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Convergence of cubic spline interpolation with the not-a-knot condition

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Abstract

It is shown that cubic spline interpolation with the not-a-knot side condition converges to any C^2-interpoland without any mesh-ratio restriction as the mesh size goes to zero.

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The cubic spline interpolant to a given function agrees with that function at the knots $t_1 < \cdots < t_n$ of the spline; its complete specification requires two additional conditions. Originally, the slope at the end points t_1 and t_n was used, giving the complete spline interpolant. But, in the absence of such information, one has to try something else. One proposal is the **not-a-knot** condition [B_3: Ch. IV] in which the third derivative of Pg is made continuous at t_2 and at t_{n-1}. In effect, these next-to-boundary knots are not knots. But, whereas the convergence of complete spline interpolation has been established in various ways, there doesn't seem to be a proof of the convergence of not-a-knot cubic spline interpolation in the literature (except for [SV] in which it appears as a special case but is only treated for a uniform mesh).

The argument here follows the standard path (see [B_1], [B_2]): One considers the derived projector P'' obtained from P by the rule

(1) $$P'' D^2 g := D^2(Pg)$$

and shows it to be bounded in the uniform norm independently of the mesh, i.e., one shows that

(2) $$M := \sup_t \|P''\| < \infty$$

with

$$\|P''\| := \sup_f \|P'' f\| / \|f\|$$

and

$$\|f\| := \|f\|_\infty := \sup_{t_1 \le x \le t_n} |f(x)|.$$

Since P'' is a linear projector with range

$$S := \$_{2,s},$$

i.e., with range the span of the linear B-splines on the knot sequence

$$s := (s_i)_1^{m+2} := (t_1, t_1, t_3, t_4, \ldots, t_{n-3}, t_{n-2}, t_n, t_n), \quad \text{hence} \;\; m := n-2,$$

this implies that

$$\|D^2(g - Pg)\| = \|D^2 g - P''(D^2 g)\| \le \|1 - P''\| \; \mathrm{dist}\,(D^2 g, S) \le (1+M)\omega_{D^2 g}(|s|)$$

with ω_f the modulus of continuity of f (on $[t_1, t_n]$) and

$$|s| := \max_i \Delta s_i.$$

Consequently,

$$\|g - Pg\| \le \big((1+M)|t|^2/8\big)\,\big(|s|^2/8\big)\,\|D^4 g\|,$$

using the fact that Pg agrees with g at t and using the standard estimate for dist (D^2g, S) in case $D^2(D^2g)$ is bounded.

This leaves the hard part of the argument, namely a proof of (2), but here, too, the approach is standard, as follows.

Since Pg agrees with g at t, $P''D^2g$ agrees with D^2g at

$$\lambda_i := \int M_i \cdot , \quad i = 1, \ldots, m,$$

with $M_i := M_{i,2,t}$ the linear B-splines for the knot sequence t (and normalized to have $\int M_i = 1$). This follows from the fact that

$$2![t_1, t_{i+1}, t_{i+2}]g = \int M_i(x) D^2 g(x)\, \mathrm{d}x.$$

Hence, with $N_i := N_{i,2,s}$ the linear B-splines for the knot sequence s (and normalized to sum to 1), and $f := D^2g$, we obtain $P''f$ in the form

$$P''f =: \sum a_i\, N_i,$$

where a is the unique solution of the linear system

(3) $$Aa = (\lambda_i f)$$

with

(4) $$A := \Big(\int M_i N_j\Big).$$

Consequently,

$$\|P''f\| = \|a\|_\infty \le \|A^{-1}\|_\infty \max{}_i \Big|\int M_i f\Big| \le \|A^{-1}\|_\infty \|f\|_\infty.$$

This shows that

(5) $$\|P''\| = \|A^{-1}\|_\infty$$

and so reduces the proof of (2) to finding a uniform lower bound for A.

Since A is totally positive, a lower bound for A is obtained from any vector x for which

$$d := \min{}_j\ (-)^j (Ax)_j > 0.$$

For, the total positivity of A implies (by Cramer's Rule) that its inverse is checkerboard, i.e.,

$$(-)^{i+j} A^{-1}(i,j) > 0.$$

Therefore,

$$|x(i)| = |\sum A^{-1}(i,j)(Ax)_j| = \sum |A^{-1}(i.j)|(-)^j(Ax)_j \geq \sum |A^{-1}(i,j)|\, d$$

hence

$$\|x\|_\infty = \max{}_i |x_i| \geq \|A^{-1}\|_\infty\, d$$

or

$$\|A^{-1}\|_\infty \leq \|x\|_\infty / d. \tag{6}$$

The matrix $3A$ is tridiagonal, with typical row

$$\Delta s_i/(s_{i+2}-s_i), \quad 2, \quad \Delta s_{i+1}/(s_{i+2}-s_i), \qquad i = 3,\ldots,m-2,$$

and only the first two and last two rows deviate from this. Here are the first two rows:

$$\begin{array}{llll} 1+(\Delta t_2)/(s_3-s_1), & 2-(\Delta t_2)/(s_3-s_1) & & \\ \dfrac{(\Delta t_2)^2}{(s_3-s_1)(t_4-t_2)}, & \dfrac{1}{(t_4-t_2)}[\dfrac{2(\Delta t_2)^2+3\Delta t_2\Delta t_1}{(s_3-s_1)}+2\Delta t_3], & \Delta s_3/(t_4-t_2) & \end{array}$$

