This chapter promotes, details and exploits the fact that (univariate) splines, i.e., smooth piecewise polynomial functions, are weighted sums of B-splines.

1. Piecewise polynomials

A **piecewise polynomial** of **order** k with **break sequence** $\boldsymbol{\xi}$ (necessarily strictly increasing) is, by definition, any function f that, on each of the half-open intervals $[\xi_j \mathinner{..} \xi_{j+1})$, agrees with some polynomial of degree $< k$. The term 'order' used here is not standard but handy.

Note that this definition makes a piecewise polynomial function **right-continuous**, meaning that, for any x, $f(x) = f(x+) := \lim_{h\downarrow 0} f(x+h)$. This choice is arbitrary, but has become standard. Keep in mind that, at its break ξ_j, the piecewise polynomial function f has, in effect, two values, namely its limit from the left, $f(\xi_j-)$, and its limit from the right, $f(\xi_j+) = f(\xi_j)$.

The set of all piecewise polynomial functions of order k with break sequence $\boldsymbol{\xi}$ is denoted here

$$\Pi_{<k,\boldsymbol{\xi}}.$$

2. B-splines defined

B-splines are defined in terms of a **knot sequence** $\mathbf{t} := (t_j)$, meaning that

$$\cdots \le t_j \le t_{j+1} \le \cdots.$$

The j**th B-spline of order 1 for the knot sequence t** is the characteristic function of the half-open interval $[t_j \mathinner{.\,.} t_{j+1})$, i.e., the function given by the rule

$$\mathrm{B}_{j1}(x) := \mathrm{B}_{j,1,\mathbf{t}}(x) := \begin{cases} 1, & \text{if } t_j \le x < t_{j+1}; \\ 0, & \text{otherwise.} \end{cases}$$

Note that each of these functions is piecewise constant, and that the resulting sequence (B_{j1}) is a **partition of unity**, i.e.,

$$\sum_j \mathrm{B}_j(x) = 1, \quad \inf_j t_j < x < \sup_j t_j.$$

In particular,

$$t_j = t_{j+1} \text{ implies } \mathrm{B}_{j1} = 0.$$

From these first-order B-splines, B-splines of higher order can be derived inductively by the following **B-spline recurrence**.

(2.1) Property (i): Recurrence relation. *The jth B-spline of order $k > 1$ for the knot sequence* $\mathbf{t}$ *is*

(2.2) $$\mathrm{B}_{jk} := \mathrm{B}_{j,k,\mathbf{t}} := \omega_{jk}\mathrm{B}_{j,k-1} + (1 - \omega_{j+1,k})\mathrm{B}_{j+1,k-1},$$

with

(2.3) $$\omega_{jk}(x) := \omega_{j,k,\mathbf{t}}(x) := \frac{x - t_j}{t_{j+k-1} - t_j}.$$

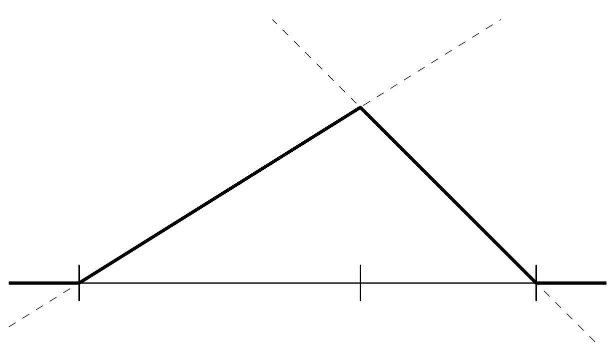

Figure 2.4 The functions ω_{j2} and $1-\omega_{j+1,2}$ (dashed), and the linear B-spline B_{j2} (solid) formed from them.

For example, the jth **second-order** or **linear** B-spline is given by

$$\mathrm{B}_{j2} = \omega_{j2}\mathrm{B}_{j1} + (1 - \omega_{j+1,2})\mathrm{B}_{j+1,1},$$

and so consists of two nontrivial linear pieces and is continuous, unless there is some equality in the inequalities $t_j \le t_{j+1} \le t_{j+2}$.

In order to appreciate just how remarkable the recurrence relation is, consider the 3rd-order B-spline

$$\mathrm{B}_{j3} = \omega_{j3}\mathrm{B}_{j2} + (1 - \omega_{j+1,3})\mathrm{B}_{j+1,2}$$

in the generic case, i.e., when $t_j < t_{j+1} < t_{j+2} < t_{j+3}$. As is illustrated in Figure (2.5), both summands have corners (i.e., jumps in their first derivative), but these corners appear to be perfectly matched so that their sum is smooth.

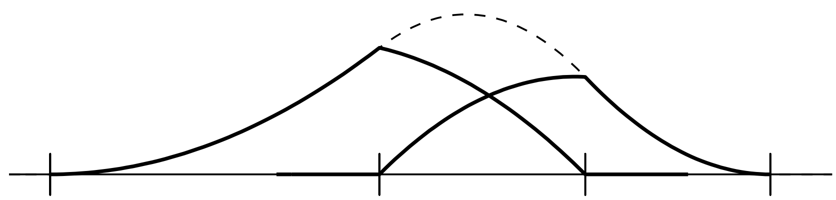

Figure 2.5 The two functions $\omega_{j3}\mathrm{B}_{j2}$ and $(1 - \omega_{j+1,3})\mathrm{B}_{j+1,2}$ have corners, but their sum, B_{j3}, does not.

3. Support and Positivity

Directly from (2.2) by induction on k,

$$\mathrm{B}_{jk} = b_j\mathrm{B}_{j1} + \cdots + b_{j+k-1}\mathrm{B}_{j+k-1,1},$$

with each b_r a product of $k-1$ polynomials of (exact) degree 1, hence a polynomial of (exact) degree $k-1$. This shows B_{jk} to be a piecewise polynomial of order k, with breaks at $t_j, \ldots, t_{j+k}$.

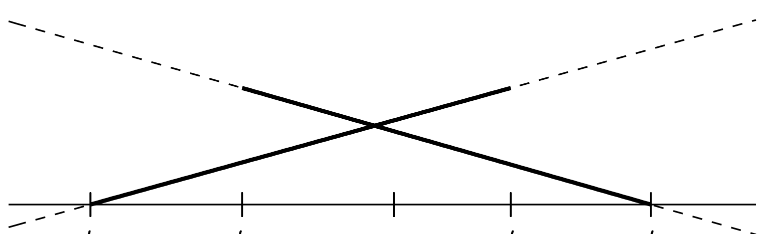

Figure 3.1 The two weight functions, ω_{jk} and $1-\omega_{j+1,k}$, in (2.2) are positive on $\mathrm{supp}(\mathrm{B}_{jk}) = (t_j \mathinner{..} t_{j+k})$.

Further, B_{jk} is zero off the interval $[t_j \mathinner{..} t_{j+k}]$. Hence, since both ω_{jk} and $1 - \omega_{j+1,k}$ are positive on the interval $(t_j \mathinner{..} t_{j+k})$, it follows, by induction on k, that B_{jk} is positive there.

