Error Bounds for Spline Interpolation

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1. Introduction. In [1], Ahlberg and Nilson proved the convergence of the second derivatives of (cubic) spline interpolations to a given periodic function $f(x) \in C^2$, as the mesh-lengths tend to zero on asymptotically uniform meshes. Using different methods, convergence of the third derivatives is proved below when $f'''(x)$ is absolutely continuous. Moreover the assumption of periodicity is dispensed with, and the hypothesis of asymptotically uniform mesh-spacing is relaxed.

Specifically, let $f'''(x)$ be absolutely continuous on [0, 1]. For any partition $\pi : 0 = x_0 < x_1 < \cdots < x_n = 1$ of [0, 1], let the "piecewise cubic" interpolating *spline* function (for π) be defined as usual ([2], [3]) by the condition that $s(x) \in C^2$ and

$$(1) \qquad f(x_i) = s(x_i), \qquad i = 0, \cdots, n; \qquad f'(0) = s'(0), \qquad f'(1) = s'(1).$$

(A *spline function* with joints x_i is a function $s(x) \in C^2$ which is equal to a cubic polynomial on each interval $[x_{i-1}, x_i]$ between successive joints.) We define the *cardinal functions* $C_i(x)$ for spline interpolation on π as the spline functions which satisfy

$$(2) \qquad C_i(x_j) = \delta_{ij}, \qquad C_i'(0) = C_i'(1) = 0, \qquad i = 1, \cdots, n-1.$$

By definition, the *error* in spline interpolation is

$$(3) \qquad e(x) = f(x) - s(x).$$

If $p_f(x)$ denotes the cubic polynomial which satisfies $p_f(x_k) = f(x_k)$ and $p_f'(x_k) = f'(x_k)$ for $k = 0, n$, then the error in spline interpolation to $f(x)$ is the same as that in spline interpolation to $g(x) = f(x) - p_f(x)$, which satisfies $dg'''(x) = df'''(x)$ and $g(0) = g'(0) = g(1) = g'(1) = 0$; hence we can assume

$$(4) \qquad f(0) = f'(0) = f(1) = f'(1) = 0,$$

without essential loss of generality.

For such functions, we have

$$(5) \qquad f(x) = \int_0^1 G(x, y)\, df'''(y),$$

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where $G(x, y)$ is the Green's function for the boundary value problem defined by $f^{iv}(x) = h(x)$ and (4). Explicitly, $G(x, y)$ is given by

$$G(x, y) = (x - y)_+^3/3! - P(x, y), \tag{6}$$

where, for fixed y, $P(x, y) = x^2(1 - y)^2(x + 2xy - 3y)/6$ is the cubic polynomial in x such that $G(0, y) = G_x(0, y) = G(1, y) = G_x(1, y) = 0$. The function $(x)_+^k$ is defined by

$$(x)_+^k = \begin{cases} x^k, & x \geqq 0 \\ 0, & x < 0 \end{cases}.$$

Hence, considered as a function of x or y alone, $G(x, y)$ is a spline function with exactly one joint at $x = y$.

Likewise, for functions satisfying (4) we have

$$s(x) = \sum_{i=1}^{n-1} f(x_i)C_i(x), \tag{7}$$

whence, by (3), (5), and the preceding paragraph,

$$e(x) = \int_0^1 \left[G(x, y) - \sum_{i=1}^{n-1} C_i(x)G(x_i , y) \right] df'''(y). \tag{8}$$

Using (8), and special properties of the cardinal functions, we will bound the r^{th} derivatives $e^{(r)}(x)$ of orders $r = 0, 1, 2, 3$.

2. Properties of cardinal functions. For convenience, we will define the *mesh-ratio bound* M_π by

$$M_\pi = |\pi|/\min \Delta x_i , \qquad |\pi| = \max \Delta x_i , \qquad \Delta x_i = x_{i+1} - x_i , \tag{9}$$

and we will write $\|f\| = \max |f(x)|$ on $[0, 1]$. The main result of this section will be that each cardinal function $C_i(x)$ decays exponentially away from x_i, and that $|C_i(x)|$ is bounded in $[x_{i-1} , x_{i+1}]$ by a constant K' depending only on M_π. The proof will use some qualitative properties of the signs of the $C_i(x)$ and their derivatives, which will be established in a series of lemmas.

