# Truncated power splines: acyclic resolutions of cone polynomials

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ABSTRACT

Cone polynomials in $n > 1$ variables, also known as volume polynomials and/or spline polynomials, are the polynomials that appear in the local structure of the truncated powers, hence in the local structure of any derived construction such as box splines, simplex splines, partition functions, character formulas and moment maps. The underlying geometry is determined by a real linear matroid, i.e., a real matrix $X$ $n \times N$ of rank $n$. The polynomial space itself is defined as the kernel $\mathcal{D}(X)$ of an ideal $\mathcal{J}(X)$ of differential operators, whose generators, each, are products of linear forms. While important statistics on $\mathcal{D}(X)$ (e.g., its Hilbert series) are classically known, its algebraic structure is considered to be hopelessly involved. In particular, as of today, and save a handful of truly rudimentary cases, basis constructions for $\mathcal{D}(X)$ are scarce, and provide, perhaps, neither an insight into the polynomials that make $\mathcal{D}(X)$, nor an aid in pertinent applications.

We study the above setup when $X$ is the incidence matrix of a graph $G$, and focus only on the *socle* $\mathrm{soc}(\mathcal{D}(X))$ of $\mathcal{D}(X)$, which is comprised of the top-degree homogeneous polynomials in $\mathcal{D}(X)$: the polynomial pieces that make the truncated powers span that socle only. We first resolve the ideal $\mathcal{J}(X)$ by representing it as the intersection of larger ideals, each of which a much simpler one: a complete intersection (CI) ideal. Each CI ideal $\mathcal{J}_{G_\iota}$ is induced by an acyclic directed version $G_\iota$ of the graph $G$. Kernels of CI ideals have 1-dimensional socles, and the final outcome is a resolution of $\mathrm{soc}(\mathcal{D}(X))$ into a direct sum of these 1-dimensional socles:

“lababs (0.1)
$$\mathrm{soc}(\mathcal{D}(X)) = \oplus_{G_\iota} \mathrm{soc}(\mathcal{J}_{G_\iota}\perp).$$

This decomposition can be thought of as an algebraic realization of a known combinatorial graph identity, i.e., that the number of spanning trees of the graph with 0 external activity (which is known to be equal of $\dim \mathrm{soc}(\mathcal{D}(X))$) is the same as the number of acyclic orientations of $G$ with one fixed source.

We then provide an explicit combinatorial algorithm for the construction of the 1-dimensional socles of $\mathcal{J}_{G_\iota}\perp$. This explicit construction leads to the following core, surprising, observation: when writing each polynomial in the basis provided in (0.1) as a combination of monomials, the monomial coefficients are determined by a discrete truncated power (i.e., a partition function) in dimension $n-1$. That means that not only truncated powers in $n$ dimensions are piecewise in the polynomial space $\mathrm{soc}(\mathcal{D}(X))$, but also, in a suitable sense, this latter polynomial space is canonically isomorphic to a suitable discrete truncated power space of a lower dimension. In short, cone polynomials underlie the structure of truncated powers, while truncated powers underlie the structure of cone polynomials!

## 1. Introduction

### 1.1. Outline of main results

We are interested in classes of multivariate polynomials that underlie mainstream spline approximation, most notably box spline approximation. Such splines, together with their pertinent polynomials, appear, either explicitly or implicitly, in a host of other mathematical areas: wavelet representations and CAGD are typical areas in *analysis*. Representation theory and symplectic geometry are typical examples in *algebra* and *geometry*.

The fundamental notion we tackle, indirectly, is known as *truncated power function* in analysis and as *partition function* outside analysis: these are (essentially) synonyms. These functions are piecewise-analytic (piecewise-polynomial in this article) in $n$ (real) variables $t = (t(1), \ldots, t(n))$, and are defined with the aid of a finite multiset $X \subset \mathbb{R}^n \backslash 0$: each such multiset defines a finite collection $\mathrm{TP}(X)$ of truncated powers that are intimately related one to the other. Of interest to us is the interplay between these truncated powers on the one hand, and associated classes of "volume polynomials" on the other hand.

**The interplay between polynomials and truncated powers I:** *One aspect of this interplay is the underlying polynomial structure of truncated powers.* The local pieces of any fixed truncated power function in $\mathrm{TP}(X)$ span, in the piecewise-polynomial case, the socle of a polynomial space that is denoted herein by $\mathcal{D}(X)$, and which depends only on $X$ (hence independent of the choice of the particular member in $\mathrm{TP}(X)$). The polynomials in $\mathcal{D}(X)$ are sometimes referred to as "volume polynomials", sometimes as "box spline polynomials" and sometimes as "cone polynomials". The space $\mathcal{D}(X)$ is referred to a "box spline space" as well as a "central Zonotopal Algebra space". Let us describe this space in the case when $X$ is the incidence matrix of a connected (undirected) graph $G$ with vertex set $[0{:}n]$. In what follows,

$$\Pi$$

stands for the polynomial ring $\mathbb{R}[t(1), \ldots, t(n)]$, and

$$(e_i)_{i=1}^n$$

is the standard basis for $\mathbb{R}^n$, $e_0 := 0 \in \mathbb{R}^n$. An edge $(i, j)$ in a graph with vertex set $[0{:}n]$ is commonly identified with one of the two vectors $x = \pm(e_i - e_j)$. Whenever the graph is *directed*, the edge direction $i \to j$ corresponds to the choice $x = e_j - e_i$. The matrix $X := X(G)$ whose columns are in bijection with the edges of a given graph is *the incidence* matrix of it.[1] In what follows, we identify the graph $G$ with its edge set $X(G)$; thus the statement $x \in G$ (or $x \in X(G)$) means that $x$ is an edge of $G$ (a column of $X(G)$, respectively).

One starts by defining the following polynomial ideal $\mathcal{J}(X) \subset \Pi$. An edge $x = e_j - e_i \in G$ induces the linear polynomial $p_x(t) := t(j) - t(i)$ $(t(0) := 0)$, and a (multi)subset of edges $Y \subset X$ gives rise to

$$p_Y := \prod_{x \in Y} p_x.$$

[1] If $G$ is undirected, the orientation of each edge $x \in X$ may be chosen arbitrarily.

The ideal

$$\mathcal{J}(G) := \mathcal{J}(X)$$

is then generated by (all) the *cut polynomials* $p_Y$ of $G$. A single cut $Y \subset X$ is created by a non-trivial partitioning of $[0{:}n]$ into disjoint $V_0, V_1$, with the edges in the cut $Y$ are those that connect between $V_0$ and $V_1$. Alternatively, a cut is a minimal (multi)subset of edges $Y \subset X$ whose removal from $G$ makes it disconnected. The polynomial space $\mathcal{D}(G) := \mathcal{D}(X)$ is then defined as the kernel of $\mathcal{J}(G)$:

$$\mathcal{D}(G) := \mathcal{J}(G)\perp.$$

Here, and hereafter, given any ideal $\mathcal{J} \subset \Pi$, its **kernel** $\mathcal{J}\perp$ is defined as

$$\mathcal{J}\perp := \{q \in \Pi : p(D)q = 0, \ \forall p \in \mathcal{J}\}.$$

It is well-known that $\deg q \le \#X - n$, for $q \in \mathcal{D}(G)$, with $\#X - n$ being sharp here. Actually, we are interested here only in the homogeneous polynomials of degree $\#X - n$ in $\mathcal{D}(G)$: whether graphical or not, $\mathcal{D}(X)$ is the translation-invariant closure of its top degree polynomials. Thus, in algebraic jargon, these polynomials form its socle, with the remainder of $\mathcal{D}(X)$ recovered via the differentiation of the socle polynomials. Altogether, we have just identified the socle

$$\mathrm{soc}(\mathcal{D}(X))$$

of $\mathcal{D}(X)$ as the homogeneous polynomials in that space of the degree $\#X-n$. Importantly, the truncated power is piecewise in the socle only.

Going back to the ideal $\mathcal{J}(G)$ and its kernel $\mathcal{D}(G)$, a simple count tells us that there are nearly $2^n$ cuts in the graph, and each edge appears in half of the cuts. So, $\mathcal{J}(G)$ is generated by about $2^n$ polynomials, with the sum of their degrees being at the order of $\#X \times 2^{n-1}$. There are syzygies galore in part since different cuts may surely have large intersection; so it may seem at first glance that this ideal, let alone its kernel, are brutally involved. Unfortunately, this is indeed the case: with the exception of a few special cases, the polynomial space $\mathcal{D}(X)$ is considered, more than 30 years after its introduction, hopelessly complicated. We review some pertinent literature later.

Since our object of interest is the collection of truncated powers $\mathrm{TP}(X)$ (to be defined later), and since each member in $\mathrm{TP}(X)$ is piecewise in $\mathrm{soc}(\mathcal{D}(X))$, we study in what follows that latter space.

Our analysis of the space $\mathrm{soc}(\mathcal{D}(X))$ for an incidence matrix $X$ of a graph $G$, is underlined by the collection

“defOGO (1.1) $$\overline{\mathrm{O}}(G) = \{G_\iota\}$$

of the *acyclic orientations* of $G$. Actually, this set is usually too large for our purposes, and we mostly work instead with the following smaller set:

“assump **Definition 1.2.** *Assume that the graph $G$ satisfies*

*(1) $I \subset X$, i.e., the (undirected) edges $\pm(e_i)_{i=1}^n$ appear, each, in $X(G)$ with positive multiplicity (i.e., the vertex 0 is directly connected to all other vertices).*

*We define then the subset*

$$\mathrm{O}(G)$$

*as the collection of directed versions $G_\iota$ of $G$ that satisfy:*

*(2) $G_\iota \in \overline{\mathrm{O}}(G)$: it should not contain a directed cycle (i.e., edges cannot sum up to 0).*

*(3) The edge $-e_i$, $i \in [1{:}n]$ (i.e., an edge directed into 0) is not allowed in $G_\iota$.* □

Next, it is known, [24], [3], that[2]

“dimsoc (1.3)
$$\dim(\mathrm{soc}(\mathcal{D}(G))) = \#\mathrm{O}(G).$$

We will then associate each $G_\iota \in \mathrm{O}(G)$ with an ideal

$$\mathcal{J}_{G_\iota} \subset \Pi,$$

for which $\mathcal{J}(G) \subset \mathcal{J}_{G_\iota}$. Therefore,

$$\mathcal{J}_{G_\iota}\perp \subset \mathcal{D}(G), \quad \forall G_\iota \in \mathrm{O}(G).$$

While $\mathcal{J}(G)$ has an involved structure, the ideal $\mathcal{J}_{G_\iota}$ is a simple one: it is both complete intersection and zonotopal: the former is a standard notion in ideal theory, see, e.g., §2.3 in [13], and the latter is defined in the sequel. Importantly, being a (locally) complete intersection implies that $\mathcal{J}_{G_\iota}$ has a single polynomial (up to normalization) in its socle (i.e., there is a single (in our case: homogeneous) polynomial in the kernel whose derivatives comprise the *entire* kernel). We will find explicitly that polynomial. It will be shown to be of degree $\#X - n$; thus:

“socsoc (1.4).
$$\mathrm{soc}(\mathcal{J}_{G_\iota}\perp) \subset \mathrm{soc}(\mathcal{D}(G)), \quad \forall G_\iota \in \mathrm{O}(G).$$

We will verify that the socle polynomials from different orientations $G_\iota$ are linearly independent. In this way, we will establish the complete intersection decomposition (CID):

(1.5)
$$\mathcal{J}(G) = \bigcap_{G_\iota \in \mathrm{O}(G)} \mathcal{J}_{G_\iota},$$
“complete

and will obtain a direct sum decomposition of $\mathrm{soc}(\mathcal{D}(G))$ into 1-dimensional summands:

(1.6)
$$\mathrm{soc}(\mathcal{D}(G)) = \bigoplus_{G_\iota \in \mathrm{O}(G)} \mathrm{soc}(\mathcal{J}_{G_\iota}\perp).$$
“dirsumsoc

[2] Neither reference states (1.3) explicitly. However, the first reference shows that the left hand side coincides with the number of bases in $X$ with 0 external activity, and the other reference proves the same on the right hand side.