(3.2)Property (ii): Support and positivity. *The B-spline* $B_{j,k} = B_{j,k,\mathbf{t}}$ *is piecewise polynomial of order* k *with at most* k *nontrivial polynomial pieces, and breaks only at* $t_j, \ldots, t_{j+k}$*, vanishes outside the interval* $[t_j \mathinner{\ldotp\ldotp} t_{j+k})$*, and is positive on the interior of that interval, that is,*

(3.3) $$B_{j,k}(x) > 0, \quad t_j < x < t_{j+k},$$

while

(3.4) $$t_j = t_{j+k} \quad \Longrightarrow \quad B_{jk} = 0.$$

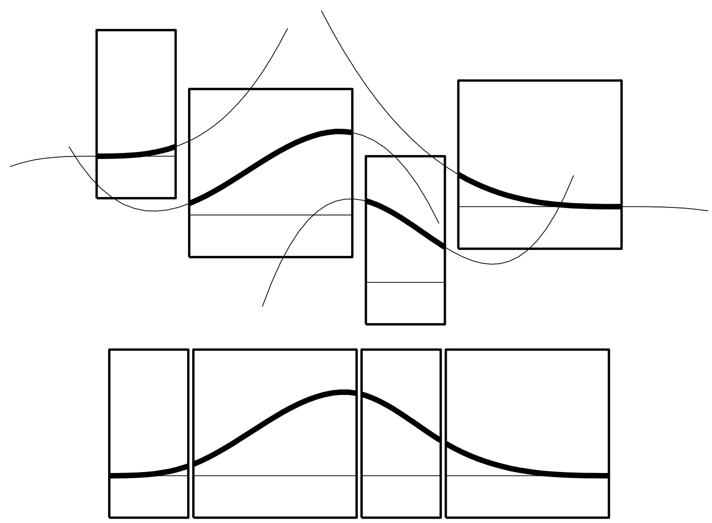

Figure 3.5 The four cubic polynomials whose pieces join to form a certain cubic B-spline.

Notice that B_{ik} is completely determined by the $k+1$ knots $t_i, \ldots, t_{i+k}$. For this reason, the notation

$$B(\cdot|t_i, \ldots, t_{i+k}) \ := \ B_{i,k,\mathbf{t}} = B_{ik}$$

is sometimes used. Other notations in use include

$$N_{ik} := B_{ik} \ \text{ and } M_{ik} := \bigl(k/(t_{i+k} - t_i)\bigr)B_{ik}.$$

The latter is special in that

$$\int_{\mathbb{R}} M_{ik} = 1,$$

as follows from (11.4).

The many other properties of B-splines are derived most easily by considering not just one B-spline but the linear span of *all* B-splines of a given order k for a given knot sequence $\mathbf{t}$. This brings us to splines.

4. Spline spaces defined

A **spline of order k with knot sequence $\mathbf{t}$** is, by definition, a linear combination of the B-splines B_{ik} associated with that knot sequence. We denote by

$$S_{k,\mathbf{t}} := \{\sum_i a_i B_{ik} \ : \ a_i \in \mathbb{R}\} \tag{4.1}$$

the collection of all such splines. .

It has become customary in CAGD to use the term 'B-spline' for what has just been defined to be a spline. This unfortunate mistake will not be made in this chapter, particularly since, once made, one has to make up another term (such as 'B-spline basis function' and the like) for what is called here by its original name, namely a 'B-spline'.

So far, the knot sequence $\mathbf{t}$ has been left unspecified except for the requirement that it be nondecreasing. In any practical situation, $\mathbf{t}$ is necessarily a *finite* sequence. But, since on any nontrivial interval $[t_j \,..\, t_{j+1})$ at most k of the B_{ik} are nonzero, namely $B_{j-k+1,k}, \ldots, B_{jk}$ (see Figure (4.2)), it does not really matter whether $\mathbf{t}$ is finite, infinite, or even bi-infinite; the sum in (4.1) always makes pointwise sense, meaning that $\sum_i a_i B_{ik}(x)$ is well-defined for any x, since at most k of its summands are not zero.

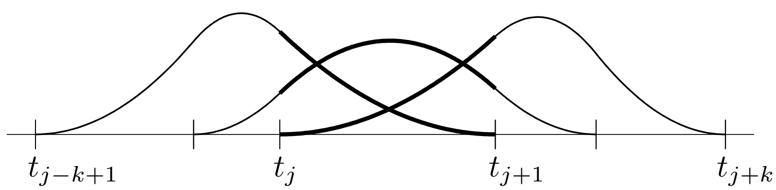

Figure 4.2 The k B-splines whose support contains $[t_j \,..\, t_{j+1})$; here $k = 3$.

However, while each B_{jk} is defined on the entire real line, $\mathbb{R}$, it is convenient to restrict all claims concerning the spline space $S_{k,\mathbf{t}}$ to its **basic interval**

$$I_{k,\mathbf{t}}$$

which, by definition, is the union of all knot intervals $[t_j \,..\, t_{j+1}]$ on which the full complement of k different B-splines from (B_{ik}) have some support. Correspondingly, if $I_{k,\mathbf{t}}$ has a finite right endpoint, it is very convenient to modify the earlier definition of B-splines to make them *left-continuous at that right endpoint.*

At times, it will be convenient to assume that

$$t_i < t_{i+k}, \text{all} \quad i,$$

which can always be achieved by removing from $\mathbf{t}$ its ith entry as long as $t_i = t_{i+k}$. This does not change the space $S_{k,\mathbf{t}}$ since the only kth order B-splines removed thereby are zero anyway. In fact, another way to state this condition is:

$$B_{ik} \neq 0, \text{all} \quad i.$$

5. Specific knot sequences

The following two 'extreme' knot sequences have received special attention:

$$\mathbb{Z} := (\ldots, -2, -1, 0, 1, 2, \ldots), \qquad \mathbb{B} := (\ldots, 0, 0, 0, 1, 1, 1, \ldots).$$

A spline associated with the knot sequence $\mathbb{Z}$ is called a **cardinal** spline. This term was chosen by Schoenberg [10] because of a connection to Whittaker's Cardinal Series. This is not to be confused with its use in earlier spline literature where it refers to a spline that vanishes at all points in a given sequence except for one at which it takes the value 1. The latter splines, though of great interest in spline interpolation, do not interest us here.

Because of the uniformity of the knot sequence $\mathbf{t} = \mathbb{Z}$, formulæ involving cardinal B-splines are often much simpler than corresponding formulæ for general B-splines. To begin with, *all cardinal B-splines (of a given order) are translates of one another*. With the natural indexing $t_i := i$, all i, for the entries of the uniform knot sequence $\mathbf{t} = \mathbb{Z}$, we have

$$\mathrm{B}_{ik} = N_k(\cdot - i),$$

with

$$N_k := \mathrm{B}_{0k} = \mathrm{B}(\cdot|0, \ldots, k). \tag{5.1}$$

The recurrence relation (2.2) simplifies as follows:

$$(k-1)N_k(t) = tN_{k-1}(t) + (k-t)N_{k-1}(t-1). \tag{5.2}$$

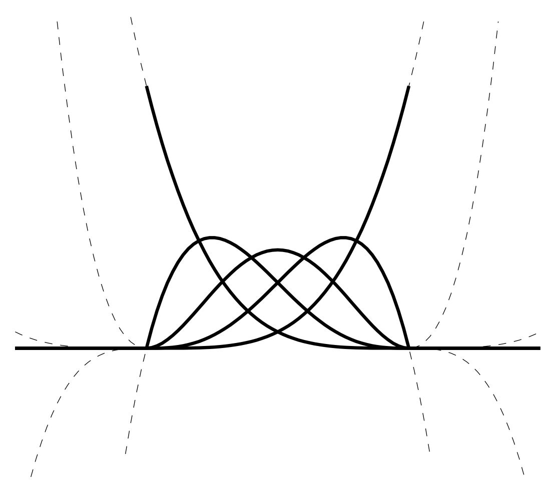

Figure 5.3 Bernstein basis of degree 4 or order 5

The knot sequence $\mathbf{t} = \mathbb{B}$ contains just two points, namely the points 0 and 1, but each with infinite multiplicity. The only nontrivial B-splines for this sequence are those that have both 0 and 1 as knots, i.e., those B_{ik} for which $t_i = 0$ and $t_{i+k} = 1$; see Figure (5.3). There seems to be no natural way to index the entries in the sequence $\mathbb{B}$. Instead, it is customary to index the corresponding B-splines by the multiplicities of their two distinct knots. Precisely,