Lemma 1. *If $p(x)$ is a cubic polynomial which vanishes at* 0 *and $h \neq 0$, then*

$$p'(h) = -2p'(0) - h(p''(0)/2), \quad \textit{and} \quad p''(h)/2 = -\frac{3}{h}p'(0) - 2(p''(0)/2). \tag{10}$$

Indeed, $p(x) \equiv p'(0)x + (p''(0)/2)x^2 - h^{-2}[p'(0) + (p''(0)/2)h]x^3$, from which (10) follows.

Corollary 1. *For $i \neq j + 1, j$, $C_i(x)$ satisfies*

$$\begin{aligned} C_i'(x_{j+1}) &= -2C_i'(x_j) - \Delta x_j(C_i''(x_j)/2), \\ C_i''(x_{j+1})/2 &= -(3/\Delta x_j)C_i'(x_j) - 2(C_i''(x_j)/2). \end{aligned} \tag{11}$$

The significance of equations (11) is clear: they are *recursive relations* on the vectors $\{C_i'(x_j), C_i''(x_j)/2\}$, whose coefficients constitute the *negative* matrix

$$\begin{bmatrix} -2 & -\Delta x_j \\ -3/\Delta x_j & -2 \end{bmatrix}.$$

Corollary 2. *For* $i = 1, \cdots, n-1$, $C_i(x)$ *satisfies*

$$C_i'(x_j)C_i''(x_j) \geqq 0, \quad \textit{for} \quad j < i, \tag{12a}$$

$$C_i'(x_j)C_i''(x_j) \leqq 0, \quad \textit{for} \quad j > i. \tag{12b}$$

The proof for $j = 0, 1, \cdots, i-1$ is by induction on j. For $j = 0$, it follows from (2). Since the coefficients in (11) are all negative, the condition (12a) that $C_i'(x_j)$ and $C_j''(x_j)$ have the same sign implies that $C_i'(x_{j+1})$ and $C_i''(x_{j+1})$ have the same sign, namely, the reverse of that of $C_i'(x_j)$ and $C_i''(x_j)$. The proof for $j > i$ is obtained by changing x to $-x$, which reverses the sign of $C_i'(x)C_i''(x)$.

Corollary 3. *For* $i = 1, \cdots, n-1$, $C_i(x)$ *satisfies*

$$|C_i'(x_j)| < \tfrac{1}{2}\,|C_i'(x_{j+1})|, \quad j < i-1, \qquad |C_i'(x_{j+1})| < \tfrac{1}{2}\,|C_i'(x_j)|, \quad j > i. \tag{13}$$

The first inequality follows from (12a) and (11), with the observation that $C_i''(x_0) \neq 0$ (otherwise, by (11), $C_i(x) \equiv 0$), hence $C_i''(x_j) \neq 0$, $j < i$. The second inequality follows then by symmetry about x_i.

The exponential decay of each $|C_i(x)|$ away from x_i follows from Corollary 3 unless Δx_j increases exponentially away from x_i as a function of $|j-i|$, at a rate comparable with the exponential decrease of $|C_i'(x_j)|$.

Lemma 2. *Let* $S(x)$ *be any spline function with joints at the* x_i, *which satisfies*

$$S_{i-1} = S_{i+1} = 0, \qquad S_i = h > 0, \qquad S_{i-1}'\cdot S_{i-1}'' \geqq 0, \qquad S_{i+1}'\cdot S_{i+1}'' \leqq 0,$$

where $S_{i-1} = S(x_{i-1})$, $S_{i+1}'' = S''(x_{i+1})$, *etc. Then* $S_i'' < 0$, $S_{i-1}' \geqq 0$, $S_{i+1}' \leqq 0$, *and* $S(x) \geqq 0$ *on* $[x_{i-1}, x_{i+1}]$.