The fact that the ideals $\mathcal{J}_{G_\iota}$ are complete intersection (and not merely Gorenstein) is important, and surely means that our basis is special.[3] Intuitively, it means that the kernels of the different ideals are "as separated from one another as possible". Examples show that the basis polynomials we obtain in this way have small monomial support, and maximal separation (one from the other) in that support. The linear algebra analog is "finding an eigenbasis which is as close to orthonormal as possible".

A short disclaimer before we continue. We highlighted above that the ideals $\mathcal{J}_{G_\iota}$ are complete intersection. We have done so merely for the benefit of our readers who are familiar with such ideals and their simplicity. However, nowhere in this paper we define this notion, and nowhere in this paper we use in our proofs any known property of such ideals. We take a different route when analysing the ideals of type $\mathcal{J}_{G_\iota}$, and use the (less common) property of *zonotopality*. To take this discussion to an extreme: a reader who is merely interested in our algorithm for constructing a basis for $\mathrm{soc}(\mathcal{D}(X))$, may shortcut the ideal construction part altogether. Our algorithm provides, for a given $G_\iota \in \mathrm{O}(G)$, a homogeneous polynomial $\mathbf{M}_{G_\iota}$ of degree $\#X - n$ which is proved to be annihilated by the differential operator $p_Y(D)$ whenever $Y$ is a cut of $G$. Ideal theory, from that perspective, provides a convenient framework for understanding and describing the results, but plays no simplifying role in the derivation of them.

Going back to our main discussion, here are some details on (1.6): given $G_\iota \in \mathrm{O}(G)$, we define a partition $(Y_i)_{i=1}^n := (Y_{i,G_\iota})_i$ of the edge set $X(G_\iota)$ in the following way

$$x = e_i - e_j,\ i \in [1{:}n],\ j \in [0{:}n] \quad \Longrightarrow \quad x \in Y_i.$$

Note that the assignment $x \to Y_i$ is determined by the orientation of $x \in G_\iota$. Then:

$$\mathcal{J}_{G_\iota} := \mathrm{Ideal}(p_{Y_i} : i \in [1{:}n]).$$

We show that the acyclicity of $G_\iota$ guarantees that each cut of $G$ contains one of the above $Y_i$ sets, hence that $\mathcal{J}(G) \subset \mathcal{J}_{G_\iota}$.

Next, it can be verified (again by invoking the acyclicity) that each $\mathcal{J}_{G_\iota}$ in (1.5) has the smallest possible kernel:

$$\dim \mathcal{J}_{G_\iota}\perp = \prod_{i=1}^{n} (\#Y_i),$$

and that $\mathcal{J}_{G_\iota}\perp$ has a socle of (linear) dimension 1, with that single polynomial being of degree $\#X - n$.

In order to make use of the decomposition (1.6), it will be valuable to find the polynomial $\mathbf{M}_{G_\iota}$ that spans $\mathrm{soc}(\mathcal{J}_{G_\iota}\perp)$. We solve this latter problem by associating each $G_\iota$ with (i) a parking function, which, for the discussion here, is nothing more than a special vector

$$\mathbf{s}(G_\iota) \in \mathbb{Z}_+^n.$$

[3] If one chooses a random non-zero polynomial in $\mathrm{soc}(\mathcal{D}(G))$, and defines an ideal $I$ to be the annihilator of this polynomial, then $I$ in general will be neither complete intersection nor zonotopal.

The vector essentially records the cardinality of the generators $(Y_i)_i$ of $\mathcal{J}_{G_\iota}$,[3]:

“defpark
$$\mathbf{s}(G_\iota) : i \mapsto \#Y_i - 1, \quad i \in [1{:}n]. \tag{1.7}$$

We then induce: (ii) a ‘flow scheme’ over the directed graph $G_\iota$. Once the parking $\mathbf{s}(G_\iota)$ flows according to the scheme, it generates a multiset

$$\mathrm{Flow}(G_\iota, \mathbf{s}(G_\iota)) \subset \mathbb{Z}_+^n.$$

Each $\beta \in \mathrm{Flow}(G_\iota, \mathbf{s}(G_\iota))$ appears there with multiplicity that is denoted as

“defmultitp
$$\mathrm{tp}(\beta) := \mathrm{tp}_{G_\iota, \mathbf{s}(G_\iota)}(\beta). \tag{1.8}$$

The parking function itself is in the flow: $\mathbf{s}(G_\iota) \in \mathrm{Flow}(G_\iota, \mathbf{s}(G_\iota))$, and is simple there: $\mathrm{tp}(\mathbf{s}(G_\iota)) = 1$. Then, with

$$\mathbf{m}_\beta(t) := t^\beta / \beta!,$$

the normalized monomial, the socle polynomial of $\mathcal{J}_{G_\iota}\perp$ is proved to be

“defmog
$$\mathbf{M}_{G_\iota} := \sum_{\beta \in \mathrm{Flow}(G_\iota, \mathbf{s}(G_\iota))} \mathbf{m}_\beta. \tag{1.9}$$

In particular, it follows that the polynomials $\mathbf{M}_{G_\iota}$ are, each, combinations of (normalized) monomials with *integer, positive* coefficients.

**The interplay between polynomials and truncated powers II:** *The other aspect of this interplay is the underlying truncated power structure of the socle polynomials of $\mathcal{D}(G)$.* It is fundamental in Approximation Theory that two related spline systems may be connected by discrete splines: box splines of large support are expressed by box splines of small support with the aid of *discrete box splines*, [8]. Truncated powers are expressed by box splines (of any support) with the aid of *discrete truncated powers*, [22]. Our case, however, is different: we express *polynomials by monomials* with the aid of discrete splines. To the best of our knowledge, this is the first time, within the spline theory literature, that such linkage is discovered.

The definition of the space $\mathcal{D}(G)$ does not require us to orient the graph $G$ and is independent of any chosen orientation. The orientation set $\mathrm{O}(G)$ is a *tool*, a “mere” tool that allows us to resolve the structure of $\mathcal{D}(G)$. In contrast, truncated powers cannot be *defined* without an orientation of $G$: there is a bijection

$$\mathcal{BT} : \overline{\mathrm{O}}(G) \to \mathrm{TP}(G) := \mathrm{TP}(X(G))$$

between the acyclic orientations of $G$ and the truncated powers associated with $X(G)$. We denote those truncated powers as $\mathrm{TP}_{G_\iota}$, $G_\iota \in \overline{\mathrm{O}}(G)$; i.e., $\mathrm{TP}_{G_\iota} := \mathcal{BT}(G_\iota)$. As said, the various truncated powers, $\mathrm{TP}_{G_\iota}$, $G_\iota \in \overline{\mathrm{O}}(G)$, are piecewise in the same polynomial space, viz. $\mathrm{soc}(\mathcal{D}(G))$.

So, any acyclic $G_\iota \in \overline{\mathrm{O}}(G)$ leads to a truncated power $\mathrm{TP}_{G_\iota}$; if, further, the acyclic $G_\iota$ lies in $\mathrm{O}(G)$, it also induces a flow polynomial $\mathbf{M}_{G_\iota}$. Then, *what is the connection, if any, between the socle polynomial* $\mathbf{M}_{G_\iota}$*, and the truncated power* $\mathrm{TP}_{G_\iota}$*?*

There are two answers to the above, of different nature: one interesting and the other fundamental. The former is that

“interest (1.10) $$\mathrm{TP}_{G_\iota}|_{\mathbb{R}^n_+} = \mathbf{M}_{G_\iota}, \quad G_\iota \in \mathrm{O}(G).$$

So each truncated power registers its “own” socle polynomial in its positive octant. The positive octant receives this “recognition” since we have chosen it as the “stabilizer” of the set $\mathrm{O}(G)$ (via the fixed orientation condition on $(e_i)_i$).

The latter connection is more fundamental. Let us define a reduction rd[4] on the graph $G$ (hence on its edge set $X(G)$): in this reduction, we remove from $G$ the vertex 0 and all the edges connected to it, reenumerate the remaining vertices ($\gamma : [1{:}n] \to [0{:}n{-}1]$ is the enumeration function), and obtain a graph

$$\mathrm{rd}(G),$$

with vertex set $[0{:}n{-}1]$. The reduction map induces naturally a map

$$\mathrm{rd} : \mathrm{O}(G) \to \overline{\mathrm{O}}(\mathrm{rd(G)}).$$

It is easy to see that this new map is a (set) bijection. The map thus induces a bijection between the flow polynomials $(\mathbf{M}_{G_\iota})$ and $\overline{\mathrm{O}}(\mathrm{rd(G)})$:

$$\mathbf{M}_{G_\iota} \mapsto \mathrm{rd}(\mathbf{M}_{G_\iota}) := \mathrm{rd}(G_\iota).$$

Our claim is that this association is not just a formality: the graph $\mathrm{rd}(G_\iota)$ records the information needed for the complete recovery of the polynomial $\mathbf{M}_{G_\iota}$. Let us explain.

An acyclic graph ($\mathrm{rd}(G_\iota)$ here) is associated not only with a truncated power, but also with a *discrete truncated power* (= a partition function): a cone function $\mathrm{tp}_{\mathrm{rd}(G_\iota)}$ defined on the $\mathbb{Z}^{n-1}$-lattice. The cone structure (i.e., the cones of polynomiality) of $\mathrm{TP}_{\mathrm{rd}(G_\iota)}$ and $\mathrm{tp}_{\mathrm{rd}(G_\iota)}$ is the same, and, in the graph case like here, $\mathrm{tp}_{\mathrm{rd}(G_\iota)}$ is also piecewise-polynomial and also in the same $\mathcal{D}(\mathrm{rd}(G))$ space.