(5.4) $$\mathrm{B}_{(\mu,\nu)} := B(\cdot|\underbrace{0,\ldots,0}_{\mu+1 \text{ times}},\underbrace{1,\ldots,1}_{\nu+1 \text{ times}}).$$

With this, the recurrence relations (2.2) simplify as follows:

(5.5) $$\mathrm{B}_{(\mu,\nu)}(x) = x\mathrm{B}_{(\mu,\nu-1)}(x) + (1-x)\mathrm{B}_{(\mu-1,\nu)}(x).$$

This gives the formula

(5.6) $$\mathrm{B}_{(\mu,\nu)}(x) = b_\nu^{\mu+\nu}(x) := \binom{\mu+\nu}{\mu}(1-x)^\mu x^\nu \quad \text{for } 0 < x < 1$$

for the one nontrivial polynomial piece of $\mathrm{B}_{(\mu,\nu)}$, as one verifies by induction. The formula enables us to determine the *smoothness* of the B-splines in this simple case: Since $\mathrm{B}_{(\mu,\nu)}$ vanishes identically outside $[0\,..\,1]$, it has exactly $\nu - 1$ continuous derivatives at 0 and $\mu - 1$ continuous derivatives at 1. This amounts to ν **smoothness conditions** at 0 and μ smoothness conditions at 1. Since the order of $\mathrm{B}_{(\mu,\nu)}$ is $\mu + \nu + 1$, this is a simple illustration of the generally valid formula

(5.7) $$\#\text{smoothness conditions at knot} + \text{multiplicity of knot} = \text{order}.$$

For fixed $\mu + \nu$, the polynomials in (5.6) form the so-called **Bernstein** basis (for polynomials of degree $\le \mu + \nu$) and, correspondingly, the representation

(5.8) $$p = \sum_{\mu+\nu=h} a_{(\mu,\nu)}\mathrm{B}_{(\mu,\nu)}$$

is the **Bernstein-Bézier** form for the polynomial $p \in \Pi_h$. It may be simpler to use the short term **BB-form** instead.

6. The polynomials in the spline space: Marsden's identity

Directly from the recurrence relation,

(6.1) $$\sum a_j \mathrm{B}_{jk} = \sum \big((1-\omega_{jk})a_{j-1} + \omega_{jk}a_j\big)\mathrm{B}_{j,k-1}.$$

On the other hand, for the special sequence

$$a_j := \psi_{jk}(\tau) := (t_{j+1} - \tau)\cdots(t_{j+k-1} - \tau),$$

one finds for $\mathrm{B}_{j,k-1} \ne 0$, i.e., for $t_j < t_{j+k-1}$ that

$$(1-\omega_{jk})a_{j-1} + \omega_{jk}a_j = (\cdot - \tau)\psi_{j,k-1}(\tau).$$

Hence, induction on k establishes the following.

(6.2) B-spline Property (iii): Marsden's Identity. *For any* $\tau \in \mathbb{R}$,

$$(\cdot - \tau)^{k-1} \;=\; \sum_j \psi_{jk}(\tau) \mathrm{B}_{jk} \quad \textit{on} \;\; I_{k,\mathbf{t}}, \tag{6.3}$$

with

$$\psi_{jk}(\tau) := (t_{j+1} - \tau) \cdots (t_{j+k-1} - \tau). \tag{6.4}$$

Since τ here is arbitrary, it follows that $S_{k,\mathbf{t}}$ contains all polynomials of degree $< k$. More than that, differentiation of (6.3) with respect to τ leads to the following explicit B-spline expansion of an arbitrary $p \in \Pi_{<k}$:

$$p = \sum_i \mathrm{B}_{ik} \lambda_{ik} p \;, \quad \text{on} \;\; I_{k,\mathbf{t}}, \tag{6.5}$$

with λ_{ik} given by the rule

$$\lambda_{ik} f := \sum_{\nu=1}^{k} \frac{(-D)^{\nu-1} \psi_{ik}(\tau)}{(k-1)!} \, D^{k-\nu} f(\tau). \tag{6.6}$$

For the particular choice $p = 1$, this gives

(6.7) B-spline Property (iv): (positive and local) partition of unity. *The sequence* (B_{jk}) *provides a positive and local partition of unity, that is, each* B_{jk} *is positive on* $(t_j \,..\, t_{j+k})$*, is zero off* $[t_j \,..\, t_{j+k}]$*, and*

$$\sum_j \mathrm{B}_{jk} \;=\; 1 \qquad \textit{on} \;\; I_{k,\mathbf{t}}. \tag{6.8}$$

Further, by considering $p = \ell \in \Pi_2$, one obtains

(6.9) B-spline Property (v): Knot averages. *For* $k > 1$ *and any* $\ell \in \Pi_2$,

$$\ell = \sum_j \ell(t^*_{jk}) \, \mathrm{B}_{jk} \qquad \textit{on} \;\; I_{k,\mathbf{t}},$$

with t^*_{jk} *the* **Greville sites***:*

$$t^*_{jk} := \frac{t_{j+1} + \cdots + t_{j+k-1}}{k-1}, \quad \text{all } j. \tag{6.10}$$

7. The piecewise polynomials in the spline space

Each $s \in S_{k,\mathbf{t}}$ is piecewise polynomial of order k, with breaks only at its knots. If $\boldsymbol{\xi}$ is the strictly increasing sequence of *distinct* knots, then we can write this as

$$S_{k,\mathbf{t}} \subseteq \Pi_{<k,\boldsymbol{\xi}}.$$

But $S_{k,\mathbf{t}}$ is usually a proper subset of $\Pi_{<k,\boldsymbol{\xi}}$. Which subset exactly depends on the **knot multiplicities**

$$\#t_j := \#\{i : t_i = t_j\}$$

according to the rule (5.7). This is usually proved by showing that $S_{k,\mathbf{t}}$ contains the **truncated power** function $(\cdot - t_i)_+^{k-r}$ if and only if $r \le \#t_i$. Here

$$\alpha_+^j := \begin{cases} \alpha^j, & \alpha > 0; \\ 0, & \alpha < 0, \end{cases}$$

with its value at 0 determined by whatever convention is adopted with respect to right or left continuity at a break.

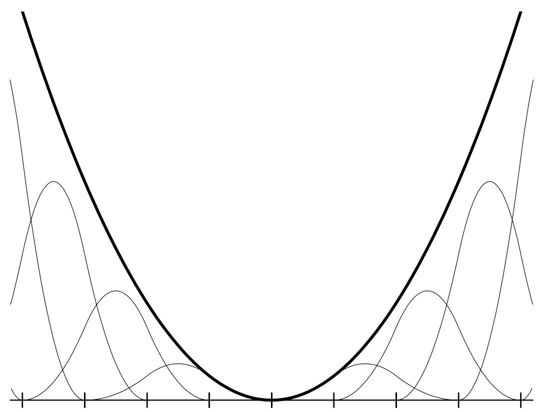

Figure 7.1 The terms $\psi_{jk}(t_i)\mathrm{B}_{jk}$ and their sum, $(\cdot - t_i)^{k-1}$, for $k = 3$. Note that *all* these terms are zero at t_i. Hence, by summing only the terms that are nonzero somewhere to the right of t_i, one gets instead the *truncated* power, $(\cdot - t_i)_+^{k-1}$.

Figure (7.1) gives an illustration of how Marsden's Identity can be used to prove that the truncated power function $(\cdot - t_i)_+^{k-1}$ is in $S_{k,\mathbf{t}}$. For $r > 1$, one may use the $(r-1)$st derivative with respect to τ of that identity in the same way, provided only that $\mathrm{B}_{jk}(t_i) \ne 0$ implies that $D^{r-1}\psi_{jk}(t_i) = 0$, i.e., provided $\#t_i \ge r$.

