In ODEs, it is important to understand the solutions to the first-order ODE
\[ Dx(t) = Ax(t), \quad x(0) = c, \]
in which \( A \) is a linear map on some finite-dimensional vector space, \( V \), and, correspondingly, \( x : \mathbb{R} \to V \) is a \( V \)-valued function to be determined.

The formal solution is
\[ x(t) = \exp(tA)c, \]
with
\[ \exp(B) := \sum_{j=0}^{\infty} B^j / j! \]
well-defined for every linear map \( B \) on \( V \), but such a formal expression doesn’t give much insight.

Let \( \oplus_j V_j \) be a finest \( A \)-invariant direct sum decomposition for \( V \), and let \( A_j \) be the restriction of \( A \) to \( V_j \). Assuming the underlying field to be algebraically closed, there is some \( \lambda_j \) in the spectrum of \( A_j \), hence \( B := A - \lambda_j \) has a nontrivial kernel, while the sequence \( \ker B^r : r = 0, 1, \ldots \) is increasing, hence must become stationary. If \( q \) is the smallest natural number for which \( \ker B^q = \ker B^{q+1} \), then ran \( B^q \cap \ker B^q \) is trivial, hence \( V_j \) is the direct sum of \( \ker B^q \) and \( \ker B^q \) and, the direct sum decomposition being finest, it follows that ran \( B^q \) must be trivial, i.e., \( B \) is nilpotent. Thus, on \( V_j \), \( A = \lambda_j + B \), the sum of a constant (hence diagonalizable trivially and also commuting with any linear map on \( V_j \)) and a nilpotent. Since the direct sum decomposition is \( A \)-invariant, it follows that \( A = D + N \), with \( D \) diagonalizable, and \( N \) nilpotent, and \( DN = ND \).

It follows that, on \( V_j \) and with \( N_j := A_j - \lambda_j \) nilpotent of order \( q_j \),
\[ \exp(tA) = \exp(t\lambda_j + tN_j) = \exp(t\lambda_j) \exp(tN_j) = \exp(t\lambda_j) \sum_{i < q_j} (tN_j)^i / i!. \]

In particular, if \( q_j = 1 \), then \( \exp(tA) \) reduces on \( V_j \) to multiplication by the number \( \exp(t\lambda_j) \).

To this, Mike Crandall has the following to say.

Let \( p \) be any monic polynomial that annihilates \( A \) and factor it, i.e.,
\[ p =: \prod_{j=1}^{m} (\cdot - \lambda_j)^{m_j} =: p_1 \cdots p_m. \]

(If \( p \) is of minimal degree, then the spectrum of \( A \) is necessarily the set \( \{ \lambda_j : j = 1:m \} \), but that matters only when we are looking for \( m \) and the \( m_j \) here to be as small as possible). Define
\[ V_j := \ker p_j(A), \quad j = 1:m, \]
and
\[ \ell_i := \prod_{j \neq i} p_j, \quad i = 1:m. \]
Since the $\ell_i$ have no zeros in common, any nontrivial polynomial of minimal degree in

$$I(\ell_i : i = 1:m) := \sum_i \ell_i \Pi$$

must be of degree 0 (since, otherwise, by the Euclidean algorithm, there would be a polynomial of positive degree dividing each of the $\ell_i$, hence its zeros (sure to exist since we are over $\mathbb{F}$) would be common to all the $\ell_i$). In particular,

$$1 = \sum_i \ell_i q_i$$

for some $q_i \in \Pi$.

It follows that

$$1 = P_1 + \cdots + P_m,$$

with

$$P_i := \ell_i(A)q_i(A), \quad i = 1:m,$$

linear maps that commute with $r(A)$ for any $r \in \Pi$. Further, $P_i$ vanishes on each $V_j = \ker p_j(A)$ for $j \neq i$ (since $\ell_i$ contains the factor $p_j$ for each such $j$), hence $P_i = 1$ on $V_i$. On the other hand, $\text{ran} P_i \subset V_i$ since

$$p_i(A)P_i = p(A)q_i(A) = 0.$$ 

Consequently, $\text{ran} P_i = V_i$ and $P_i = 1$ on its range, hence $P_i$ is a linear projector, onto $V_i$, all $i$, and so,

$$P_i P_j = \delta_{ij}.$$ 

It follows that

$$V = \oplus_i V_i.$$ 

Further,

$$N := A - \sum_i \lambda_i P_i = \sum_i (A - \lambda_i)P_i$$

is nilpotent since $P_i P_j = 0$ for $i \neq j$, hence

$$N^q = \sum_i (A - \lambda_i)^q P_i = 0$$

for $q \geq \max_i m_i$. 

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