The last two rows can be obtained from this by the appropriate change in variables. The typical row of $3A$ suggests the vector $x := (-1,1,-1,1,\ldots)$ as suitable since this gives

$$b := 3Ax = (b_1, b_2, -1, 1, -1, 1, \ldots, b_{m-1}, b_m),$$

i.e., an appropriate result except, perhaps, for the first two and last two components. One can check that, as t_2 varies in the interval $[s_1, s_3]$, the first component of b becomes positive and this destroys the desired alternation. This can be helped, though, by choosing x_1 and x_2 in dependence on t_2. Specifically, I choose x in the form

$$x = (x_1, x_2, -1, 1, -1, 1, \ldots)$$

so that

$$b = (-1, 1, b_3, 1, -1, 1, \ldots),$$

with the analogous happening at the other end. This requires that $m > 4$, to avoid destructive interference of the machinations on one end with those at the other. With this assumption, though, it is sufficient to run through the details for the left end only. This means that I solve the linear system

$$\begin{aligned} A(1,1)x_1 + A(1,2)x_2 &= -1 \\ A(2,1)x_1 + A(2,2)x_2 &= 1 - A(2,3)*(-1) \end{aligned}$$

Since the system is invariant under a linear change in the independent variable, I normalize the situation to one where

$$s_1 = 0, \quad s_3 = 1$$

and define

$$k := \Delta t_2, \quad h := \Delta s_3.$$

This means that I am interested in the range $k \in [0,1]$, $h \ge 0$. In these terms, the system reads

$$\begin{bmatrix} 1+k & 2-k \\ k^2 & 3k-k^2+2h \end{bmatrix} x \;=\; (-1, k+2h).$$

Its solution is

$$x = (-5k + 2k^2 - 6h + 2hk, k + 2k^2 + 2h + 2hk)/D,$$

with

$$D := (3k + 2h + 2hk) \ge 0.$$

Now observe that, for $(k,h) \in [0,1] \times \mathbb{R}_+$,

$$\begin{aligned} x_1 - (-3) &= (4k + 2k^2 + 8hk)/D \;\ge\; 0 \quad \text{with equality only if } k = 0, \\ -1 - x_1 &= 2(1-k)(k+2h)/D \;\ge\; 0 \quad \text{with equality only if } k = 1. \end{aligned}$$

Finally,

$$x_2 = 1 - 2k(1-k)/D \;\le\; 1 \quad \text{with equality only if } k = 0 \text{ or } 1.$$

I conclude that $x_1 \in [-3,-1]$ and $x_2 \in [2/3, 1]$, hence

$$\|x\|_\infty \le 3.$$

Also, $b_3 \le -1$, since $x_2 \le 1$, while $b_i = (-)^i$ otherwise, hence

$$3d := \min{}_i (-)^i 3b_i \ge 1.$$

This shows that

$$\|A^{-1}\|_\infty \le 9 \tag{7}$$

independently of t (as long as $n \ge 6$).

Remarks (i) As $t_2 \to t_1$ and $t_{n-1} \to t_n$, not-a-knot cubic spline interpolation reduces to complete cubic spline interpolation, hence P'' becomes least-squares approximation from $\$_{2,t}$. The above argument then reduces to the standard one. In particular, the computed x is just the alternating vector $(-1, 1, -1, 1, \ldots)$ and the resulting bound is the customary one: $\|A^{-1}\|_\infty = 3$.

(ii) It may be possible to carry out the argument by perturbation, starting off with the known stability of P'' for complete cubic spline interpolation and showing that the change in the side conditions to the not-a-knot conditions is gentle enough (at least for large n) to change $\|P''\|$ by a bounded amount.

(iii) When I wrote this note in August of 1984, I wrote that "It should be possible to show the boundedness, independently of t, of the derived linear projector P', given by the rule

$$P'(Dg) \;:=\; D(Pg).$$

But the argument via total positivity would be a bit messier.' In the meantime, R.K. Beatson, who had earlier brought my attention to the fact that there did not seem to be a convergence proof for spline interpolation with the not-a-knot condition, has independently studied this question and has shown that, strictly speaking, P' is not boundable independent of the mesh. The difficulty lies in the first and last data interval. If one measures the size of $P'f$ by the number

$$\|P'f\|' \;:=\; \sup_{t_2 \le t \le t_{n-1}} |P'f(t)|,$$

then he obtains a mesh-independent bound for $\|P'\| := \sup_f \|P'f\|'/\|f\|$. See Beatson's forthcoming paper "On the convergence of some cubic spline interpolation schemes", ms., Feb. 85, Mathematics, University of Connecticut, Storrs CN 06268.

Further remarks (1986) Beatson's paper has meanwhile appeared; see [Be].

Further remarks (20 sep 95) Since the matrix in question here is tridiagonal, use of the alternating vector $x = (-1, 1, -1, 1, \ldots)$ is equivalent to using the diagonal dominance of the matrix for purposes of a bound on its inverse. Use of the modified vector $x = (x_1, x_2, -1, 1, \ldots)$ amounts to postmultiplication by an invertible diagonal matrix followed by an argument based on diagonal dominance, hence still needs no reference to total positivity.

Also, Donald Kershaw has recently pointed out to me that he proposes the not-a-knot condition in [K], though with a rather different motivation; the fact that it is actually the not-a-knot condition is stated only at the very end of the paper. In that paper, Kershaw also states, without proof, the convergence, for $g \in C^{(5)}$ and for $\Delta t_1/\Delta t_2$ bounded.

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