We claim that the coefficients tp in the formula (1.8) are the values of a suitable discrete truncated power (and that is the reason we denoted them in this way). Moreover, the truncated power tp inherits the orientation of $G_\iota$![5]

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[4] The reduction map is merely a technical step. We could have avoided it by defining volume polynomials and truncated powers on linear manifolds. This is equivalent to embedding the edge set $X$ of a graph $G$ with vertex set $[0{:}n]$ in $\mathbb{R}^{n+1}$, rather than in $\mathbb{R}^n$. We have avoided this choice in the definition of $X(G)$ since we need, in any event, to select a special vertex 0 in Definition 1.2. Therefore, for consistency, we describe the current part in a similar setup. For some applications, when the initial $G$ satisfies various symmetry relations, the avoidance of the reduction map allows one to preserve better the symmetry relationships.

[5] For simplicity, we assume in the theorem that the enumeration $\gamma$ of the vertices $[1{:}n]$ is $\gamma(i) = i$, $i \in [1{:}n-1]$, $\gamma(n) = 0$.

“corintro **Theorem 1.11.** *Let $G$ be a (connected) graph with edge set $X$. Set:*

$$\mathcal{A}_G := \{\alpha \in \mathbb{Z}_+^n : \|\alpha\|_1 = \#X - n\}.$$

*Let $G_\iota \in \mathrm{O}(G)$, with parking function* $\mathbf{s}$*. Then*

$$\mathbf{M}_{G_\iota}(t) = \sum_{\alpha \in \mathcal{A}_G} \mathrm{tp}_{\mathrm{rd}(G_\iota)}(\mathrm{rd}(\alpha - \mathbf{s}))\,[t^\alpha].$$

*In particular, the monomial coefficients of $\mathbf{M}_{G_\iota}$ coincide with the values of $\mathrm{tp}_{\mathrm{rd}(G_\iota)}$ on the set $\mathrm{rd}(\mathcal{A}_G - \mathbf{s})$, and the multiplicity function $\mathrm{tp}_{G_\iota,\mathbf{s}}$ (that appears in (1.8)) coincides, on $\mathcal{A}_G$, with $\mathrm{tp}_{\mathrm{rd}(G_\iota)}(\mathrm{rd}(\cdot - \mathbf{s}))$.* □

Note that the last theorem does more than capturing the multiplicities of the flow vectors: it determined also the vectors that are not the flow. These are the vectors $\alpha \in \mathcal{A}_G$ on which $\mathrm{tp}_{\mathrm{rd}(G_\iota)}(\mathrm{rd}(\alpha - \mathbf{s}))$ vanishes.

## 1.2. Two examples

We provide here the concrete details of two examples. The first deals with the case known as two-dimensional three-directional splines. This is the most commonly used setup in bivariate spline approximations (on regular grids). Then we consider the case of complete graphs (with $n$ arbitrary). This latter setup is pertinent to the structure of the character of the irreducible representations of Lie algebras with root system $A_n$, as well as to suitable generalizations in the context of Jack polynomials.

### 1.2.1 Bivariate 3-directional spaces

Let $G$ be a graph supported on $[0{:}2]$ with the edges $(0,1), (0,2), (1,2)$ appearing with positive multiplicities $k+1, l+1, m$ respectively. The underlying $\mathcal{D}(G)$ is known as a “3-directional box spline space”, or “3-directional central zonotopal space of type $\mathcal{D}$”. Recall that we deal here with polynomials in two variables. It is essentially well-known that

$$\dim(\mathrm{soc}(\mathcal{D}(G))) = 2,$$

regardless of the values of $k, l, m$.

We begin the presentation of this case by noting that the three polynomials that are pertinent here are

$$p_1(t) = t(1)^{k+1}, \quad p_2(t) = t(2)^{l+1}, \quad p_3(t) = (t(1) - t(2))^m.$$

The ideal $\mathcal{J}(G)$ is generated by the three cut polynomials which are

$$p_1p_2, \quad p_1p_3, \quad p_2p_3.$$

In order to compute the decomposition (1.5), we need to find the acyclic orientations of $G$. The orientations of the edges $e_1, e_2$ are fixed. As for the $m$ edges $e_1 - e_2$, we must use the same orientation for all to avoid a cycle. So, we have two acyclic orientations, capturing the fact that the socle is of dimension 2. We compute the polynomial $\mathbf{M}_{G_\iota}$ corresponding to the orientation $e_2 - e_1$, i.e., $1 \to 2$.

This orientation partitions the edge set $X$ into the following two subsets: $Y_1$ consists of the $k+1$ copies of $e_1$, and $Y_2$ is the complement. Thus, the ideal $\mathcal{J}_{G_\iota}$, for this particular orientation, is generated by the two polynomials $p_1, p_2p_3$. Trivially, this ideal contains $\mathcal{J}(G)$. Note that the other orientation, $e_1 - e_2$, leads to an ideal that is generated by $p_1p_3, p_2$. It is not hard to argue then that the two new ideals are complete intersection, or to compute their Hilbert series, for example. It is further not too hard to argue directly that their intersection is $\mathcal{J}(G)$.

Now, we want to compute the polynomial $\mathbf{M}_{G_\iota}$ that spans the socle of $\mathcal{J}_{G_\iota}\perp$, for the first $G_\iota$. For this, we associate $G_\iota$ with a valuation on [1:2] that is commonly known as a *parking function*:

$$\mathbf{s} := \mathbf{s}(G_\iota) := \mathbf{s}_G(G_\iota) : i \mapsto \#Y_i - 1.$$

In words, $\mathbf{s}(G_\iota) + 1$ counts the number of edges in $G_\iota$ that are directed *into* each vertex $i > 0$. In our case, the valuation is $\mathbf{s}(1) = k,\ \mathbf{s}(2) = l+m$. The valuation defines the initial monomial

$$[t^{\mathbf{s}}] := t^{\mathbf{s}}/\mathbf{s}!.$$

(The symbol $[\cdot]$ is used to denote the fact that $t^{\mathbf{s}}$ is normalized). Additional monomials are obtained by *flowing* over $G_\iota$, i.e., given any normalized monomial $[t^\alpha]$ that is already in the flow and an edge $x \neq e_i$, the normalized monomial $[t^{\alpha+x}]$ is also in the flow, provided that $\alpha + x \in \mathbb{Z}_+^2$.

In our particular case, the value $k$ at vertex 1 can flow from that vertex to vertex 2 over the edge $e_2 - e_1$. Let $1 \leq s \leq k$. If we want to flow $s$ times from 1 to 2 over the $m$ channels (=edges) between 1 and 2, we have $\mathrm{tp}(s) := \binom{m+s-1}{s}$ different ways to execute such flow: All resulting in the flow vector $(k-s, l+m+s)$, hence in the monomial

$$[t^{(k-s,l+m+s)}].$$

So, the flow polynomial, in total, is

“socthreedir
$$\mathbf{M}_{G_\iota}(t) = \sum_{s=0}^{k} \binom{m+s-1}{s} [t^{(k-s,l+m+s)}]. \tag{1.12}$$

The other polynomial, from the other orientation $G_\iota{}'$, is then

“socthreedira
$$\mathbf{M}_{G_\iota{}'}(t) = \sum_{s=0}^{l} \binom{m+s-1}{s} [t^{(k+m+s,l-s)}]. \tag{1.13}$$

Note that these two polynomials are linearly independent; actually, their monomial supports are disjoint. So, we just found a basis for $\mathrm{soc}(\mathcal{D}(G))$.

Now, suppose that we are interested in finding lower degree polynomials in $\mathcal{D}(G)$. One option is to differentiate the above socle polynomials. The other option is to compute flows over reduced graphs. I.e., if we remove the edge $x$ from $G_\iota$ and compute the new flow, we are computing $D_x\mathbf{M}_{G_\iota}$. So for example, for $x = e_2 - e_1$,

$$D_x^r \mathbf{M}_{G_\iota} = \sum_{s=0}^{k} \binom{m+s-r-1}{s} [t^{(k-s,l+m-r+s)}] :$$

we have just replaced $m$ by $m-r$, which is all we need to account for the removal of $r < m$ copies of $x$. If $r = m$, then the edge $x$ disappears and we obtain the monomial $[t^{(k,l)}]$.

Having completed the computation of the socle polynomials, we turn our attention to their coefficients. We claimed before that those are obtained by evaluating suitably a lower-dimensional discrete truncated power. Continuing with the first $G_\iota$, we remove from $X(G_\iota)$ all edges connected to 0. We are left with $m$ copies of $e_2 - e_1$. We remove thus the first row, and obtain the matroid

$$X' = [1, \ldots, 1]_{1\times m}.$$

The discrete truncated power $\mathrm{tp}_{X'}$ is the following:

$$\mathrm{tp}_{X'}(s) := \begin{cases} q(s) := \frac{\prod_{i=1}^{m-1}(i+s)}{(m-1)!}, & s > -m, \\ 0, & s < 0. \end{cases}$$

Note that we have not erred above: the two definitions of $\mathrm{tp}_{X'}$ agree at $-1, \ldots, 1-m$: this is the "smoothness" of the discrete spline $\mathrm{tp}_{X'}$, known in other fields as "reciprocity relations". Now, $\mathrm{tp}_{X'}$ is (discrete) piecewise-polynomial of degree $m-1$. It is a fundamental solution of the difference operator $R_+^m$, with

$$R_+ f := f - f(\cdot - 1),$$

and that implies the formal power series identity

$$\sum_{s\in\mathbb{Z}} \mathrm{tp}_{X'}(s)\exp(ts) = \frac{(1-\exp(t))^{-m}}{(m-1)!},$$

where $\exp : t \mapsto e^t$.

Now, the socle polynomial is homogeneous of degree $k+l+m$, so it can be written as

$$\mathbf{M}_{G_\iota}(t) = \sum_{s=-m-l}^{k} c(s)[t^{k-s,m+l+s}],$$

for some coefficients $(c(s))_s$. Theorem 1.11 asserts that

$$c(s) = \mathrm{tp}_{X'}(\mathrm{rd}((k-s, m+l+s) - \mathbf{s})),$$

with $\mathbf{s}$ the parking function, i.e., $\mathbf{s} = (k, l+m)$. Since

$$\mathrm{rd}((k-s, l+m+s) - (k, l+m)) = \mathrm{rd}((-s, s)) = s,$$

the claim is actually that $c(s) = \mathrm{tp}_{X'}(s)$, for $s \in [-l-m{:}k]$, which is evidently the case. Note that the range of summation $s \in [0{:}k]$ in (1.12) is determined by the identity itself: whenever $s \notin [0{:}k]$, either $\mathrm{tp}_{X'}(s) = 0$ (hence creating a term that can be dismissed) or the exponent $(k-s, l+m+s)$ has a negative entry (hence creating a term which is not an acceptable monomial).