Now, the truncated power function $f := (\cdot - t_i)_+^{\nu}$ satisfies exactly ν **smoothness conditions** across t_i in the sense that $D^{j-1}f$ is continuous across t_i for $j = 1, \ldots, \nu$. This, finally, leads to the following B-spline property.

(7.2) B-spline Property (vi): Local linear independence. *For any knot sequence* $\mathbf{t}$, *and any interval* $I = [a\,..\,b] \subseteq I_{k,\mathbf{t}}$ *containing finitely many of the* t_i, *the sequence*

$$\mathcal{B} := (\mathrm{B}_{j,k}|_I : \mathrm{B}_{j,k}|_I \neq 0) \tag{7.3}$$

is a basis for the restriction to I *of the space*

$$\Pi^{(\boldsymbol{\nu})}_{<k,\boldsymbol{\xi}}$$

of all piecewise polynomials of order k *with break sequence* $\boldsymbol{\xi}$ *the strictly increasing sequence containing* a, b, *as well as every* $t_i \in I$, *and satisfying* $\nu_i := k - \min(k, \#\{r : t_r = \xi_i\})$ *smoothness conditions across each such* t_i. *In particular,* $\mathcal{B}$ *is linearly independent.*

It is worthwhile to think about this the other way around. Suppose we start off with a partition

$$a =: \xi_1 < \xi_2 < \cdots < \xi_\ell < \xi_{\ell+1} := b$$

of the interval $I := [a\,..\,b]$ and wish to consider the space

$$\Pi^{(\boldsymbol{\nu})}_{<k,\boldsymbol{\xi}}$$

of all piecewise polynomial functions of degree $< k$ on I with breaks ξ_i that satisfy ν_i smoothness conditions at ξ_i, i.e., are $\nu_i - 1$ times continuously differentiable at ξ_i, all i. Then a B-spline basis for this space is provided by (7.3), with the knot sequence $\mathbf{t}$ constructed from the break sequence $\boldsymbol{\xi}$ in the following way: To the sequence

$$(\,\underbrace{\xi_2,\ldots,\xi_2}_{k-\nu_2 \text{ terms}}\,,\ \underbrace{\xi_3,\ldots,\xi_3}_{k-\nu_3 \text{ terms}}\,,\ldots,\ \underbrace{\xi_\ell,\ldots,\xi_\ell}_{k-\nu_\ell \text{ terms}}\,), \tag{7.4}$$

adjoin at the beginning k points $\le a$ and at the end k points $\ge b$. While the knots in (7.4) have to be exactly as shown to achieve the specified smoothness at the specified breaks, the $2k$ additional knots are quite arbitrary. They are often chosen to equal a resp. b, and this has certain advantages (among other things that of simplicity). In any case, the basic interval $I_{k,\mathbf{t}}$ for the resulting spline space is $[a\,..\,b]$, and, on this interval, it coincides with the piecewise polynomial space $\Pi^{(\boldsymbol{\nu})}_{<k,\boldsymbol{\xi}}$ we started out with, – keeping in mind that we agreed earlier to make all elements of $S_{k,\mathbf{t}}$ be left-continuous at the right endpoint of $I_{k,\mathbf{t}}$.

The fact that, in this way, B-splines can be used to staff a basis for any of the spaces $\Pi^{(\boldsymbol{\nu})}_{<k,\boldsymbol{\xi}}$ is also known as the **Curry-Schoenberg Theorem** and has led their creator, Schoenberg, to call them 'B-splines' or 'basic splines'.

The representation of a piecewise polynomial f as a weighted sum of B-splines is called a **B-form** for f.

Any basis $\Phi = (\varphi_1, \ldots, \varphi_n)$ of a linear space F provides a unique (linear) representation $f = \sum_j \alpha_j \varphi_j$ for each $f \in F$. The usefulness of such a representation of $f \in F$ is judged in many ways.

(i) *How robust is the representation in floating-point arithmetic with its inevitable rounding errors?* This is a question of the **condition** of the basis.

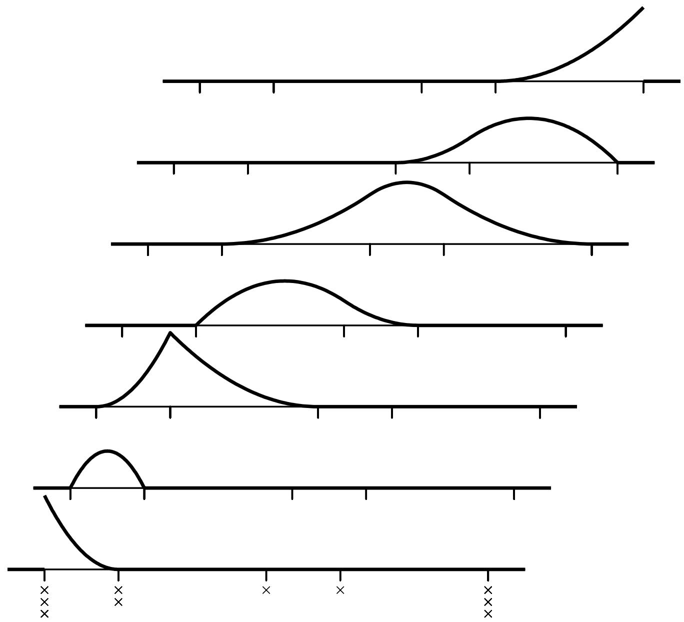

Figure 7.5 The B-spline basis for $\Pi^{(\boldsymbol{\nu})}_{<3,\boldsymbol{\xi}}$ with $\boldsymbol{\xi} = (0,1,3,4,6)$ and $\boldsymbol{\nu} = (1,2,2)$ is, by the recipe, the quadratic B-spline sequence for the knot sequence $\mathbf{t} = (0,0,0,1,1,3,4,6,6,6)$. Note how the smoothness of each B-spline exactly mirrors the multiplicity of each of its 4 knots.

(ii) *How easy is it to derive from the coefficient vector α the information about f that one is really interested in?* In our particular case, this concerns evaluation, differentiation, and integration, determination of zeros, etc, of a spline given in B-form.

(iii) *How easy is it to determine the coefficient vector from some other information about f?* In our particular case, this concerns the construction of a B-form for f by interpolation, discrete least-squares, smoothing, and the like.

In terms of these questions, the B-spline basis for $\Pi^{(\boldsymbol{\nu})}_{<k.\boldsymbol{\xi}}$ does remarkably well, as is explored in the remaining sections.

8. Dual functionals and blossoms

Information about a basis is not complete without some information about its 'inverse', i.e., about the map that associates an element of the space spanned by that basis with its coordinates with respect to that basis. For the B-spline basis, this information is provided by the following explicit formula which we already met in (6.6).

(8.1) B-spline Property (vii): Dual functionals. *For any $f \in S_{k,\mathbf{t}}$,*

$$f = \sum_j \lambda_{jk} f \, \mathrm{B}_{jk},$$

with

$$\text{(8.2)}\qquad \lambda_{jk} f := \sum_{\nu=1}^{k} \frac{(-D)^{k-\nu}\psi_{jk}(\tau_j)}{(k-1)!} D^{\nu-1} f(\tau_j)$$

and $t_j{+} \le \tau_j \le t_{j+k}{-}$, *all* j. *Hence*

$$\text{(8.3)}\qquad \lambda_{ik}\big(\sum_j \alpha_j \, \mathrm{B}_{jk}\big) \;=\; \alpha_i, \quad \text{all } i.$$

To be sure, there are many different dual functionals for B-splines available, but these particular ones have proven quite useful in various contexts.

