On the (bi)infinite case of Shadrin's theorem concerning the L_∞-boundedness of the L_2-spline projector

Carl de Boor

Dedicated to Professor Yurii Nikolaevich Subbotin on the occasion of his 75th birthday

Abstract. Some loose ends in Shadrin's remarkable paper are tied up.

Shadrin's theorem [S] settles a problem posed first in [B73] in the following setting.
For $k \in \mathbb{N}$, let $t := (t_i : i \in \mathbb{Z})$ be nondecreasing with $t_i < t_{i+k}$, all i, and set

$$a := \inf_i t_i, \quad \sup_i t_i =: b.$$

For each i, let

$$N_{ik}(x) := (t_{i+k} - t_i)\mathbf{\Delta}(t_i, \ldots, t_{i+k})(x - \cdot)_+^{k-1} = (\mathbf{\Delta}(t_{i+1}, \ldots, t_{i+k}) - \mathbf{\Delta}(t_i, \ldots, t_{i+k-1}))(x - \cdot)_+^{k-1}$$

be the **ith L_∞-normalized B-spline of order k for the knot sequence t**. For an arbitrary coefficient sequence $c = (c_i)$, the biinfinite sum $\sum_i N_{ik} c_i$ makes sense pointwise, i.e.,

$$\sum_i N_{ik}\, c_i : \mathbb{R} \to \mathbb{R} : x \mapsto \sum_{N_{ik}(x) \neq 0} N_{ik}(x) c_i$$

is well-defined since the last sum is finite due to the fact that $\operatorname{supp} N_{ik}$ lies in the interval $[t_i \,..\, t_{i+k}]$. Any such function $\sum_i N_{ik} c_i$ is called a **polynomial spline of order k with knot sequence t**. I'll denote the collection of all such functions by

$$\$_{kt}.$$

In particular, with

$$\ell_\infty := \ell_\infty(\mathbb{Z}), \quad L_\infty := L_\infty(a \,..\, b),$$

the map

$$\mathbf{N}_{kt} : \ell_\infty \to L_\infty : c \mapsto \sum_i N_{ik}\, c_i$$

is wellknown to be well-defined, of norm 1, and bounded below, hence boundedly invertible on its range

$$\$_{kt\infty} := \$_{kt} \cap L_\infty.$$

Now consider the linear map

$$\mathbf{M}_{kt}^{\mathrm{t}} : L_\infty \to \ell_\infty : f \mapsto (\int M_{ik} f : i \in \mathbb{Z})$$

with

$$M_{ik} := \frac{k}{t_{i+k} - t_i} N_{ik}$$

the **ith L_1-normalized B-spline of order k for the knot sequence t**, so called because

$$\int M_{ik} = \|M_{ik}\|_1 = 1,$$

hence $\|\mathbf{M}_{kt}^{\mathrm{t}}\| = 1$.

Now consider least-squares approximation to $g \in L_\infty$ from $\$_{kt\infty}$. To be sure, $g \in L_\infty$ need not have finite L_2-norm, hence it makes, offhand, no sense to talk about least-squares approximation to such g. But we can look for $f \in \$_{kt\infty}$ for which

$$\mathbf{M}_{kt}^{\mathrm{t}}(g - f) \;=\; 0,$$

a condition that characterizes f as the unique least-squares approximation to g from $\$_{kt\infty}$ for $g \in L_2(a\,..\,b)$. This raises the question whether the data map $\mathbf{M}_{kt}^{\mathrm{t}}$ is 1-1 on $\$_{kt\infty}$ which, in turn, raises the question whether the biinfinite matrix

$$(1) \qquad A_{kt} := \mathbf{M}_{kt}^{\mathrm{t}}\mathbf{N}_{kt} = [M_{ik} : i \in \mathbb{Z}]^{\mathrm{t}}[N_{jk} : j \in \mathbb{Z}] = (\int M_{ik}N_{jk} : i, j \in \mathbb{Z})$$