Finally, one may note that the truncated power $\mathrm{tp}_{X'}$ depends only on $m$, while the homogeneous degree $k+l+m$ of the socle polynomials surely depends on $k, l$, too. This is not a contradiction: in order to find the flow polynomial via its discrete truncated power representation, we need not only to know what discrete truncated power to construct, but also where to evaluate it. This latter evaluation interval, $[-l-m, k]$, depends, as expected, on all the three parameters. In summary, (here, as well as in the general case) the discrete truncated power $\mathrm{tp}_{\mathrm{rd}(G_\iota)}$ that appears in Theorem 1.11 is independent of the multiplicities of the edges $(e_i)_i$; those multiplicities, however, determine where to evaluate $\mathrm{tp}_{\mathrm{rd}(G_\iota)}$ for the sake of recovering the flow polynomial $\mathbf{M}_{G_\iota}$.

### 1.2.2 Complete graphs and the symmetric group

We assume in this example that every edge $(i, j)$ appears in $G$ with fixed (but arbitrary) multiplicity $k > 0$. It follows from Cayley's formula that $\dim \mathcal{D}(G) = k^n(n+1)^{n-1}$. On the other hand, the socle size is independent of $k$:

$$\dim(\mathrm{soc}(\mathcal{D}(G))) = n!.$$

In fact, this latter dimension is true even when the (positive) multiplicities of the edges are chosen in complete arbitrariness. The polynomials that make up the socle are (homogeneous) of degree $\binom{n+1}{2}k - n$. We analyse the acyclic orientation $G_\iota$ obtained by orienting each edge forward:

$$x = e_i - e_j, \quad 0 \le j < i \le n:$$

it corresponds to the order $[0{:}n]$ of the vertices. The other acyclic orientations in $\mathrm{O}(G)$ are obtained by permuting the order of the vertices $[1{:}n]$.

For the acyclic orientation that we have just chosen, the set $Y_i := Y_{i,G_\iota}$ contains $k$ copies of each edge $e_i - e_j$, $j < i$, hence $\deg p_{Y_i} = ki$. The parking valuation is thus

“defparkcomplete (1.14) $$\mathbf{s}(i) := \mathbf{s}_{n,k}(i) = ki - 1, \quad i \in [1:n],$$

and the initial monomial of the flow is then

(1.15) $$\mathbf{m}_{\mathbf{s}}(t) := [t^{\mathbf{s}}] = \prod_{i=1}^{n} \frac{t(i)^{ki-1}}{(ki-1)!}.$$

“parkcomp

The socle polynomial $\mathbf{M} := \mathbf{M}_{n,k} := \mathbf{M}_{G_\iota}$ is obtained by flowing the parking function over $G_\iota$, which means that the weights only move forward.

**Examples.** *For $n = 2$, the discussion in §1.2.1 provides complete details for the more general case of non-equal multiplicities. If we choose here $k+1 = l+1 = m$ there, we obtain that*

$$\mathbf{M}(t) = \sum_{s=0}^{m-1} \binom{m+s-1}{s} [t^{(m-s-1, 2m+s-1)}].$$

*For $n = 3$, $k = 1$ results in*

$$\mathbf{M}_{3,1}(t) = [t^{(0,1,2)}] + [t^{(0,0,3)}],$$

*while* $n = 3$, $k = 2$, *is*

“completem

$$\mathbf{M}_{3,2}(t) = \sum_{s=0}^{3}(s+1)[t^{(1,3-s,5+s)}] + 2\sum_{s=0}^{4}(2s+1)[t^{(0,4-s,5+s)}]. \tag{1.16}$$

*Finally, for* $n = 4$ *and* $k = 1$,

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$$\mathbf{M}_{4,1}(t) = [t^{(0,1,2,3)}] + [t^{(0,1,1,4)}] + [t^{(0,1,0,5)}] + [t^{(0,0,3,3)}] +$$

$$2\left([t^{(0,0,2,4)}] + [t^{(0,0,1,5)}] + [t^{(0,0,0,6)}]\right). \tag{1.17}$$

Let us examine (1.16). Let $X_{3\times 12}$ be the incidence matrix of this $(n,k) = (3,2)$ case. Removing all the $e_i$-columns and the first row, we obtain the $2 \times 6$ matrix

$$X' = [e_1, e_1, e_2, e_2, e_2 - e_1, e_2 - e_1].$$

This is the incidence matrix of the case $(n,k) = (2,2)$ (which is always the case, i.e., the reduction of general $(n,k)$ is to the setup corresponding to $(n-1,k)$). The results of §1.2.1 imply that the basis for the socle polynomials of $\mathcal{D}(X')$ is

$$Q_1(\tau) := [\tau^{(1,3)}] + 2[\tau^{(0,4)}], \quad Q_2(\tau) = Q_1(\tau(2), \tau(1)).$$

The pieces of the truncated power $\mathrm{TP}_{X'}$ are $Q_1$ and

$$Q_3(\tau) := Q_1(\tau) - Q_2(\tau) = Q_1(-\tau(1), \tau(1) + \tau(2)).$$

As to the discrete case, denoting, for $a, k, j \in \mathbb{Z}^2$,

$$a_j^k := \prod_{\ell=1}^{2} \frac{\prod_{i=j(\ell)}^{j(\ell)+k(\ell)-1}(a(\ell)+i)}{k(\ell)!},$$

we define

$$q_1(\tau) = \tau_{(1,1)}^{(1,3)} + 2\tau_{(0,0)}^{(0,4)}, \quad q_2(\tau) = \tau_{(-1,1)}^{(3,1)} + 2\tau_{(-1,0)}^{(4,0)}, \quad q_3(\tau) := q_1(\tau) - q_2(\tau) = -r_{(-1,1)}^{(1,3)} + 2r_{(0,0)}^{(0,4)},$$

with $r := (\tau(1), \tau(1) + \tau(2))$. The polynomial pieces that make $\mathrm{tp}_{X'}$ are now $q_1$ and $q_3$. One can now check that $\mathrm{tp}_{X'}$ provides the correct coefficients: with

$$\mathcal{A}_{3,9} := \{\alpha \in \mathbb{Z}_+^3 : \|\alpha\|_1 = 9\},$$

we have that

$$\mathbf{M}_{3,2}(t) = \sum_{\alpha \in \mathcal{A}_{3,9}} \mathrm{tp}_{X'}(\alpha(2) - 3, \alpha(3) - 5))\,[t^\alpha].$$

While $\#\mathcal{A}_{3,9} = 55$, the restriction of $\text{tp}_{X'}(\text{rd}(\cdot) - (3,5))$ to $\mathcal{A}_{3,9}$ is non-zero at only 9 locations, as explicitly shown in (1.16).

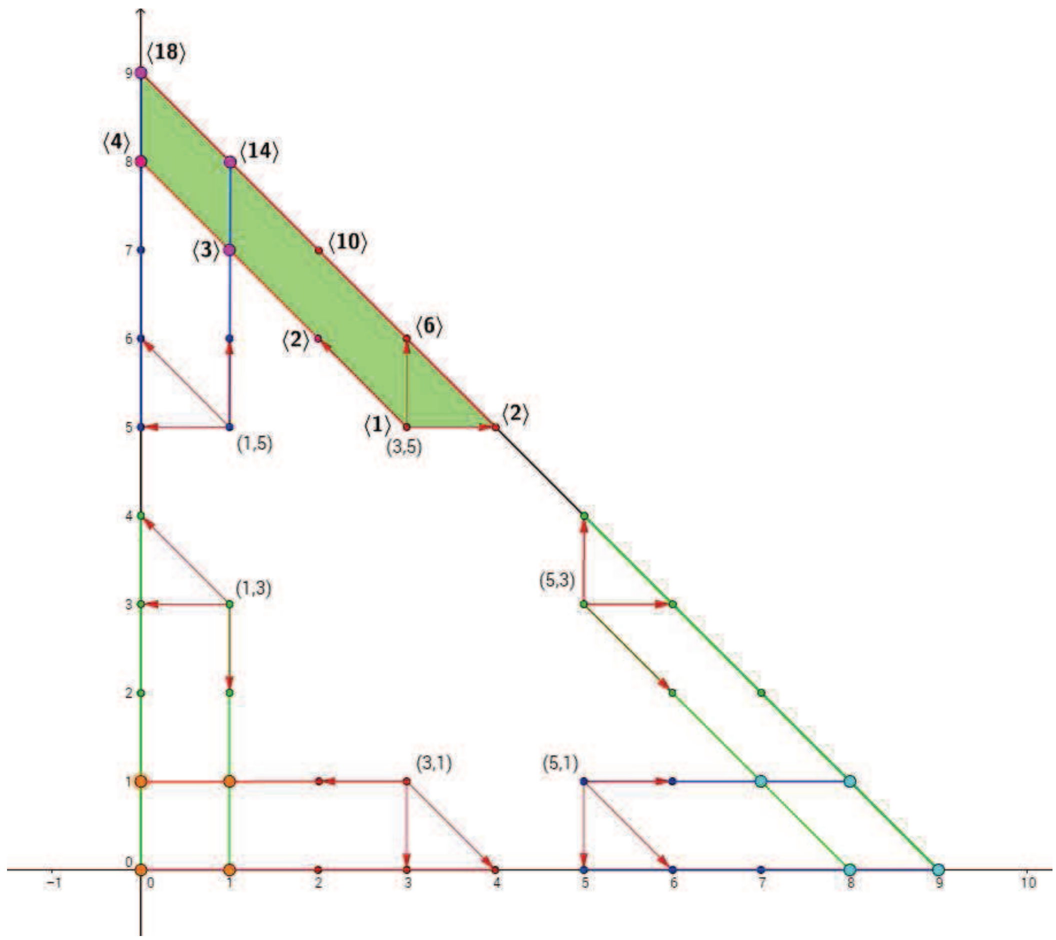


**Figure 1**: The six truncated powers that appear in the case $(n,k) = (3,2)$.

Figure 1 provides an illustration. There are six polynomials in the basis for the socle, all homogeneous of degree 9. The grid points in the big triangle $T$ with vertices $(0,0), (0,9), (9,0)$ comprise the range of the reduction of $\mathcal{A}_{3,9}$: $\text{rd}(\mathcal{A}_{3,9}) = \{\beta \in \mathbb{Z}_+^2 : \beta(1) + \beta(2) \leq 9\}$. The six parking functions are obtained by permuting the vector $(1,3,5)$, and their reduced vectors are labeled in the figure. There are six truncated powers in $\text{TP}(\text{rd}(G))$, each shifted (per Theorem 1.11) to the locations of the parking functions. The three rays emanating from the parking function form two cones of polynomiality for each truncated power: the truncated power vanishes outside those cones. The area shaded green shows the support of $\text{tp}_{X'}(\cdot - (3,5))$ within $T$: there are nine points, marked 'red' and 'pink' (with the four pink points are in the support of two different truncated powers). There are, thus, nine different monomials in the flow, and the coefficients are determined by $\text{tp}_{X'}$: the coefficients are enclosed in $\langle\rangle$. The truncated power vanishes at all other points of $T$, hence provides the correct coefficients of the flow polynomial at *all* the points of $\mathcal{A}_{3,9}$ (hence for all the monomials).