Proof. By direct computation:

$$S_i' = 3h/\Delta x_{i-1} - 2S_{i-1}' - \tfrac{1}{2}S_{i-1}''\Delta x_{i-1}, \tag{14a}$$

$$S_i' = -3h/\Delta x_i - 2S_{i+1}' + \tfrac{1}{2}S_{i+1}''\Delta x_i, \tag{14b}$$

$$\tfrac{1}{2}\Delta x_{i-1}S_i'' = 3h/\Delta x_{i-1} - 3S_{i-1}' - S_{i-1}''\Delta x_{i-1}, \tag{14c}$$

$$\tfrac{1}{2}\Delta x_i S_i'' = 3h/\Delta x_i + 3S_{i+1}' - S_{i+1}''\Delta x_i, \tag{14d}$$

and so

$$S_i' + \tfrac{1}{2}S_i''\Delta x_i = S_{i+1}' - \tfrac{1}{2}S_{i+1}''\Delta x_i. \tag{15}$$

Now suppose $S_{i-1}' < 0$. Then $S_{i-1}'' \leqq 0$ by assumption, hence by (14a), (14c), $S_i' > 0$ and $S_i'' > 0$. If now $S_{i+1}' > 0$, then $S_{i+1}'' \leqq 0$, so by (14a), $S_i' < 0$, a contradiction. Likewise, if $S'_{i+1} \leqq 0$, then $S''_{i+1} \geqq 0$, so $S'_i + \frac{1}{2} \Delta x_i S''_i \leqq 0$ by (15), again a contradiction. Hence $S'_{i-1} \geqq 0$. By symmetry about x_i it follows that $S'_{i+1} \leqq 0$. Hence S''_{i-1} and S''_{i+1} are nonnegative. Since the second divided difference $S(x_{i-1}, x_i, x_{i+1})$ is negative, one has $S''_i < 0$.

Suppose next that for some $x \in [x_{i-1}, x_i]$, $S(x) < 0$. If $S''_{i-1} = 0$, then $S''(x) < 0$ in (x_{i-1}, x_i), but $S(x_{i-1}, x, x_i) > 0$, a contradiction. If, on the other hand, $S''_{i-1} > 0$, then, since $S'_{i-1} \geqq 0$, there exists $y \in (x_{i-1}, x_i)$ such that $S(t) > 0$ for $t \in (x_{i-1}, y)$. But then $S(x_{i-1}, y, x) < 0$, $S(y, x, x_i) > 0$, which implies that the linear function $S''(x)$ has two distinct zeros in (x_{i-1}, x_i) without being identically zero, a contradiction. Hence $S(x) > 0$, $x \in (x_{i-1}, x_i]$. By symmetry about x_i, it follows that $S(x) > 0$ identically on (x_i, x_{i+1}).

Lemma 3. *Let $T(x)$ be a spline with a joint at x_i, such that*

$$T_{i-1} = T_i = T_{i+1} = 0, \qquad T''_{i-1} \leqq 0, \qquad T'_{i+1} \geqq 0. \tag{16}$$

Then $T(x) \geqq 0$ in $[x_{i-1}, x_i]$.

Proof. With $h = 0$, since $T_i = 0$, (14a)–(14c) give

$$\Delta x_i T'_{i-1} + 2(\Delta x_{i-1} + \Delta x_i) T'_i + \Delta x_i T'_{i+1} = 0. \tag{17}$$

If $T'_{i-1} < 0$, then it follows as in Corollary 2 of Lemma 1, that $T'_i > 0$, $T''_i \geqq 0$, so $T'_{i+1} < 0$, a contradiction. Therefore $T'_{i-1} \geqq 0$. Hence, if now $T'_i = 0$, then by (17), $T'_{i-1} = T'_{i+1} = 0$, so $T(x) \equiv 0$, which completes the proof for this case. Otherwise, by (17), $T'_i < 0$, and so since $T_i = 0$, there is a $y \in (x_{i-1}, x_i)$ such that $T(x) > 0$, $x \in (y, x_i)$. But then the assumption that $T(x) < 0$ for some $x \in (x_{i-1}, y)$ would imply $T(x_{i-1}, x, y) > 0$, $T(x, y, x_i) < 0$, hence with $T''_{i-1} \leqq 0$, the linear function $T''(x)$ had two distinct zeros in $[x_{i-1}, x_i]$ without being identically zero, which is impossible.