is invertible on ℓ_∞. If it is, then the linear map

$$P_{kt} := \mathbf{N}_{kt}(A_{kt})^{-1}\mathbf{M}_{kt}^{\mathrm{t}}$$

is a well-defined linear projector on L_∞ with range $\$_{kt\infty}$ and, for any $g \in L_\infty$, $P_{kt}g$ is the unique element f of $\$_{kt\infty}$ for which $g - f$ is **orthogonal to** $\$_{kt\infty}$ in the sense that $\mathbf{M}_{kt}^{\mathrm{t}}(g - f) = 0$. For this reason, I will call P_{kt} an L_2**-spline projector** and note that

$$\|P_{kt}\| \le \|\mathbf{N}_{kt}\|\,\|(A_{kt})^{-1}\|_\infty\,\|\mathbf{M}_{kt}^{\mathrm{t}}\| = \|(A_{kt})^{-1}\|_\infty,$$

hence the L_∞-boundedness of P_{kt} is assured once we know that A_{kt} is boundedly invertible as a map on ℓ_∞.

To be sure, while [B73] starts in this general setting, it considers L_2-spline projectors only in the *finite-dimensional* setting in which t is a *finite* knot sequence and, correspondingly, the invertibility of the Gramian is a standard result, and conjectures that

$$(2) \qquad \forall\{k \in \mathbb{N}\} \quad \sup_t \|(A_{kt})^{-1}\|_\infty < \infty,$$

with the supremum taken over all finite knot sequences t.

It is this conjecture that Shadrin settles in [S] in the sense that he proves the following

Shadrin's Theorem ([S]). *For all $k \in \mathbb{N}$,*

$$s_k := \sup_t \|(A_{kt})^{-1}\|_\infty < \infty,$$

*with the supremum taken over all finite knot sequences t that are k-***complete***, meaning that the first and the last knot appear with maximal multiplicity k.*

Shadrin [S] also considers, in his Corollary II, the biinfinite case described above and deduces (2) for that case from the finite-dimensional case, using the observation that, therefore, all principal submatrices of A_{kt} are uniformly boundedly invertible, hence so must A_{kt} be, and with the same bound. It is the purpose of the present note to clarify this argument.

There are two points of concern.

(i) The bounded invertibility of A_{kt} is deduced from the uniformly bounded invertibility of its finite principal submatrices, but no reference is given for this (nontrivial) result.

(ii) Shadrin's Theorem is proved only for k-complete finite knot sequences hence says, offhand, nothing about finite principal submatrices of A_{kt} since there is no reason for any finite section $(t_i, \ldots, t_{n+k})$ of t to be k-complete. This was completely overlooked by me twelve years ago and only recently realized by Shadrin while trying to make the point that his paper [S] covers the periodic case.

Point (i) is taken care of by the following proposition which, while not explicitly stated, is established in [B82] during the proof of Theorem 4.1 there.

Proposition. *Let A be an ℓ_∞-bounded bi-infinite matrix that maps the closed subspace*

$$c_0 := \{a \in \mathbb{R}^{\mathbb{Z}} : \lim_{|i|\to\infty} a_i = 0\}$$

of the $\|\cdot\|_\infty$-normed space $\ell_\infty = \ell_\infty(\mathbb{Z})$ of bounded bi-infinite sequences into itself.

If for some r and for all sufficiently large integer intervals J, the submatrix

$$A_J := (A_{i,j} : i \in J, j \in r + J) \in \mathbb{R}^{J\times(r+J)}$$

is invertible and

$$s := \limsup_{J\to\mathbb{Z}} \|(A_J)^{-1}\|_\infty < \infty,$$

then also A is boundedly invertible as a map on ℓ_∞; in fact, $\|A^{-1}\|_\infty = s$.

Settling Point (ii) was the start of the present note. The argument given here for it also provides a proof of the Proposition for a totally positive matrix A.