Now, there are a total of 42 monomials that are ‘active’, i.e., appear in the monomial support of some polynomial in $\mathrm{soc}(\mathcal{D}(G))$. The six polynomials in our basis are the only polynomials in $\mathrm{soc}(\mathcal{D}(G))$ with minimal monomial support. $\square$

Let us continue the discussion of the general complete graph case. With $S_n$ the symmetric group (acting on $[1:n]$, or on $[t(1),\ldots,t(n)]$, per the context), we conclude that

$$\sigma^*\mathbf{M}_{n,k} \in \mathrm{soc}(\mathcal{D}(G)), \quad \forall \sigma \in S_n.$$

One then verifies that for $\sigma \neq 1$, the initial monomial $[t^{\sigma(\mathbf{s})}]$ does not appear in the monomial support of $\mathbf{M}_{n,k}$, which implies that

$$\sigma^*\mathbf{M}_{n,k}, \quad \sigma \in S_n,$$

are linearly independent, hence form the requisite basis for $\mathrm{soc}(\mathcal{D}(G))$.

We have thus recovered the fact (well-known, at least for $k = 1$) that the entire socle here is isomorphic to the group algebra, i.e., to the vector space $V(S_n)$, whose basis elements are the members of the group $S_n$: for fixed $n$ and $k$, the socle of the corresponding $\mathcal{D}(G)$ is isomorphic to $V(S_n)$, with the bijection being

$$V(S_n) \ni v \mapsto v^*\mathbf{M}_{n,k} \in \mathrm{soc}(\mathcal{D}(G)).$$

In that bijection, the basis $S_n$ for $V(S_n)$ is mapped to the basis we constructed for $\mathrm{soc}(\mathcal{D}(G))$.

Note that the above applies, for example, to the Weyl's character of $A_n$ root system (the case $k = 1$ here): the polynomials in the various chambers of the partition function are in the same socle space, hence are determined, via the above bijection, by a suitable choice of a transformation in $V(S_n)$. That said, one will need to modify a bit the socle: the truncated power in Weyl's character formula is discrete, and we need to switch from differential operators to difference ones. The overall changes are minor, primarily since the incidence matrices are unimodular.

Finally, let us state, without proof, the following theorem which follows directly from the above discussion when combined with Zonotopal Algebra basics.

“thman **Theorem 1.18.** *Let $\mathbf{M}_{n,k}$ be the flow polynomial associated with the complete graph $G$ with $n+1$ vertices, and multiplicity $k$ on the edges, let $T$ be the path tree $0 \to 1 \to \ldots \to n$, and let $X$ be the edge set of $G$. Given $f \in \mathrm{soc}(\mathcal{D}(G))$, we have, with $\mathbf{s} := \mathbf{s}_{n,k}$ as in (1.14),*

$$f = \sum_{\sigma\in S_n} \left(p_{X\setminus\sigma(T)}(D)f\right)\ \sigma^*\mathbf{M}_{n,k}. = \sum_{\sigma\in S_n} \left(D^{\sigma(\mathbf{s})}f\right)\ \sigma^*\mathbf{M}_{n,k}.$$

$\square$

A reader who is familiar with this subject may notice how divergent the above representation is from previous efforts. Past efforts usually invoke the external activity count by Tutte: this will require an order of the edges of $X(G)$, and then the identification of the spanning trees of 0 external activity (those will strongly depend on the order). These latter trees will appear in an analog of the above theorem, with the (implicit, i.e., defined only by duality) basis for $\mathrm{soc}(\mathcal{D}(G))$ that appears there being skewed by the order we impose on $X(G)$. In our case, the spanning trees that are used in the representation are all the *path* trees (with 0 the initial vertex of the path), a collection that obviously has nothing to do with an ordering of the edges.

### 1.3. Literature

Special types of multivariate splines appear in mathematics at least as early as 1925, [31], long before Schoenberg coined the term in 1946, [30]. Multivariate splines made their official debut in Approximation Theory in 1976, [5], where de Boor proposed to define multivariate splines as inverse volumes. Micchelli, [27], and Dahmen, [14], each, provided mechanisms for computing such inverse volumes. Of particular relevance is Dahmen's approach, since he introduced and utilized to this end *multivariate truncated powers*.

It would take a few more years for the space $\mathcal{D}(X)$ to appear, concurrently with the introduction of *Box Splines*. The special bivariate 3-direction (see $1.2.1 above) appears first in the work of de Boor and DeVore in 1983, [6], with the general $\mathcal{D}(X)$ showing first in the seminal de Boor-Höllig, 1983, [7]. The local approximation properties of $\mathcal{D}(X)$ were translated by the authors there to general results on the approximation powers of box spline spaces.

The combinatorial and algebraic treatment of $\mathcal{D}(X)$ was mainly the subsequent joint work of Dahmen and Micchelli, [16],[17],[18],[20]. Of particular notice was their formula, [17], for $\dim \mathcal{D}(X)$ (the dimension is the number of bases in the matroid), which, back in 1985, was an exceptionally deep result. Also of important notice is their treatment of discrete setups, discrete truncated powers in particular, [18],[20],[19]. Twenty years later, when De Concini and Procesi rediscovered box spline theory, they were impressed by these contributions to the degree of naming $\mathcal{D}(X)$ "Dahmen-Micchelli's space", cf. [21] as well as [22].

The dual space $\mathcal{P}(X)$ was introduced by Dyn et al. in [24] (where a homogeneous basis for it was constructed, revealing thereby the Hilbert function of $\mathcal{D}(X)$), and independently by H. Hakopian. De-homogenization techniques (i.e., capturing $\mathcal{J}(X)$ as the homogenization of a radical ideal) started in [29] and by Ben-Artzi et al. in [2], and culminated by de Boor et al. in [9]. Subsequent work focused on zonotopality of generalizations of $\mathcal{D}(X)$, [10],[11],[15].

The introduction of 'Zonotopal Algebra' as an umbrella for all the efforts in the area, together with the addition of "internal" and "external" versions appear in Holtz et al., [25]. For a review of more recent work, mostly on algebraic and geometric aspects of the theory, one may consult [26],[1],[23],[28] and the references within.

Nowadays, some statistics of $\mathcal{D}(X)$ spaces are well understood. For example, the Hilbert series of $\mathcal{D}(X)$ was established, [24], via the same algorithm that computes the external activity in the Tutte polynomial, [4]. In turn this means that we have a tight hold on the *dimension* (linear dimension as a vector space) of $\mathrm{soc}(\mathcal{D}(X))$. However, with the exception of very special cases, the structure of neither $\mathcal{D}(X)$ nor its socle have been accessible to any degree.

One notable exception to the above is the use of de-homogenization to this end. Within Approximation Theory, this is related to *exponential box splines* (introduced in [29]) and some follow-up efforts that were already noted above. Another relevant effort within Approximation Theory, directly related to truncated powers, is [32]. In Algebra, the technical term is "the parametric case", [22]. The treatise of Brion-Vergne, [12], fits this paradigm: they used the explicit geometry of exponential truncated powers to compute the limit polynomial case. While the results of the present paper can also be connected to the 'de homogenization approach'[6], we forwent that discussion since that approach is essentially tangential to the principles established in the current work.

## 2. Main results

$G$ is a connected undirected graph with vertex set $[0{:}n]$ and incidence matrix $X := X(G)$. $X$ is considered also as the multiset of its columns (viz. the edges of $G$). A **cut** $L$ of $G$ is a minimal set of edges whose removal renders the graph disconnected; equivalently, these are the minimal sets in

$$L(G) := \{L \subset X : \mathrm{rank}(X \backslash L) < n\}.$$

We assume that

$$(e_i)_{i=1}^n \subset X.$$

The cut polynomials $p_L,\ L \in L(G)$, generate a polynomial ideal in $\Pi$ which is denoted $\mathcal{J}(G)$. Its kernel is denoted $\mathcal{D}(G)$. The subspace of $\mathcal{D}(G)$ of homogeneous polynomials of degree $\#X - n$ is isomorphic here to the *socle* of $\Pi/\mathcal{J}(G)$, i.e., to the minimal submodules of the latter, when considered over the ring $\Pi$. With that in mind, this top homogeneous degree subspace is denoted

$$\mathrm{soc}(G) := \mathrm{soc}(\mathcal{D}(G)).$$

Define further

$$S(G) := \{S \subset X : \mathrm{rank}(X \backslash S) = n\} :$$

these are the edge sets who removal from $G$ still leaves it connected. The space $\mathcal{D}(G)$ is complemented with the space

$$\mathcal{P}(G) := \mathrm{span}\{p_S : S \in S(G)\}.$$

It is known, [24], that $\mathcal{D}(G)$ and $\mathcal{P}(G)$ are dual to each other via the pairing

"defpairing (2.1) $$(p, q) := p(D)q(0).$$

[6] For example, we could describe explicitly the radical ideal that corresponds to each $\mathcal{J}_{G_\iota}$; the union of the varieties of the different radical ideals of the various $G_\iota$ is the variety of the ideal that appears in the context of exponential box splines.

In particular,

$$\Pi = \mathcal{P}(G) \oplus \mathcal{J}(G).$$

This decomposition is particularly elegant since each $p_Y$, $Y \subset X$, appears in one of the two summands. The decomposition stars implicitly or explicitly in several domains: for example, the Brion-Vergne interpretation of the Jeffrey-Kirwan residue theory, [12], is based on a geometric version of the above decomposition, with $\mathcal{P}(G)$ corresponding to the "free part" and $\mathcal{J}(G)$ to the "torsion part".