As a particular example, notice that, according to (8.2), $\lambda_{jk} f$ depends only on part of the knot sequence $\mathbf{t}$, namely only on $t_{j+1}, \ldots, t_{j+k-1}$, and on these it depends linearly (since ψ_{jk} does). Further, for $f \in \Pi_{<k}$, $\lambda_{jk} f$ is independent of τ_j. In other words,

$$\lambda_{jk} p = \lambda_k(t_{j+1}, \ldots, t_{j+k-1}) p, \quad p \in \Pi_{<k},$$

with the precise algebraic structure of λ_k neatly captured by the following notion.

Associated with each $p \in \Pi_r$, there is a unique symmetric r-affine form called its **polar form** (in Algebra) or its **blossom** (in CAGD), denoted therefore here by

$$\overset{\omega}{p},$$

for which

$$\forall\{x \in \mathbb{R}\}\; p(x) = \overset{\omega}{p}(x, \ldots, x).$$

E.g., the blossom of $(\cdot - \tau)^r \in \Pi_r$ is $s \mapsto (s_1 - \tau) \cdots (s_r - \tau)$. If $p = \sum_j ()^j c_j \in \Pi_r$, then

$$\overset{\omega}{p}(s_1, \ldots, s_r) := \sum_j c_j \sum_{I \subset \{1,\ldots,r\}, \#I = j} (\prod_{i \in I} s_i) / \binom{r}{j}.$$

We deduce from the above that

$$\overset{\omega}{p}(t_1, \ldots, t_{k-1}) = \lambda_k(t_1, \ldots, t_{k-1})\, p, \quad p \in \Pi_{<k}.$$

In particular, the jth B-spline coefficient of a kth order spline with knot sequence $\mathbf{t}$ is the value at $(t_{j+1}, \ldots, t_{j+k-1})$ of the blossom of *every* of the k polynomial pieces associated with the intervals $[t_i \,..\, t_{i+1})$, $i = j, \ldots, j+k-1$. This observation was made, in language incomprehensible to the uninitiated, by *de Casteljau* in the sixties. It was discovered independently and made plain (and given the nice name of 'blossom') by *Lyle Ramshaw* in the early eighties.

9. Good condition

(9.1) B-spline Property (viii): Good condition. ($B_i : i = 1{:}n$) *is a relatively well conditioned basis for* $S_{k,\mathbf{t}}$ *in the sense that there exists a positive constant* $D_{k,\infty}$*, which depends only on* k *and not on the particular knot sequence* $\mathbf{t}$*, so that for all* i*,*

$$|\alpha_i| \le D_{k,\infty} \Big\| \sum_j \alpha_j B_j \Big\|_{[t_{i+1} \,..\, t_{i+k-1}]}. \tag{9.2}$$

Smallest possible values for $D_{k,\infty}$ *are*

k	2	3	4	5	6
$D_{k,\infty}$	1	3	$5.5680\cdots$	$12.0886\cdots$	$22.7869\cdots$

Based on numerical calculations, it is conjectured that, in general,

$$D_{k,\infty} \sim 2^{k-3/2}.$$

As of 2000, the best result concerning this conjecture is to be found in [9]: $D_{k,\infty} \le k2^{k-1}$.

10. Convex hull

Since the $B_{j,k}$ are nonnegative, sum to 1, yet at most k are nonzero at any particular x, the next property is immediate.

(10.1) B-spline Property (ix): Convex hull. *For* $t_i < x < t_{i+1}$*, the value of the spline function* $f := \sum_j \alpha_j B_j$ *at the site* x *is a strictly convex combination of the* k *numbers* $\alpha_{i+1-k}, \ldots, \alpha_i$*.*

On the other hand, by (9.1), the B-spline coefficients cannot be too far from the nearby function values. Precisely, if, on the interval $[t_{i+1} \,..\, t_{i+k-1}]$, the spline $f = \sum_j \alpha_j B_{jk}$ is bounded from below by m and from above by M, then

$$|\alpha_i - (M+m)/2| \le D_{k,\infty}(M-m)/2. \tag{10.2}$$

Much more precise estimates have become available more recently. See, for example, [LP].

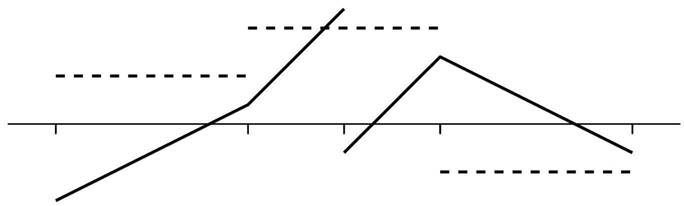

Figure 11.3 A piecewise linear function (solid) and its derivative (dashed) taken in the piecewise polynomial sense.

11. Differentiation and integration

The striking structure of the dual functionals (8.2) readily provides the following formula for the derivative of a spline.

(11.1) B-spline Property (x): Differentiation.

$$D\Big(\sum_j \alpha_j \mathrm{B}_{jk}\Big) = (k-1)\sum_j \frac{\alpha_j - \alpha_{j-1}}{t_{j+k-1}-t_j}\mathrm{B}_{j,k-1}. \tag{11.2}$$

To be sure, the derivative of a spline f is taken here *in the piecewise polynomial sense*, meaning that the derivative, Df, is the piecewise polynomial whose jth polynomial piece is the derivative of the jth polynomial piece of f. In particular, if, e.g., $t_j = t_{j+k-1} < t_{j+k}$, then $\mathrm{B}_{jk}(t_j-) = 0 < 1 = \mathrm{B}_{jk}(t_j+)$, i.e., B_{jk} has a jump across t_j and is certainly not differentiable there. However, in this case (see (3.4)), $\mathrm{B}_{j,k-1}$ is just the zero function and, sticking to the useful maxim that *anything times zero is* 0, we won't have to worry about the fact that, in this case, the coefficient of $\mathrm{B}_{j,k-1}$ in (11.2) involves division by zero since there is no need to compute it. In practical terms, this means that, in this case, the knot sequence for Df has one less knot (see the discussion at the end of Section 4).

By taking derivatives in this piecewise polynomial sense, we ensure that, for every $f \in S_{k,\mathbf{t}}$, $Df \in S_{k-1,\mathbf{t}}$, making (11.2) possible. However, this has the following, perhaps negative, consequence: When we integrate Df, we may not recover f itself since, after all, the integral of a piecewise continuous function is continuous.

It follows from (11.1) that $\sum_j \beta_j \mathrm{B}_{j,k+1}$ is the *antiderivative* or *primitive* of $\sum_j \alpha_j \mathrm{B}_{jk}$ provided

$$\beta_j = c + \begin{cases} \sum_{i=j_0}^{j} \alpha_i (t_{i+k}-t_i)/k, & j \ge j_0; \\ \sum_{i=j}^{j_0-1} \alpha_i (t_{i+k}-t_i)/k, & j < j_0, \end{cases} \tag{11.4}$$

with c and j_0 arbitrary. However, this is strictly true only in case the knot sequence is biinfinite. In the contrary case, it is only locally true since, in general, it requires infinitely many B-splines to write down the integral of a spline; see Figure 11.5.

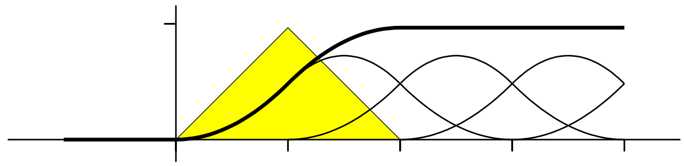

Figure 11.5 It takes infinitely many B-splines to express the integral of one B-spline, as is illustrated here for

$$\int_0^x \mathrm{B}(\cdot|0,1,2) = \sum_{j\ge 0} \mathrm{B}(x|j,j+1,j+2,j+3).$$

12. Evaluation

The recurrence relations (2.2) lead directly to a stable algorithm for the evaluation of a spline

$$s = \sum_i a_i \mathrm{B}_{ik}$$

from its B-spline coefficients (a_i).