It is well-known that the Gramian A_{kt} is **totally positive** (or **tp**, for short), meaning that all its minors are nonnegative. A good up-to-date reference regarding total positivity is [P]. We need only one of the many properties of invertible totally positive matrices A of order n, namely that *their inverse is a* **checkerboard matrix**, meaning that,

$$(-1)^{i-j} A^{-1}(i,j) \geq 0, \quad i,j = 1,\ldots,n. \tag{3}$$

This follows at once from Cramer's rule which gives

$$A^{-1}(i,j) = (-1)^{i-j} \det A(\backslash j, \backslash i)/\det A, \quad i,j = 1,\ldots,n.$$

Here, and in the following, it is convenient to denote the (i,j)-entry $A_{i,j}$ of the matrix A `MATLAB`-fashion by $A(i,j)$, and use the notation $A(\backslash j, \backslash i)$ for the matrix obtained from A by omitting the jth row and ith column.

The checkerboard nature of the inverse of a tp matrix implies the following remarkable property of tp matrices which was stated in [BJP] as a known fact, but without a reference and only a hint for how to prove it. Because of the importance of this property in the present context, I give a detailed and elementary proof.

Lemma ([BJP]). *If $B \in \mathbb{R}^{n\times n}$ is invertible and tp, then, for any integer interval $\mathbf{m} \subseteq \{1,2,\ldots,n\}$, so is the principal submatrix $C := B(\mathbf{m},\mathbf{m}) \in \mathbb{R}^{\mathbf{m}\times\mathbf{m}}$ of B involving only the rows and columns of B with index $i \in \mathbf{m}$, and*

$$0 \leq (-1)^{i+j} C^{-1}(i,j) \leq (-1)^{i+j} B^{-1}(i,j), \quad i,j \in \mathbf{m}. \tag{4}$$

Proof: By the generalized Hadamard inequality for tp matrices (see, e.g., Theorem 1.21 of [P]),

$$\det B \leq \det C \det B(\backslash\mathbf{m}, \backslash\mathbf{m}),$$

with $B(\backslash\mathbf{m}, \backslash\mathbf{m})$ the principal submatrix of B complementary to C; hence the invertibility of B implies the invertibility of its principal submatrix C. The inequalities (4) follow by repeated application of the special case $\mathbf{m} = \{1,2,\ldots,n-1\}$ which in turn, by the checkerboard nature of the inverse of a tp matrix, follows from the formula

$$C^{-1} = B^{-1}(\mathbf{m},\mathbf{m}) - \frac{B^{-1}(\mathbf{m},n)B^{-1}(n,\mathbf{m})}{B^{-1}(n,n)}, \tag{5}$$

valid for that choice of $\mathbf{m}$. Finally, (5) for such $\mathbf{m}$ can be proved as follows. Since

$$\begin{bmatrix} \mathrm{id}_{\mathbf{m}} & 0 \\ 0 & 1 \end{bmatrix} = BB^{-1} = \begin{bmatrix} C & B(\mathbf{m},n) \\ B(n,\mathbf{m}) & B(n,n) \end{bmatrix} \begin{bmatrix} B^{-1}(\mathbf{m},\mathbf{m}) & B^{-1}(\mathbf{m},n) \\ B^{-1}(n,\mathbf{m}) & B^{-1}(n,n) \end{bmatrix},$$

therefore,

$$CB^{-1}(\mathbf{m},\mathbf{m}) = \mathrm{id}_{\mathbf{m}} - B(\mathbf{m},n)B^{-1}(n,\mathbf{m})$$