“degadmiss **Definition 2.2.** *Let $G_\iota$ be a directed version of $G$. We say that $G_\iota$ is* **admissible** *if the following hold:*
*(1) The orientation of the edges $x = (0, i)$ is $x = e_i$ (i.e., $0 \to i$).*
*(2) $G_\iota$ is acyclic: no edges of it sum up to zero.*
*The collection of all admissible orientations of $G$ is denoted by $\mathrm{O}(G)$.*

Given $G_\iota \in \mathrm{O}(G)$, we associate it with an ideal $\mathcal{J}_{G_\iota}$ as follows: we partition first the edge set $X$ into $n$ subsets $(Y_i := Y_{i,G_\iota})_i$ as follows:

$$x = e_i - e_j \Longrightarrow x \in Y_i.$$

Thus, the edge $x = e_i$ is always in $Y_i$, and an edge $x = \pm(e_i - e_j)$ lies in either $Y_i$ or $Y_j$, depending on its orientation. Then

“defjog (2.3) $$\mathcal{J}_{G_\iota} := \mathrm{Ideal}(p_{Y_i} : i \in [1{:}n]).$$

“joglarge **Proposition 2.4.** *$\mathcal{J}(G) \subset \mathcal{J}_{G_\iota}$, for every $G_\iota \in \mathrm{O}(G)$. Therefore,*

$$\mathcal{J}(G) \subset \cap_{G_\iota \in \mathrm{O}(G)} \mathcal{J}_{G_\iota}, \quad \textit{and} \quad \mathcal{D}(G) \supset \sum_{G_\iota \in \mathrm{O}(G)} \mathcal{J}_{G_\iota}\perp.$$

**Proof:** Let $L$ be a cut of $G$. If there exists $i$ such that $Y_i \subset L$, then $p_L \in \mathcal{J}_{G_\iota}$. Otherwise, let $V_0, V_1$ be the partition of $G$ induced by the cut, with $0 \in V_0$. Let $i_1 \in V_1$. Since, by assumption, $Y_{i_1} \not\subset L$, there exists $i_2 \in V_1$ such that $x = e_{i_1} - e_{i_2} \in G_\iota$. Continuing in this fashion we obtain an infinite sequence $(i_l)_l \subset V_1$ such that $e_{i_l} - e_{i_{l+1}} \in G_\iota$. Since $V_1$ is finite, some index appears twice, creating an oriented cycle in $G_\iota$, which is impossible, since $G_\iota$ is acyclic.

Thus, every generator $p_L$ of $\mathcal{J}(G)$ is divisible by a generator $p_Y$ of $\mathcal{J}_{G_\iota}$, and the claim follows. □

Now, the ideal $\mathcal{J}_{G_\iota}$ is *zonotopal* in the following sense: Let

$$\mathbb{B}(G)$$

be the collection (i.e., multiset) of all spanning trees of $G$. Note that $L \in L(G)$ iff $L \cap T \neq \emptyset$, for all $T \in \mathbb{B}(G)$. Let $\mathcal{B}$ be a subset of $\mathbb{B}(G)$, and define

$$L(G, \mathcal{B}) := \{L \subset X : L \cap T \neq \emptyset,\ \forall T \in \mathcal{B}\}.$$

This gives rise to

$$\mathcal{J}(G,\mathcal{B}) := \text{Ideal}(p_L : L \in L(G,\mathcal{B})), \quad \mathcal{D}(G,\mathcal{B}) := \mathcal{J}(G,\mathcal{B})\perp.$$

It follows from general arguments, [9], that

$$\dim(\mathcal{D}(G,\mathcal{B})) \geq \#\mathcal{B}.$$

One calls the setup (i.e., $\mathcal{B}$, $\mathcal{J}(G,\mathcal{B})$, $\mathcal{D}(G,\mathcal{B})$) **zonotopal** if

$$\dim(\mathcal{D}(G,\mathcal{B})) = \#\mathcal{B}.$$

In general, if one is only given an ideal $J$, then (by definition) for $J$ to be zonotopal, one needs to come up with a graph $G$ (or more generally a linear matroid $X$), and a subset $\mathcal{B} \subset \mathbb{B}(G)$ such that the resulting setup is zonotopal.

“lemzon **Lemma 2.5.** *Let $G$ be a connected graph with vertex set $[0{:}n]$ and edge multiset $X$. Let $(Y_i)_{i=1}^n$ be a partition of $X$, and assume $\mathcal{B} \subset \mathbb{B}(G)$, with*

$$\mathcal{B} := Y_1 \times Y_2 \times \ldots \times Y_n.$$

*(With '$\times$' being cartesian product). Then:*

*(1) The ideal $\mathcal{J}(G,\mathcal{B})$ is zonotopal, and hence*

$$\dim \mathcal{D}(G,\mathcal{B}) = \prod_{i=1}^{n} \#Y_i.$$

*(2) Up to normalization, $\mathcal{D}(G,\mathcal{B})$ contains a unique homogeneous polynomial of maximal degree $\#X - n$.*

We will prove the lemma shortly. We note that a similar result with an identical proof applies to a general linear matroid $X$, i.e., the result is valid beyond the graph case. Let's see first how the result applies in our case.

“propcomint **Proposition 2.6.**

*(1) The ideal $\mathcal{J}_{G_\iota}$ is zonotopal, and hence*

$$\dim \mathcal{J}_{G_\iota}\perp = \prod_{i=1}^{n} \#Y_i.$$

*(2) Up to normalization, $\mathcal{J}_{G_\iota}\perp$ contains a unique homogeneous polynomial of maximal degree $\#X - n$.*

**Proof:** Let $(Y_i)_i$ be the partition of $X(G)$ induced by $G_\iota$, and define

$$\mathcal{B} := Y_1 \times Y_2 \times \ldots \times Y_n.$$

It is then easy to see that $\mathcal{J}_{G_\iota} = \mathcal{J}(G,\mathcal{B})$. So, the stated result here will follow from Lemma 2.5, once we show that $\mathcal{B} \subset \mathbb{B}(G)$.

To this end, let $T := (y_i)_{i=1}^n \in \mathcal{B}$. Starting with some $i_0 \in [1{:}n]$, define a sequence $i_0, i_1, \ldots$ by the condition $e_{i_{l+1}} = e_{i_l} - y_{i_l}$. The sequence stops iff $i_l = 0$ (since then there is no vector $y_{i_l}$), and it must stop, since $G_\iota$, hence $T$, do not have an oriented cycle. Therefore, there is a path in $T$ from 0 to every $i \in [1{:}n]$, hence $T$ is connected, and since $\#T = n$, it is a spanning tree.

So we verified that $\mathcal{B} \subset \mathbb{B}(G)$, and the result follows. □

**Proof of Lemma 2.5.** The proof is based on the notion of "placibility" from [10]. Given any $\mathcal{B} \subset \mathbb{B}(G)$, let $x \in X$, and define

$$\mathcal{B}_1 := \{T \in \mathcal{B} : x \notin T\}, \quad \mathcal{B}_2 := \mathcal{B}\backslash\mathcal{B}_1.$$

Assume that $\mathcal{B}_i$, $i = 1, 2$ are both zonotopal. Then [10] defines the operator $M$ to be the restriction of $p_x(D)$ to $\mathcal{D}(G, \mathcal{B})$, and assumes that $x$ is placible in $\mathcal{B}$ in the sense that "for every $T \in \mathcal{B}$, there exists $y \in T$ such that $\{x \cup (T\backslash y)\} \in \mathcal{B}$". It then proves that $\ker M = \mathcal{D}(G, \mathcal{B}_2)$ while $\operatorname{ran} M = \mathcal{D}(G, \mathcal{B}_1)$.

With this result in hand, we prove the claims in the lemma by induction on $\#X$. If $\#X = n$, then $\mathcal{B}$ is a singleton, and the result is nearly trivial. Assume $\#X > n$, hence, without loss, that $\#Y_1 > 1$. It is quite obvious to check that every $x \in X$ is placible in the present $\mathcal{B}$, so let $x \in Y_1$ be arbitrary. Then we have

$$\mathcal{B}_1 = (Y_1\backslash x) \times Y_2 \times \ldots \times Y_n,$$

and

$$\mathcal{B}_2 = \{x\} \times Y_2 \times \ldots \times Y_n.$$

Since both $\mathcal{B}_i$ have the structure assumed in the lemma, the induction hypothesis applies to both (both use fewer than $\#X$ edges, since $Y_1$ is not a singleton). So, we first obtain that the two reduced setups are zonotopal, and then that $\operatorname{ran} M = \mathcal{D}(G, \mathcal{B}_1)$ and $\ker M = \mathcal{D}(G, \mathcal{B}_2)$. By induction, $\mathcal{D}(G, \mathcal{B}_2)$ has no polynomials of degree $\#X - n$, while $\mathcal{D}(G, \mathcal{B}_1)$ has a 1-dimensional socle at degree $\#X - n - 1$. This implies the statement in (2), while the statement in (1) follows from the fact that $\dim \mathcal{D}(G, \mathcal{B}) = \dim \ker M + \dim \operatorname{ran} M$ when coupled with the induction hypotheses. □

Next, we will find an explicit expression for the socle polynomial $\operatorname{soc}(\mathcal{J}_{G_\iota}\perp)$. Let us start with the definition of parking function:

"defpark **Definition 2.7.** *Let $G$ be an undirected graph as in Definition 1.2(1), and let $\mathrm{O}(G)$ be the set of all admissible orientations of $G$. Define*

$$\mathbf{s}_G : \mathrm{O}(G) \to \mathbb{Z}_+^n$$

*by*

$$\mathbf{s}_G(G_\iota) = (\deg_{G_\iota}(i) - 1 : i \in [1{:}n]),$$

*with $\deg_{G_\iota}(i)$ the in-degree of $i$ in $G_\iota$, i.e., the number of edges directed into $i$. The range $\operatorname{ran}(\mathbf{s}_G)$ is called the set of* **the (maximal) parking functions of $G$.**

Note that $\mathbf{s}_G(G_\iota)$ tracks the cardinality of the generators $(Y_i)$ of $\mathcal{J}_{G_\iota}$. The map $\mathbf{s}_G$ is injective, [3]. We actually need more than the injectivity of this map, and articulate and prove that stronger result later.

"deflabel **Definition 2.8.** *Let $G_\iota \in \mathrm{O}(G)$, and let $\alpha \in \mathbb{Z}^n$. An assignment*

$$\ell : X(G) \to \mathbb{Z}_+$$

*is called an $\alpha$-labeling of $G_\iota$ if:*
*(1)* $\ell(e_i) = 0$, $i \in [1 : n]$.
*(2)* $\alpha + \sum_{x \in G_\iota} \ell(x)x \in \mathbb{Z}_+^n$. □

We denote by

$$\mathrm{Flow}(G_\iota, \alpha)$$

the *multiset*

$$\{\alpha + \sum_{x \in G_\iota} \ell(x)x : \ell \text{ is an } \alpha\text{-label}\}.$$

We stress that the flow is a *multiset*, i.e., $\mathrm{Flow}(G_\iota, \alpha)$ is in bijection with the $\alpha$-labels.