Corollary 1. *Let $M = M_\pi$. For $i = 1, \cdots, n - 1$:*

$$0 \leqq C_i(x) \leqq L \quad \text{on} \quad [x_{i-1}, x_{i+1}], \quad \text{where} \quad L = 3\,\frac{M(M+1)^2}{3 + 4M}, \tag{18}$$

$$|C'_i(x_{i-1})| \leqq L/\Delta x_{i-1}, \qquad |C'_i(x_{i+1})| \leqq L/\Delta x_i. \tag{19}$$

By Corollary 2 of Lemma 1, $C_i(x)$ satisfies the hypotheses on $S(x)$ in Lemma 2, hence the first inequality in (18) follows from that lemma. To prove the second inequality for $x \in [x_{i-1}, x_i]$, let $U(x)$ be the spline with a joint at x_i such that $U_{i-1} = U_{i+1} = U''_{i-1} = U'_{i+1} = 0$, $U_i = 1$. Then $T(x) = U(x) - C_i(x)$ satisfies the hypotheses of Lemma 3, since by Lemma 2, as applied to $C_i(x)$, $C''_{i-1} \geqq 0$, $C'_{i+1} < 0$. Hence $0 \leqq C_i(x) \leqq U(x)$ on $[x_{i-1}, x_i]$. Since $U_{i-1} = 0$, one has

$$U(x) \leqq \Delta x_{i-1} \max_{[x_{i-1}, x_i]} U'(y).$$

Applying Lemma 2 to $U(x)$ gives $U''_i < 0$. But $U''_{i-1} = 0$, hence $U''(x) < 0$

in (x_{i-1}, x_i), and so

$$\max_{[x_{i-1},x_i]} U'(y) = U'_{i-1} = \frac{3}{\Delta x_{i-1}\Delta x_i} \frac{(\Delta x_{i-1} + \Delta x_i)^2}{3\Delta x_i + 4\Delta x_{i-1}} \leqq \frac{1}{\Delta x_{i-1}} 3 \frac{(M+1)^2}{(3/M) + 4},$$

and (18) follows now for $x \in [x_{i-1}, x_i]$. The first inequality of (19) is an immediate consequence. The remaining statements follow from symmetry about x_i.

Corollary 2. For $i = 1, \cdots, n-1$,

$$|C_i(x)| \leqq |C_i'(x_j)|\, \Delta x_j \quad \textit{on} \quad [x_j, x_{j+1}], \qquad j > i, \tag{20a}$$

$$|C_i(x)| \leqq |C_i'(x_j)|\, \Delta x_{j-1} \quad \textit{on} \quad [x_{j-1}, x_j], \qquad j < i-1. \tag{20b}$$

Let $j > i$, and assume without loss of generality that $C_i''(x_j) < 0$. Then by Corollary 2 to Lemma 1, $C_i'(x_j) \geqq 0$, $C_i'(x_{j+1}) \leqq 0$, and the proof of Lemma 3 shows that $C_i(x) \geqq 0$ on $[x_j, x_{j+1}]$. Moreover $C_i'(x_j) \geqq C_i'(x)$, $x \in [x_j, x_{j+1}]$; hence (20a) follows. By symmetry about x_i, (20b) then follows from (20a).

3. More inequalities. We can now prove our first main result.

Theorem 1. *There exists a constant* $K = K(M_\pi)$ *depending on* M_π *alone such that*

$$\int_0^1 |C_i(x)|\, dx \leqq K\, |\pi|. \tag{21}$$

Proof. For $j < i$, by (20b) and (13),

$$|C_i(x)| \leqq |C_i'(x_j)|\, \Delta x_{j-1} \leqq 2^{j-i+1}\, |C_i'(x_{i-1})|\, \Delta x_{j-1},$$

for $x \in [x_{j-1}, x_j]$. Hence, by (19),

$$|C_i(x)| \leqq 2^{j-i+1} L \Delta x_{j-1}/\Delta x_{i-1} \leqq 2^{j-i+1} L M_\pi, \tag{22}$$

where $L = L(M_\pi)$ is given by (18). Consequently

$$\int_0^{x_i} |C_i(x)|\, dx = \sum_{j=1}^{i} \int_{x_{j-1}}^{x_j} |C_i(x)|\, dx$$

$$\leqq \sum_{j=1}^{i-1} 2^{j-i+1} L M_\pi \Delta x_{j-1} + L \Delta x_{i-1} \leqq (2M_\pi + 1) L\, |\pi|,$$

since $\sum_0^\infty 2^{-k} = 2$ and $\Delta x_{j-1} \leqq |\pi|$. Combining the preceding inequality with a like inequality for $j > i$, we get (21) with $K = (4M_\pi + 2)L$. Since L is given by (18), we have $K \leqq 3M_\pi(M_\pi + 1)^2$.