The recurrence relations imply

$$s = \sum_i a_i \mathrm{B}_{ik} = \sum_i a_i^{[1]} \mathrm{B}_{i,k-1},$$

with

$$a_i^{[1]} := (1-\omega_{ik})a_{i-1} + \omega_{ik} a_i. \tag{12.1}$$

Note that $a_i^{[1]}$ is not a constant, but is the straight line through the points (t_i, a_{i-1}) and (t_{i+k-1}, a_i). In particular, $a_i^{[1]}(t)$ *is a convex combination of* a_{i-1} *and* a_i *if* $t_i \le t \le t_{i+k-1}$.

After $k-1$-fold iteration of this procedure, we arrive at the formula

$$s = \sum_i a_i^{[k-1]} \mathrm{B}_{i1},$$

which shows that

$$s = a_i^{[k-1]} \quad \text{on} \quad [t_i \mathinner{..} t_{i+1}).$$

(12.2) Evaluation Algorithm. *From given constant polynomials* $a_i^{[0]} := a_i, i = j-k+1, \ldots, j$, *(which determine* $s := \sum_i a_i \mathrm{B}_{ik}$ *on* $[t_j \mathinner{..} t_{j+1})$*), generate polynomials* $a_i^{[r]}, r = 1, \ldots, k-1$, *by the recurrence*

$$a_i^{[r+1]} := (1-\omega_{i,k-r})a_{i-1}^{[r]} + \omega_{i,k-r} a_i^{[r]}, \quad j-k+r+1 < i \le j. \tag{12.3}$$

Then $s = a_j^{[k-1]}$ *on* $[t_j \,..\, t_{j+1})$*. Moreover, for* $t_j \le t \le t_{j+1}$*, the weight* $\omega_{i,k-r}(t)$ *in (12.3) lies between 0 and 1. Hence the computation of* $s(t) = a_j^{[k-1]}(t)$ *via (12.3) consists of the repeated formation of convex combinations.*

In the *cardinal* case (5.1-5.2), the algorithm simplifies, as follows. Now

$$s =: \sum_i N_k(\cdot - i) a_i = \sum_i N_{k-1}(\cdot - i) a_i^{[1]}/(k-1),$$

with

$$a_i^{[1]} := (i + k - 1 - \cdot) a_{i-1} + (\cdot - i) a_i.$$

Hence

$$s = a_j^{[k-1]}/(k-1)! \quad \text{on} \quad [j \,..\, j+1), \qquad \text{with}$$

(12.3)$_{\mathbb{Z}}$

$$a_i^{[r]} := (i + k - r - \cdot) a_{i-1}^{[r-1]} + (\cdot - i) a_i^{[r-1]}, \quad j - k + r < i \le j.$$

In the *Bernstein-Bézier* case (5.4-5.5), all the nontrivial weight functions $\omega_{i,k-r}$ are the same, i.e.,

$$\omega_{i,k-r}(t) = t.$$

Thus, for

$$s = \sum_{\mu+\nu=h} a_{(\mu,\nu)} \mathrm{B}_{(\mu,\nu)},$$

we get

(12.3)$_{\mathbb{B}}$

$$s = a_{(0,0)} \quad \text{on } [0 \,..\, 1], \qquad \text{with}$$

$$a_{(\mu,\nu)}(t) = (1-t) a_{(\mu+1,\nu)} + t a_{(\mu,\nu+1)}, \quad \mu + \nu = r; \; r = h-1, \ldots, 0.$$

This is **de Casteljau's Algorithm** for the evaluation of the BB-form.

13. Spline functions *vs* spline curves

So far, we have only dealt with spline *functions*, even though CAGD is mainly concerned with spline *curves*. The distinction is fundamental.

Every spline function $f = \sum_j \alpha_j \mathrm{B}_{jk}$ gives rise to (planar) curve, namely its **graph**, i.e., the pointset

$$\{(x, f(x)) : x \in I_{k,\mathbf{t}}\}.$$

Assuming that $\# t_j < k$ for all interior knots t_j, this is indeed a **curve** in the mathematical sense, i.e., the continuous image of an interval. Its natural parametrization is the **spline curve**

$$x \mapsto (x, f(x)) = \sum_j P_j \mathrm{B}_{jk}(x), \tag{13.1}$$

with

$$P_j := (t^*_{jk}, \alpha_j)$$

its jth **control point**, and the equality in (13.1) is justified by (6.9).

However, spline curves are not restricted to control points of this specific form. By choosing the control points P_j in (13.1) in any manner whatsoever as d-vectors, we obtain a spline curve in $\mathbb{R}^d$ that smoothly follows the shape outlined by its **control polygon**, which is the broken line that connects these points P_j in order.

Note the CAGD-standard use of the term 'spline curve' to denote both, a curve that can be parametrized by a spline, and the (vector-valued) spline that provides this parametrization.

14. Knot insertion

Wolfgang Böhm (see, e.g., [3]) was the first to point out that the evaluation algorithm (12.2) can be interpreted as repeated knot insertion. This CAGD insight into B-splines has had many wonderful repercussions.

(14.1) B-spline Property (xi): Knot insertion. *If the knot sequence* $\widehat{\mathbf{t}}$ *is obtained from the knot sequence* $\mathbf{t}$ *by the insertion of just one term,* x *say, then, for any* $f \in S_{k,\mathbf{t}}$, $\sum_j \alpha_j \mathrm{B}_{j,k,\mathbf{t}} := f =: \sum_j \widehat{\alpha}_j \mathrm{B}_{j,k,\widehat{\mathbf{t}}}$ *with*

$$\widehat{\alpha}_j = (1 - \widehat{\omega}_{jk}(x))\alpha_{j-1} + \widehat{\omega}_{jk}(x)\alpha_j, \quad \text{all } j, \tag{14.2}$$

and $\widehat{\omega}_{jk} := \max\{0, \min\{1, \omega_{jk}\}\}$, *i.e.,*

$$\widehat{\omega}_{jk} : x \mapsto \begin{cases} 0, & \text{for } x \le t_j; \\ \omega_{jk}(x) = \dfrac{x - t_j}{t_{j+k-1} - t_j}, & \text{for } t_j < x < t_{j+k-1}; \\ 1, & \text{for } t_{j+k-1} \le x. \end{cases} \tag{14.3}$$

Note the need here to make the dependence of a B-spline on its knot sequence explicit in the notation. This property has the following pretty geometric interpretation, in terms of the *control polygon*

$$C_{k,\mathbf{t}} f$$

of $f \in S_{k,\mathbf{t}}$.

(14.4) Proposition. *If* $\hat{\mathbf{t}}$ *is obtained from* $\mathbf{t}$ *by the insertion of one additional knot, then, for any* $f \in S_{k,\mathbf{t}}$, $C_{k,\hat{\mathbf{t}}} f$ *interpolates, at its breakpoints, to* $C_{k,\mathbf{t}} f$ *(and is thereby uniquely determined).*

Figure 14.5 illustrates this interpretation.