and

$$CB^{-1}(\mathbf{m},n) = -B(\mathbf{m},n)B^{-1}(n,n),$$

hence, with D the right side of (5),

$$\begin{aligned} CD &= C\Big(B^{-1}(\mathbf{m},\mathbf{m}) - B^{-1}(\mathbf{m},n)B^{-1}(n,\mathbf{m})/B^{-1}(n,n)\Big) \\ &= CB^{-1}(\mathbf{m},\mathbf{m}) - CB^{-1}(\mathbf{m},n)B^{-1}(n,\mathbf{m})/B^{-1}(n,n) \\ &= \mathrm{id}_{\mathbf{m}} - B(\mathbf{m},n)B^{-1}(n,\mathbf{m}) + B(\mathbf{m},n)B^{-1}(n,n)B^{-1}(n,\mathbf{m})/B^{-1}(n,n) \\ &= \mathrm{id}_{\mathbf{m}} \end{aligned}$$

which verifies (5) since it shows the right side of (5) to be a right inverse of C, hence necessarily its inverse since C is square. □

Corollary. *For all $k \in \mathbb{N}$, and all finite knot sequences t,*

(6) $$\|(A_{kt})^{-1}\|_\infty \le s_k < \infty.$$

Proof: Any finite knot sequence t can be embedded in a k-complete knot sequence r (in many ways), and, for any such choice, $A_{kt} = A_{kr}(\mathbf{m},\mathbf{m})$ for some integer interval $\mathbf{m}$, hence

$$\|(A_{kt})^{-1}\|_\infty \le \|(A_{kr})^{-1}\|_\infty \le s_k,$$

by (4) and Shadrin's Theorem. □

Such reasoning also obviates the discussion in [S: p. 70] of the case $N < 2k$ therein.

Note that Shadrin conjectures that

$$\forall\{k \in \mathbb{N}\} \quad \|P_{kt}\| \le 2k-1 \le \|(A_{kt})^{-1}\|_\infty\,.$$

We even have (6) for arbitrary infinite or biinfinite knot sequences t, as the following theorem, applied to A_{kt} in conjunction with the Corollary, implies.

Theorem. *Let $A \in \mathbb{R}^{I\times I}$ with I equal to $\mathbb{N}$ or $\mathbb{Z}$, and assume that A is tp, and* **banded** *in the sense that*

$$h := \max_{A(i,j)\ne 0} |i-j| < \infty.$$

If, for some finite s and all finite integer intervals $\mathbf{m} \subset I$, the corresponding principal submatrix $A_{\mathbf{m}} := A(\mathbf{m},\mathbf{m})$ is invertible and $\|(A_{\mathbf{m}})^{-1}\|_\infty \le s$, then also A is boundedly invertible as a linear map on $\ell_\infty(I)$, and $\|A^{-1}\|_\infty \le s$.

Proof: For any finite integer interval $\mathbf{m} \subset I$, let

$$A^{-1}_{\mathbf{m}}(i,j) := \begin{cases} A(\mathbf{m},\mathbf{m})^{-1}(i,j), & i,j \in \mathbf{m}, \\ 0, & \text{otherwise.} \end{cases}$$

Then, by the Lemma, we know that, for any integer intervals $\mathbf{m}_1 \subset \mathbf{m}_2$,

$$0 \le (-1)^{i-j} A^{-1}_{\mathbf{m}_1}(i,j) \le (-1)^{i-j} A^{-1}_{\mathbf{m}_2}(i,j), \quad i,j \in \mathbf{m}_1,$$

which shows that, for any strictly increasing sequence $\mathbf{m}_1 \subset \mathbf{m}_2 \subset \cdots$ of integer intervals and any $i,j \in I$,

$$0 \le (-1)^{i-j} A^{-1}_{\mathbf{m}_r}(i,j), \quad r = r_0, r_0+1, r_0+2, \ldots$$

is a monotone increasing sequence bounded by s, hence converges monotonely to some limit value

$$(-1)^{i-j}D(i,j).$$

The resulting biinfinite matrix D has

$$\|D\|_\infty \le s$$

and is necessarily the inverse of A which can be worked out using the assumed bandedness of A, as follows. Since, for any $i \in I$, there is an $r_i \in \mathbb{N}$ so that $|\nu - i| \le h$ implies $\nu \in \mathbf{m}_{r_i}$, therefore