Alternatively, we can bound the integral in (21) in terms of the maximum ratio of successive mesh-lengths. Indeed,

$$\sum_{j=1}^{i-1} 2^{j-(i-1)} \Delta x_{j-1} \leqq \Delta x_{i-1} \sum_{j=1}^{i-1} 2^{j-(i-1)} \left(\frac{\Delta x_{j-1}}{\Delta x_{i-1}}\right).$$

Now choose R_π such that $R_\pi^{-1} \leqq \Delta x_i/\Delta x_{i-1} \leqq R_\pi$, $i = 1, \cdots, n$. If, for the given partition π, $R_\pi \leqq 2\rho < 2$, then

$$\sum_{j=1}^{i-1} 2^{j-(i-1)}\left(\frac{\Delta x_{j-1}}{\Delta x_{i-1}}\right) \leqq \sum_{j=1}^{i-1} 2^{j-(i-1)} R_\pi^{(i-1)-j} \leqq \frac{1}{1-\rho}.$$

Therefore

$$\int_0^{x_i} |C_i(x)|\, dx \leqq |\pi| \left(\frac{2}{2-R_\pi} + 1\right) L,$$

where L is given by (18) with $M = R_\pi$. This proves the

Corollary. *If $R_\pi < 2$, then there exists a constant K depending on R_π alone, such that* (21) *holds.*

Now recall that the interpolation error is given by (8) as

$$e(x) = \int_0^1 \left[G(x, y) - \sum_{i=1}^{n-1} C_i(x) G(x_i, y)\right] df'''(y).$$

Lemma 4. *The following identity is valid:*

$$\sum_{i=1}^{n-1} C_i(x) G(x_i, y) \equiv \sum_{i=1}^{n-1} G(x, x_i) C_i(y). \tag{23}$$

Proof. By (3), (6), and (7), one has

$$\sum_{i=1}^{n-1} C_i(x) G(x_i, y) = \sum_{j=1}^{n-1} \left(\sum_{i=1}^{n-1} C_i(x) G(x_i, x_j)\right) C_j(y) = \sum_{j=1}^{n-1} G(x, x_j) C_j(y). \tag{24}$$

Corollary 1. *The third derivative of the error exists and satisfies*

$$e'''(x) = \int_0^1 G_3(x, y)\, df'''(y), \quad \textit{where} \tag{25}$$

$$G_3(x, y) = \frac{\partial^3}{\partial x^3} G(x, y) - \sum_{i=1}^{n-1} \left[\frac{\partial^3}{\partial x^3} G(x, x_i)\right] C_i(y). \tag{25'}$$

The differentiation under the integral is justified by Leibniz' Rule (*cf.* Kaplan, *Advanced Calculus*, p. 219) since $G_3(x, y)$ is piecewise continuous. Note that here and in the following we use the normalization

$$(x)_+^0 = \begin{cases} 1, & x \geqq 0 \\ 0, & x < 0 \end{cases}.$$

Corollary 2. *For fixed $x \in (0, 1)$, there exists $j \in [1, n-2]$ such that*

$$G_3(x, y) = g(y) - g(x_j) C_j(y) - g(x_{j+1}) C_{j+1}(y), \tag{26}$$

where $\|g\| \leqq 1$, $g(y) = 0$ for $y \notin (x_{j-1}, x_{j+2})$.