“defflowpol **Definition 2.9.** *Let $G_\iota$ and $\alpha$ be as before. The sum*

$$\mathbf{M}_{G_\iota,\alpha}(t) := \sum_{\beta \in \mathrm{Flow}(G_\iota,\alpha)} [t^\beta]$$

*is called the flow polynomial of $\alpha$ over $G_\iota$. If $\alpha := \mathbf{s}_G(G_\iota)$, we may abbreviate*

$$\mathbf{M}_{G_\iota} := \mathbf{M}_{G_\iota, \mathbf{s}_G(G_\iota)}.$$

The main results of this section are collected in the following theorem.

“thmflow **Theorem 2.10.** *Let $G$ and $\mathrm{O}(G)$ be as in Definition 1.2.*

*(1) The socle of $\mathcal{J}_{G_\iota}\perp$, $G_\iota \in \mathrm{O}(G)$, is the span of the flow polynomial*

$$\mathbf{M}_{G_\iota}.$$

*(2) The flow polynomials*

$$\mathbf{M}_{G_\iota}, \quad G_\iota \in \mathrm{O}(G),$$

*form a basis for $\mathrm{soc}(\mathcal{D}(G))$.*

*(3) The ideal $\mathcal{J}(G)$ admits a complete intersection decomposition (CID):*

$$\mathcal{J}(G) = \bigcap_{G_\iota \in \mathrm{O}(G)} \mathcal{J}_{G_\iota}.$$

*(4) The (unnormalized) maximal ‘parking monomials’*

$$\mathbf{m}_{\mathbf{s}_G(G_\iota)}(t) := t^{\mathbf{s}_G(G_\iota)}, \quad G_\iota \in \mathrm{O}(G),$$

*form a monomial set which is biorthogonal to the socle basis in (2) above, via the pairing (2.1).*

We will prove the theorem in a few steps. In the first step we prove (1), and then (4). (2) will be an immediate consequence of the duality of (4), while (3) is surely implied by (2) when combined with the observations so far.[7]

[7] A zonotopal ideal $\mathcal{J}(G, \mathcal{B})$ is complete intersection iff $\mathcal{B} \subset \mathbb{B}(G)$ is the cartesian product of subsets of $X(G)$, which is the case here.

“cuttheflow **Lemma 2.11.** *Let* $\mathbf{M} := \mathbf{M}_{G_\iota,\alpha}$ *be a flow polynomial, with* $G_\iota \in \mathrm{O}(G)$ *and* $\alpha \in \mathbb{Z}^n$. *Let* $i \in [1{:}n]$, *and* $p_{Y_i}$ *the corresponding generator of* $\mathcal{J}_{G_\iota}$. *If* $\alpha(i) \le \mathbf{s}_G(G_\iota)(i)$ *then* $p_{Y_i}(D)\mathbf{M} = 0$.

Note that the statement of the lemma is local: the only property of $G_\iota$ which we use (other than the membership $G_\iota \in \mathrm{O}(G)$) is that *all its edges that flow* **into** $i$ *are a subset of* $Y_i$. The only needed condition on $\alpha$ is also merely on $\alpha(i)$.

**Proof:** Let $y \in Y_i$, $y \neq e_i$. We partition the $\alpha$-labels of $G_\iota$ into equivalence classes as follows:

$$\ell_1 \sim_y \ell_2 \iff \ell_1 = \ell_2 \text{ on } X(G)\backslash y.$$

Fix an equivalence class $< \ell >$, and let $\ell_0$ be its minimal member, i.e., the member with minimal $\ell(y)$. Assume $y = e_i - e_j$, $j \in [1{:}n]$ and that $\mathbf{m}$ is the monomial generated by the label $\ell_0$:

$$\mathbf{m}(t) := t^{\alpha + \sum_{x\in X} \ell_0(x)x} =: t^\beta.$$

Then the contribution of the equivalence class $< \ell >$ to the flow polynomial $\mathbf{M} := \mathbf{M}_{G_\iota,\alpha}$ is

$$q(t) = \sum_{m=0}^{\beta(j)} [\mathbf{m}(t)\Big(\frac{t(i)}{t(j)}\Big)^m] =: \sum_{m=0}^{\beta(j)} \mathbf{m}_m(t).$$

Now we apply the operator $p_y(D) = D_{e_i} - D_{e_j} =: D_i - D_j$ to $q$. We observe that, for $m = 0, \ldots, \beta(j) - 1$, $D_i\mathbf{m}_{m+1} - D_j\mathbf{m}_m = 0$. Also, $D_j\mathbf{m}_{\beta(j)} = 0$, too. Therefore,

$$p_y(D)q = D_i[\mathbf{m}].$$

This last identity is surely valid if $y = e_i$: in that case, each equivalence class is a singleton $\mathbf{m}$, hence $q = [\mathbf{m}]$, and therefore, trivially, $p_y(D)q = D_i[\mathbf{m}]$.

Now, assume that $\ell_0(y) > 0$. Define $\ell_1$ by $\ell_1 = \ell_0$ on $X\backslash y$, while $\ell_1(y) = \ell_0(y) - 1$. Since $\ell_0$ is minimal, $\ell_1$ is not an $\alpha$-label, which means that $\beta(i) = 0$ (the label $\ell_1$ corresponds to the flow vector $\beta - y$. If $\ell_1$ is not a label, the vector $\beta - y$ must contain a negative entry; since $\beta$ does not have such entry, it must be that $\beta(i) = 0$). Thus, in this case $D_i[\mathbf{m}] = 0$, which means that $p_y(D)$ annihilates the entire class: $p_y(D)q = 0$.

Otherwise, $\ell_0(y) = 0$, which means that the edge $y$ is not used in the labeling by $\ell_0$, and we readily conclude (by applying the above to all equivalence classes) that

$$p_y(D)\mathbf{M}_{G_\iota,\alpha} = \mathbf{M}_{G_\iota\backslash y,\alpha - e_i}.$$

Now, let $Y' \subset Y_i$ with cardinality $k := \#Y_i - 1 \ge \alpha(i)$. Repeating the above $k$ times we obtain

$$p_{Y'}(D)\mathbf{M}_{G_\iota,\alpha} = \mathbf{M}_{G_\iota\backslash Y',\alpha - ke_i}.$$

With $\gamma := \alpha - ke_i$, we have that $\gamma(i) \le 0$. Let $y$ now be the remaining vector in $Y_i$. Then the previous argument implies that $p_y(D)\mathbf{M}_{G_\iota\backslash Y',\gamma} = 0$, since, now, the minimal monomial $\mathbf{m}$ in the above argument is always missing the $i$th variable (since the only edge directed into $i$ is $y$, and the initial value $\gamma(i)$ is non-positive). □

“parklemma **Lemma 2.12.** *Let $G_\iota, G_\iota' \in \mathrm{O}(G)$ be different. Then*

$$\mathbf{s}_G(G_\iota) \notin \mathrm{Flow}(G_\iota', \mathbf{s}_G(G_\iota')).$$

Note that the flow of different parking functions over their corresponding oriented graphs can have some overlaps: such overlaps is the rule (though there are interesting exceptions). The claim above only asserts that the parking function itself cannot be captured in the flow of another $G_\iota$!

**Proof:** By induction on $n$, with the case $n = 1$ being trial.

The acyclic direction on $G_\iota'$ induces a partial ordering on the vertices: $i \prec j$ if there exists a (forward) path from $i$ to $j$ in $G_\iota'$. Let $i \in [1{:}n]$ be maximal in that order, and let $Y \subset X$ be the edges connected to $i$ in $G$. Then, with $k := \#Y - 1$, we have that

$$\mathbf{s}_G(G_\iota')(i) = k,$$

and that for every $\alpha = \mathbf{s}_G(G_\iota') + X\ell \in \mathrm{Flow}(G_\iota', \mathbf{s}_G(G_\iota'))$,

$$\alpha(i) \geq k,$$

with equality iff the label $\ell$ vanishes on $Y$. Now, necessarily, $\mathbf{s}_G(G_\iota)(i) \leq k$. So, if, with $\alpha$ as above, $\alpha = \mathbf{s}_G(G_\iota)$, then $s_G(G_\iota)(i) = k$, and $\ell$ vanishes on $Y$. The former means that $Y$ is oriented in $G_\iota$ the same way it is oriented in $G_\iota'$, i.e., each $y \in Y$ is directed in $G_\iota$ *into* $i$.

So, we remove from $G$ the vertex $i$ and the edge set $Y$, and obtain a reduced graph $\mathrm{rd}(G)$, and similarly reduced $\mathrm{rd}(G_\iota)$ and $\mathrm{rd}(G_\iota')$. The parking functions remain the same, save the fact that the $i$th entry disappears. So, with $\ell'$ the restriction of $\ell$ to $X \backslash Y$, we still have $\mathrm{rd}(\alpha) = \mathbf{s}_{\mathrm{rd}(G)}(\mathrm{rd}(G_\iota')) + (X\backslash Y)\ell'$, and that

$$\mathrm{rd}(\alpha) = \mathbf{s}_{\mathrm{rd}(G)}(\mathrm{rd}(G_\iota)).$$

Here, $\mathrm{rd}(\alpha)$ is obtained from $\alpha$ by removing the $i$th entry. By induction, this implies that $\mathrm{rd}(G_\iota) = \mathrm{rd}(G_\iota')$, hence that $G_\iota$ and $G_\iota'$ are oriented the same on $X\backslash Y$, too. □

**Proof of Theorem 2.10.** By Lemma 2.11, all the generators $p_{Y_i}$ of $\mathcal{J}_{G_\iota}$ annihilate the flow polynomial $\mathbf{M}_{G_\iota,\alpha}$ provided that $\alpha(i) < \#Y_i$, $\forall i \in [1{:}n]$. $\alpha := \mathbf{s}_G(G_\iota)$ satisfies this condition, hence $\mathbf{M}_{G_\iota,\mathbf{s}_G(G_\iota)} \in \mathcal{J}_{G_\iota}\perp$, and is clearly of degree $\#X - n$. Combining the above with (2) of Proposition 2.6, we obtain (1) of Theorem 2.10.

Having verified (1), we combine it with Proposition 2.4, to conclude that the polynomials $\mathbf{M}_{G_\iota,\mathbf{s}_G(G_\iota)}$, $G_\iota \in \mathrm{O}(G)$, are all in $\mathrm{soc}(\mathcal{D}(G))$. Lemma 2.12 then applies to show that these polynomials are linearly independent. In view of (1.3), we obtain assertion (2), which directly implies assertion (3).

