi}\) denote the set of piecewise polynomial functions of degree \(2m\) which have continuous derivatives up to and including the \((2m + 1 - d_i)\)-th at \(x_i\), \(d_i\) a positive integer not exceeding \(m\), \(i = 1, \ldots, n - 1\). If \(f \in C^{(2m+1)}[0, 1]\), then there exists \(s \in S_{\pi}\), s.t.

\[ \|f^{(k)} - s^{(k)}\|_\infty \leq N_k \|\pi\|^m \|f^{(2m+1)}\|_\infty, \quad k = 0, \ldots, m, \]

with \(N_k\) independent of \(f\) or \(\pi\).

**REFERENCES**