$$\begin{aligned}(AD)(i,j) &= \sum_{|\nu-i|\le h} A(i,\nu) \lim_{r_i\le r\to\infty} A^{-1}_{\mathbf{m}_r}(\nu,j)\\ &= \lim_{r_i\le r\to\infty} \sum_{|\nu-i|\le h} A(i,\nu)A^{-1}_{\mathbf{m}_r}(\nu,j)\\ &= \lim_{r_i\le r\to\infty} \left\{\begin{matrix}\delta_{i,j}, & j\in\mathbf{m}_r\\ 0, & \text{otherwise}\end{matrix}\right\} = \delta_{i,j}.\end{aligned}$$

Analogously, for any $j \in I$, there is an $r_j \in \mathbb{N}$ so that $|\nu - j| \le h$ implies $\nu \in \mathbf{m}_{r_j}$, therefore

$$\begin{aligned}(DA)(i,j) &= \sum_{|\nu-j|\le h} \lim_{r_j\le r\to\infty} A^{-1}_{\mathbf{m}_r}(i,\nu)A(\nu,j)\\ &= \lim_{r_j\le r\to\infty} \sum_{|\nu-j|\le h} A^{-1}_{\mathbf{m}_r}(i,\nu)A(\nu,j)\\ &= \lim_{r_j\le r\to\infty} \left\{\begin{matrix}\delta_{i,j}, & i\in\mathbf{m}_r\\ 0, & \text{otherwise}\end{matrix}\right\} = \delta_{i,j}.\end{aligned}$$

□

Note that, by the Lemma, it is sufficient to assume that $\|(A_\mathbf{m})^{-1}\|_\infty \le s$ for all sufficiently large $\mathbf{m}$. Note also that the conclusion is unchanged if $A_\mathbf{m} := A(\mathbf{m}, r+\mathbf{m})$ for some fixed r. Note finally that the argument would even work for a matrix A that is the norm-limit of banded matrices, something [L] calls **band-dominated**.

The Theorem is complementary to Theorem 1 of [BJP] which asserts that, for any ℓ_∞-invertible tp matrix $A \in \mathbb{R}^{I\times I}$, there exists r (necessarily unique) so that, for all finite intervals $\mathbf{m} \subset I$, $A_\mathbf{m} := A(\mathbf{m}, r+\mathbf{m})$ is invertible and $(A_\mathbf{m})^{-1}$ converges, monotonely in each entry, to A^{-1} as $\mathbf{m} \to I$.

Finally, the Lemma implies that, for any knot sequence t,

$$\|(A_{kt})^{-1}\|_\infty \ge 1/\inf_i \int M_{ik}N_{ik}. \tag{7}$$

Acknowledgement I am grateful to Alexei Shadrin for a lively email exchange concerning the issues discussed in this note and a very careful and constructive reading of a near-final version.

References

[B73] C. de Boor (1973), "The quasi-interpolant as a tool in elementary polynomial spline theory", in *Approximation Theory* (G. G. Lorentz *et al.* Eds.), Academic Press (New York), 269–276.

[B82] C. de Boor (1982), "The inverse of a totally positive bi-infinite band matrix", *Trans. Amer. Math. Soc.* **274**, 45–58.

[BJP] C. de Boor, Rong-Qing Jia, and A. Pinkus (1982), "Structure of invertible (bi)infinite totally positive matrices", *Linear Algebra Appl.* **47**, 41–55.

[L] Marco Lindner (2006), *Infinite Matrices and their Finite Sections. An Introduction to the Limit Operator Method*, Frontiers in Mathematics, Birkhäuser (Basel).

[P] Allan Pinkus (2010), *Totally Positive Matrices*, CUP (Cambridge, UK).

[S] A. Yu. Shadrin (2001), "The L_∞-norm of the L_2-spline projector is bounded independently of the knot sequence: A proof of de Boor's conjecture", *Acta Math.* **187**, 59–137.