By repeated insertion of the point x until its multiplicity in the resulting knot sequence $\tilde{\mathbf{t}}$ is $k-1$, we arrive at the B-form

$$\sum_j \tilde{\alpha}_j \mathrm{B}_{j,\tilde{\mathbf{t}}}$$

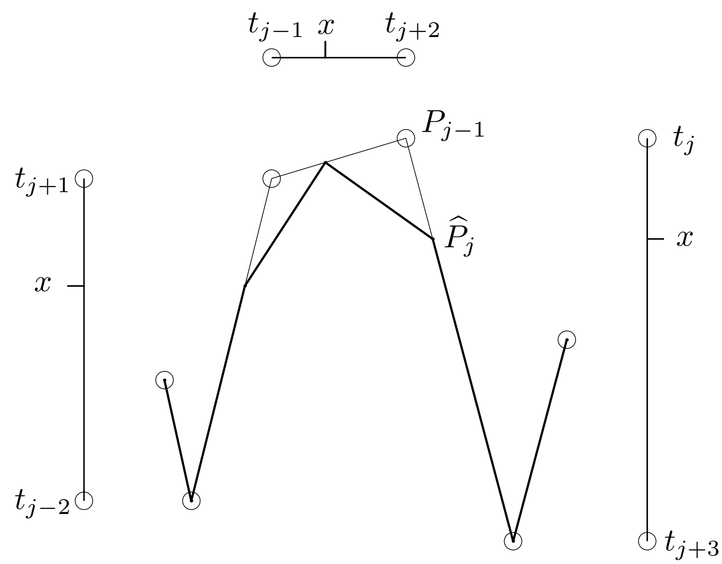

Figure 14.5 Insertion of $x = 2$ into the knot sequence $\mathbf{t} = (0,0,0,0,\mathbf{1},\mathbf{3},5,5,5,5)$, with $k = 4$.

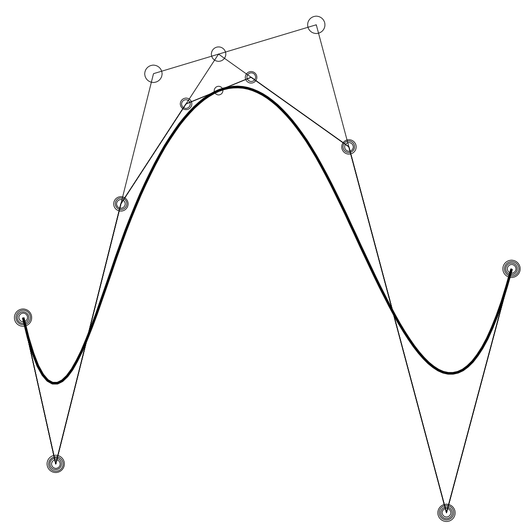

Figure 14.6 Three-fold insertion of the same knot provides a point on the graph of a cubic spline.

for $f = \sum_j \alpha_j \mathrm{B}_{j,\mathbf{t}}$, in which there is exactly one B-spline $\mathrm{B}_{j,\tilde{\mathbf{t}}}$ not zero at x. Since B-splines always sum up to 1, its coefficient must be the value of f at x. This is illustrated in Figure 14.6.

15. Variation diminution and shape preservation: Schoenberg's operator

The spline $f = \sum_j \alpha_j \mathrm{B}_{j,k,\mathbf{t}}$ can be viewed as the result of applying to its control polygon $C_{k,\mathbf{t}}$ **Schoenberg's operator** $V = V_k = V_{k,\mathbf{t}}$, as given by

$$V_k g := \sum_j g(t^*_{jk}) \mathrm{B}_{jk}.$$

Schoenberg's operator is **variation-diminishing**, meaning that, for any continuous function g, Vg crosses the x-axis no more often than does g. More than that, any crossing of Vg requires a 'nearby' crossing of g.

Here is a formal statement, in which $f := Vg$ and $\alpha_j := g(t^*_{jk})$, and which follows immediately from knot insertion.

(15.1) B-spline Property (xii): Variation diminution. *If $f = \sum_j \alpha_j \mathrm{B}_{j,k,\mathbf{t}}$ and $\tau_1 < \cdots < \tau_r$ are such that $f(\tau_{i-1})f(\tau_i) < 0$, all i, then one can find indices $1 \le j_1 < \cdots < j_r \le n$ so that*

$$\alpha_{j_i} f(\tau_i) \mathrm{B}_{j_i}(\tau_i) > 0 \qquad \text{for } i = 1, \ldots, r. \tag{15.2}$$

Further, by (6.9),

$$V\ell = \ell, \qquad \ell \in \Pi_1.$$

This implies that Vg crosses any particular straight line ℓ no more often than does g, and with the crossing of Vg closely related to the crossings of g, including the direction of the crossing. This is illustrated in Figure 15.3 for Vg a spline and g its control polygon. In particular, if g is monotone, then so is Vg; if g is convex, then so is Vg. It is in this sense that Schoenberg's operator is **shape-preserving**.

In effect, a spline is a smoothed version of its control polygon.

16. Zeros of a spline, counting multiplicity

Since a spline (function) cannot cross the x-axis more often than does its control polygon, the number of sign changes in its coefficient sequence is an upper bound on the number of its zeros.

Things are a bit more subtle when one would like to include in the zero count the **multiplicity** of a zero, defined as the maximal number of distinct nearby zeros in a nearby spline (from the same spline space), and needed when considering osculatory or Hermite interpolation by splines.

Here is one relevant result. The full story is recounted in [6].

(16.1) Proposition. *If $f = \sum_j \alpha_j B_{j,k,\mathbf{t}}$ is zero at $x_1 < \cdots < x_r$, while $f^{\mathbf{t}} := \sum_j |\alpha_j| B_{j,k,\mathbf{t}}$ is not, then $S^-(\alpha) \ge r$, with $S^-(\alpha)$ the smallest number of sign changes in the sequence α obtainable by assigning the sign of any zero entry of α appropriately.*

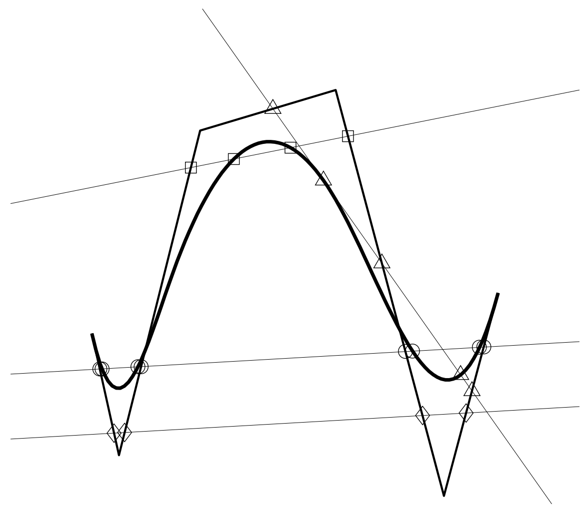

Figure 15.3 A cubic spline, its control polygon, and various straight lines intersecting them. The control polygon *exaggerates* the shape of the spline. The spline crossings are bracketed by the control polygon crossings.

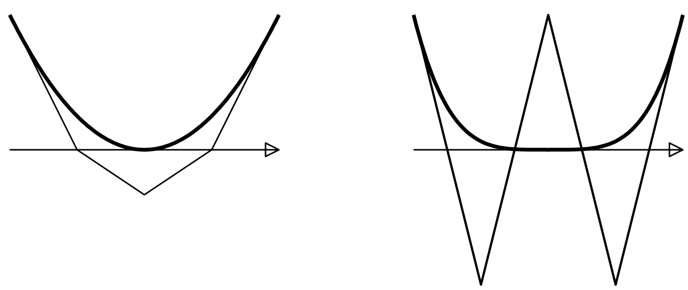

Figure A double spline zero, and a quadruple spline zero, and corresponding control polygons.

17. Spline interpolation: Schoenberg-Whitney

Spline interpolation is one ready means for constructing a spline function that satisfies certain conditions. In **spline interpolation**, one seeks a spline that matches given data values y_i at given data sites x_i, $i = 1, \ldots, n$. If the spline interpolant is to be a spline of order k with knot sequence $\mathbf{t}$, then we can write the sought-for spline in B-form, $\sum_j \alpha_j \mathrm{B}_{jk}$, hence we are looking for a solution $\boldsymbol{\alpha}$ to the linear system

$$\sum_j \alpha_j \mathrm{B}_{jk}(x_i) = y_i, \qquad i = 1, \ldots, n. \tag{17.1}$$

This linear system has exactly one solution for every choice of data values y_i exactly when its coefficient matrix is invertible. This motivates the next result, which is a ready consequence of Proposition 16.1.