Lemma 2.12 also implies that

$$D^{\mathbf{s}_G(G_\iota)}\mathbf{M}_{G_\iota',\mathbf{s}_G(G_\iota')} = 0, \quad G_\iota, G_\iota' \in \mathrm{O}(G),\ G_\iota \neq G_\iota'.$$

Furthermore, it is easy to conclude the initial value $\alpha$ in the $\mathrm{Flow}(G_\iota, \alpha)$ appears only once (i,e., with label $\ell = 0$), hence

$$D^{\mathbf{s}_G(G_\iota)}\mathbf{M}_{G_\iota,\mathbf{s}_G(G_\iota)} = 1, \quad G_\iota \in \mathrm{O}(G),$$

and (4) thus follows. □

## 3. Flow polynomials and truncated powers

We asserted in the Introduction (§1) two results that connect flow polynomials to truncated powers. We prove those results in this section: (1.10) is proved as Theorem 3.3 in the first subsection below, while Theorem 1.11 is proved in the second subsection.

### 3.1. Continuous truncated powers

Let $X$ be an $n \times N$ real matrix of full rank $n$, considered as defined only on $\mathbb{R}_+^N$, and satisfying the acyclicity condition

“acyc (3.1) $$x \subset \mathrm{pos}(X) := X(\mathbb{R}_+^N) \text{ contains no non-zero subspaces of } \mathbb{R}^n.$$

The truncated power $\mathrm{TP}_X : \mathbb{R}^n \to \mathbb{R}$ is defined by

$$\mathrm{TP}_X(t) := \mathrm{vol}(X^{-1}t),\ t \in \mathbb{R}^n.$$

Note that, by definition, only vectors with non-negative entries are included in the pre-image $X^{-1}t$. Also, “vol” is taken with respect to the Lebesgue measure in $\mathbb{R}^{N-n}$. Since the pre-image $X^{-1}t$ only lies on a linear manifold of dimension $N-n$ (rather than in $\mathbb{R}^{N-n}$), the above definition is “up to a multiplicative constant”. One can normalize $\mathrm{TP}_X$ by requiring it to be a fundamental solution of $p_X(D)$. This is related to the following properties of the truncated power:

“TPres **Result 3.2.** *The truncated power satisfies the following:*
*(1) There exists an open dense subset $O \subset \mathbb{R}^n$ such that $p_L(D)\mathrm{TP}_X = 0$ on $O$ for every $L \in L(X)$.*
*(2) $p_S(D)\mathrm{TP}_X = \mathrm{TP}_{X\backslash S}$ for every $S \in S(X)$.*
*(3)* $\mathrm{supp}\,\mathrm{TP}_X = \mathrm{pos}(X)$.
*(4) If $X = X(G)$ for some graph $G$, then $\mathrm{TP}_X|_{\mathbb{R}_+^n}$ is a polynomial.*

Given $X$ as above, the family $\mathrm{TP}(X)$ of truncated powers is comprised of truncated powers $\mathrm{TP}_\Xi$, with $\Xi$ obtained from $X$ by changing the signs of a subset of the columns of $X$, while preserving the condition (3.1) (with $X$ there replaced by $\Xi$).

In the rest of this section, we assume that $X = X(G)$ is the incidence matrix of a graph $G$, and, given $G_\iota \in \overline{\mathrm{O}}(G)$, we denote

$$\mathrm{TP}_{G_\iota} := \mathrm{TP}_{X(G_\iota)}.$$

The acyclicity assumption on $G_\iota \in \overline{\mathrm{O}}(G)$ is equivalent to the acyclicity assumption (3.1), provided that $X = X(G_\iota)$ there. Therefore, for the case here,

$$\mathrm{TP}(G) := \mathrm{TP}(X(G)) = \{\mathrm{TP}_{G_\iota} : G_\iota \in \overline{\mathrm{O}}(G)\}.$$

The fact that $\mathrm{TP}_{G_\iota}$ is piecewise in $\mathcal{D}(G)$ follows from (1) above and the definition of $\mathcal{D}(G)$. As said before, more is true: $\mathrm{TP}_{G_\iota}$ is piecewise in $\mathrm{soc}(\mathcal{D}(G))$. (4) in Result 3.2 merely asserts that in the graph case the positive octant is a domain of polynomiality for $\mathrm{TP}_{G_\iota}$, i.e., $\mathbb{R}_+^n \subset O$, with $O$ as in (1) of Result 3.2.

“tpflow **Theorem 3.3.** *Let* $G_\iota \in \mathrm{O}(G)$. *Then*

$$\mathrm{TP}_{G_\iota}|_{\mathbb{R}^n_+} = \mathbf{M}_{G_\iota}.$$

**Proof:** Set $f := \mathrm{TP}_{G_\iota}|_{\mathbb{R}^n_+}$, and $X := X(G_\iota)$. Let $(Y_i)_i$ be as in (2.3). Note that

$$t \in \mathrm{pos}(X\backslash Y_i) \Longrightarrow t(i) \le 0,$$

since $Y_i$ is comprised of all $y \in X$ with $y(i) > 0$. Thus, $\mathrm{pos}(X\backslash Y_i) \cap \mathbb{R}^n_+$ is a null set.

Now, assume $Y_i \in S(X)$. Then, by Result 3.2, $p_{Y_i}(D)\mathrm{TP}_X = \mathrm{TP}_{X\backslash Y_i}$. At the same time, $\mathrm{supp}(\mathrm{TP}_{X\backslash Y_i}) = \mathrm{pos}(X\backslash Y_i)$. Thus, $\mathrm{supp}(p_{Y_i}(D)\mathrm{TP}_X)$ is essentially disjoint of $\mathbb{R}^n_+$, and therefore $p_{Y_i}(D)f = 0$ on $\mathbb{R}^n_+$. Since $f$ is a polynomial (Result 3.2(4)), we obtain $p_{Y_i}(D)f = 0$ everywhere.

If, on the other hand, $Y_i \in L(X)$, then, by (1)&(4) of Result 3.2, $p_{Y_i}(D)\mathrm{TP}_X$ vanishes on $\mathbb{R}^n_+$, and, again, we conclude that $p_{Y_i}(D)f = 0$.

Varying $i$ over $[1{:}n]$ we obtain that $f \in \mathcal{J}_{G_\iota}\perp$. Now, since $f$ is homogeneous of degree $\#X - n$ (being in the socle of $\mathcal{D}(G)$), and since the only homogeneous polynomial of that degree in $\mathcal{J}_{G_\iota}\perp$ is (up to normalization) $\mathbf{M}_{G_\iota}$ ((1) of Theorem 2.10), it follows that $f = c\mathbf{M}_{G_\iota}$. Since $\mathrm{TP}_{G_\iota}$ is defined only up to normalization, our proof is complete. □

### 3.2. Discrete truncated powers

Let $X$ be an $n \times N$ integer matrix, satisfying the acyclicity condition (3.1). The discrete truncated power $\mathrm{tp}_X$ is then defined as

$$\mathrm{tp}_X : \mathbb{Z}^n \to \mathbb{Z}_+, \quad \mathrm{tp}_X(t) := \#\{X^{-1}t \cap \mathbb{Z}^N_+\}.$$

If $X$ is a singleton $x$ (i.e., $N = 1$), then clearly

$$\mathrm{tp}_x(t) = \begin{cases} 1, & t \in x\mathbb{Z}_+, \\ 0, & \text{otherwise.} \end{cases}$$

The general $\mathrm{tp}_X$ is the convolution of product of $\mathrm{tp}_x$, $x \in X$. If $X$ is the incidence matrix of a graph $G$, then $\mathrm{tp}_X$ is piecewise in $\mathcal{D}(G)$: furthermore, it is piecewise in a suitable discrete analog of $\mathrm{soc}(\mathcal{D}(G))$.

The fact that the flow polynomial is determined by a discrete truncated power (Theorem 1.11) follows from Theorem 2.10 almost directly: it is essentially a matter of comparing definitions, as we now explain.

Recall the definition of the index set $\mathcal{A}_G$ from Theorem 1.11. Let $G_\iota \in \mathrm{O}(G)$. Since the flow polynomial $\mathbf{M}_{G_\iota}$ is homogeneous of degree $\#X - n$, then it can be written as

$$\mathbf{M}_{G_\iota}(t) = \sum_{\beta \in \mathcal{A}_G} c(\beta)[t^\beta],$$

for some suitable coefficients $(c(\beta))$. According to (1.8), the multiplicity ($\mathrm{tp}(\beta)$, same as $c(\beta)$ here, only that we now consider also terms with 0 coefficients) of $\beta \in \mathrm{Flow}(G_\iota, \alpha)$ is defined as

$$\#\{\ell \in \mathbb{Z}_+^X : \ell \text{ is an } \alpha\text{-label and } \beta - \alpha = X\ell\}.$$

Now, let $X'$ be obtained from $X$ by removing the $e_i$-edges, and let $\ell'$ be the restriction of $\ell$ to $X'$. Then, $X\ell = X'\ell'$, and the condition '$\ell$ is an $\alpha$-label and $\beta - \alpha = X\ell$', is stated equivalently on $\ell'$ as '$\ell' \in \mathbb{Z}_+^{X'}$, and $\beta - \alpha = X'\ell'$.' Consequently,

$$c(\beta) = \#\{\ell \in \mathbb{Z}_+^X : \ell \text{ is an } \alpha\text{-label and } \beta - \alpha = X\ell\}$$

$$= \#\{\ell' \in \mathbb{Z}_+^{X'} : \beta - \alpha = X'\ell'\} = \mathrm{tp}_{X'}(\beta - \alpha),$$

with the first equality by the definition of the flow polynomial $\mathbf{M}_{G_\iota,\alpha}$, the last equality by the definition of $\mathrm{tp}_{X'}$, and the middle one argued above.

This is essentially the claim made in Theorem 1.11, only that there we reduced the matrix $X'$ to the matrix $\mathrm{rd}(X)$ by removing one the rows of $X'$ (and removing then the corresponding entry from $\beta - \alpha$, to obtain $\mathrm{rd}(\beta - \alpha)$.) This is possible, and sometimes desired, to do since $\mathrm{rank}(X') < n$. But, as we noted before, we could have stated the theorem in terms of $X'$ here rather than $\mathrm{rd}(X)$ there.

The use of the parking function $\mathbf{s}(G_\iota)$ as the initial vector in Theorem 1.11 is surely important there, since one needs to shift $\mathrm{tp}_{X'}$ correctly in order to align its values with the monomial coefficients of $\mathbf{M}_{G_\iota}$. But, as the argument above shows, all flows over $G_\iota$ result in multiplicities of the flow vectors that are determined by $\mathrm{tp}_{X'}$, with the initial seed impacting only the requisite alignment. Indeed, as we noted before too, $X'$, hence $\mathrm{tp}_{X'}$, are independent of the multiplicities of the edges $e_i$ in $G$. In contrast, the graph $G$, the flow polynomial $\mathbf{M}_{G_\iota}$, the index set $\mathcal{A}_G$, and the associated parking function $\mathbf{s}(G_\iota)$, depend, all, on those multiplicities. So, the values of $\mathrm{tp}_{X'}$ determine the coefficients of flow polynomials of multiple (closely related) setups.

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