(17.2) Schoenberg-Whitney Theorem. *Assume that all interior knots in the knot sequence* $\mathbf{t} = (t_1, \ldots, t_{n+k})$ *have multiplicity* $< k$*, hence each* $f \in S_{k,\mathbf{t}}$ *is continuous (on its basic interval,* $I_{k,\mathbf{t}}$*). Let* $\mathbf{x} := (x_1 < \cdots < x_n)$ *be a strictly increasing sequence in* $I_{k,\mathbf{t}}$*.*

Then, the **collocation matrix**

$$A_{\mathbf{x}} := (\mathrm{B}_{jk}(x_i) : i, j = 1, \ldots, n)$$

is invertible if and only if all its diagonal entries (namely the numbers $\mathrm{B}_{jk}(x_j)$*), are nonzero, i.e., if and only if*

$$t_i \le x_i \le t_{i+k}, \qquad i = 1, \ldots, n, \tag{17.3}$$

with equality occurring only if the knot in question is one of the endpoints of the basic interval, $I_{k,\mathbf{t}}$*.*

The strict ordering of the data-site sequence $\mathbf{x}$ not only leads to this neat characterization of the invertibility of the collocation matrix $A_{\mathbf{x}}$. It also ensures (see (4.2)) that $A_{\mathbf{x}}$ is a banded matrix with at most k nontrivial bands. It also ensures that $A_{\mathbf{x}}$ is **totally positive**, meaning that all its minors (i.e., determinants of submatrices) are nonnegative. This somewhat esoteric property has many consequences of practical interest. One of these is that it is numerically safe to solve the linear system (17.1) by Gauss elimination *without* pivoting, hence in no more storage than is required to store the banded matrix $A_{\mathbf{x}}$ to begin with.

If the knot sequence $\mathbf{t}$ and the order k are already chosen, then the sequence $(t_i^* : i = 1, \ldots, n)$ of Greville points is a good choice as data site sequence $\mathbf{x}$; it certainly satisfies the **Schoenberg-Whitney conditions** (17.3). It also serves as a good initial guess in the iterative process for determining the **Chebyshev-Demko** sites, $\mathbf{x}^*$. These are optimal sites for interpolation from $S_{k,\mathbf{t}}$ in that the resulting map, $f \mapsto P_{\mathbf{x}^*} f$, is the most stable among all possible such maps $f \mapsto P_{\mathbf{x}} f$. In consequence, $P_{\mathbf{x}^*} f$ is a near-best approximation to f from $S_{k,\mathbf{t}}$ in that $\|f - P_{\mathbf{x}^*} f\| \le \mathrm{const}_k \operatorname{dist}(f, S_{k,\mathbf{t}})$ for some f-independent const_k.

When the data values y_i are noisy, then spline interpolation may produce a highly oscillating spline, as is illustrated in Figure (17.4). In such a situation, one may be willing to forego exact matching in favor of a 'smoother', less wiggly, approximating spline. There are two standard procedures to accomplish this: the *smoothing spline* and *least-squares spline approximant*, and Figure (17.4) also shows a sample of both.

18. Smoothing spline

The smoothing spline is constructed as a compromise between the wish to be close to the data and the wish for a smooth approximation. Closeness of the function f to the data (x_i, y_i) is typically measured by the sum of squares of their difference:

$$E(f) := \sum_i (y_i - f(x_i))^2,$$

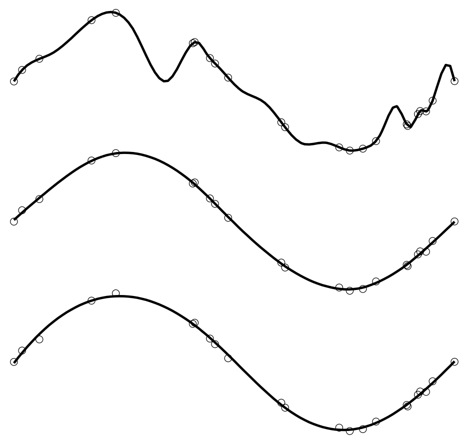

Figure 17.4 A spline interpolant (top) to noisy data (circled) may be unnecessarily wiggly. A smoothing spline (middle) or a least-squares approximant (bottom) to such data may be preferred.

while roughness of f is measured by the size of some derivative of f in the mean-square norm:

$$R(f) := \int_a^b (D^m f(t))^2 \, dt,$$

with $[a \,..\, b]$ the interval of interest. Both measures could involve some weighting function, though this is more commonly done for E than for R.

Choosing mean-square norms for both E and F ensures that, for any positive p, the minimizer $f = f_p$ of the weighted sum

$$E(f) + pR(f)$$

is a spline, of order $2m$ and with simple knots, at the data sites, and reducing to a polynomial of degree $< m$ outside the interval $(x_1 \,..\, x_n)$. As $p \to 0$, this so-called **smoothing spline** converges to the so-called 'natural' spline interpolant of order $2m$ to the given data. At the other extreme, as $p \to \infty$, the smoothing spline converges to the least-squares approximant to the data by polynomials of degree $< m$.

It is something of an art to choose the smoothing parameter 'appropriately'. The most popular choice is based on *generalized cross validation*; see [12].

19. Least-squares spline approximation

The perhaps somewhat vague notion behind least-squares approximation is to work with a spline with just enough degrees of freedom to fit the 'smooth' function underlying the noisy data, but not enough degrees of freedom to match also the noise.

In practice, this means that one must somehow choose the order, k, and the knot sequence $\mathbf{t} = (t_1, \ldots, t_{N+k})$, mindful that (1) the resulting basic interval $I_{k,\mathbf{t}}$ equal the interval $[a\,..\,b]$ of interest; and (2) that $S_{k,\mathbf{t}}$ contain a unique minimizer of E. The latter is ensured exactly when some subsequence of the data sites $\mathbf{x}$ satisfies the Schoenberg-Whitney conditions with respect to the chosen knot sequence $\mathbf{t}$. In that case, it is usually numerically safe to determine the B-spline coefficient vector $\boldsymbol{\alpha}$ of the least-squares spline approximant as the solution to the normal equations

$$A'_{\mathbf{x}} A_{\mathbf{x}} \boldsymbol{\alpha} = A'_{\mathbf{x}} \mathbf{y},$$

with $A_{\mathbf{x}}$, as before, the B-spline collocation matrix for the chosen order k and knot sequence $\mathbf{t}$ and the given data sites $\mathbf{x}$.

The least-squares fit in Figure (17.4) is a cubic spline, with 3 equally-spaced interior knots.

When approximating a function with widely varying behavior, it is tempting to choose the location of these interior knots so as to further minimize the error, but the best one can hope for is a choice that cannot be improved upon by small local variations of the knot locations.

20. Background

I have failed, except coincidentally, to supply historical comment or attribute specific results to specific authors. Nor was there any attempt to prove the results stated, not even in outline. For all these matters, consult the standard literature.

The relevant literature on (univariate) B-splines up to about 1975 is summarized in [4] which also contains hints of the most exciting developments concerning B-splines since then: knot insertion and the multivariate B-splines. Two books on splines, [5] and [11], which have appeared since 1975, cover B-splines in the traditional way. As presentations of splines from the CAGD point of view, the survey article [3] and the "Killer B's" [1,87] are particularly recommended. The revised version, [7], of [5] develops the central part of spline theory more in the spirit of CAGD and, in particular, knot insertion. All the results mentioned here are proved there.

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