Proof. It follows from (6) that

$$\frac{\partial^3}{\partial x^3} G(x, y) = (x - y)_+^0 - (1 - y)^2 (1 + 2y).$$

Let $x \varepsilon [x_{k-1}, x_k)$, and choose j such that $0 \leqq j - 1 < k \leqq j + 2 \leqq n$. Let $h(y) = L(y; x_{j-1}, x_j, x_{j+1}, x_{j+2})$ be the third divided difference in z of $L(y; z) = (z - y)_+^3$ on $\{x_{j-1}, x_j, x_{j+1}, x_{j+2}\}$. Then $h(y)$ is a spline function which is equal to $(x - y)_+^0$ for $y \notin (x_{j-1}, x_{j+2})$ and lies between 0 and 1 inside $[x_{j-1}, x_{j+2}]$. Hence with $g(y) = (x - y)_+^0 - h(y)$, one gets

$$G_3(x, y) = g(y) - \sum_{i=1}^{n-1} g(x_i)C_i(y),$$

and Corollary 2 follows.

4. Error bounds. Two bounds for $e'''(x)$, the third derivative of the error (*i.e.*, the error in the third derivative) can now be derived. Since spline functions have a piecewise continuous third derivative, the error in the third derivative will be in general bounded away from zero unless $f(x) \varepsilon C^3$. Before proving a rather sharp converse to this statement, we first establish a stronger result valid for $f(x) \varepsilon C^4$.

Theorem 2. *Let $f(x) \varepsilon C^4$, and let $e(x)$ be the error* (3), *incurred when $f(x)$ is interpolated by a spline function on a given partition $\pi : 0 = x_0 < x_1 < \cdots < x_n = 1$. Then there exists a constant $K_1(M_\pi)$ dependent on M_π alone such that*

$$\|e'''\| \leqq \|f^{iv}\| \cdot K_1(M_\pi) \cdot |\pi|. \tag{27}$$

Proof. Let $x \varepsilon (0, 1)$ be fixed. Then by Corollary 2 to Lemma 4

$$\int_0^1 |G_3(x, y)|\, dy \leqq \int_{x_{j-1}}^{x_{j+2}} dy + 2 \max_{j, j+1} \int_0^1 |C_i(y)|\, dy,$$

for some $j \varepsilon [1, n - 2]$. Let K be the constant of Theorem 1. Then with $K_1(M_\pi) = 3 + 2K$,

$$\int_0^1 |G_3(x, y)|\, dy \leqq K_1(M_\pi)\, |\pi|,$$

therefore, with (26),

$$|e'''(x)| = \left| \int_0^1 G_3(x, y)\, df'''(y) \right| \leqq \|f^{iv}\| \int_0^1 |G_3(x, y)|\, dy$$
$$\leqq \|f^{iv}\|\, K_1(M_\pi)\, |\pi|, \qquad x \varepsilon (0, 1).$$

Theorem 2 follows since $e'''(0) = e'''(0+)$, $e'''(1) = e'''(1-)$.

To find corresponding bounds for the r^{th} derivative $e^{(r)}$ when $r < 3$, observe that, by Rolle's theorem, there exist ξ_i^r with

$$0 = \xi_0^r \leqq \xi_1^r < \xi_2^r < \cdots < \xi_{n_r-1}^r \leqq \xi_{n_r}^r = 1,$$

such that $e^{(r)}(\xi_i^r) = 0$, $i = 1, \cdots, n_r - 1$, and $\max_i \Delta\xi_i^r \leqq (r + 1)\, |\pi|$. Hence

$$|e^{(r)}(x)| \leqq \int_{\xi_i^r}^{\xi_{i+1}^r} |e^{(r+1)}(y)|\, dy \leqq \Delta\xi_i^r\, \|e^{(r+1)}\|, \qquad x \varepsilon [\xi_i^r, \xi_{i+1}^r], \tag{28}$$

so

$$\|e^{(r)}\| \leqq (r+1)\,|\pi|\,\|e^{(r+1)}\|, \qquad r < 3. \tag{29}$$

Corollary. *Under the hypotheses of Theorem 2, there exist constants $K_r(M_\pi)$ depending on M_π alone such that*

$$\|e^{(r)}\| \leqq \|f^{iv}\|\cdot K_r \cdot |\pi|^{4-r}, \qquad r = 0, 1, 2, 3. \tag{30}$$

If π_n is a sequence of partitions of [0, 1] such that $|\pi_n| \to 0$ while $M_{\pi_n} \leqq M$ stays bounded, then Theorem 2 implies for the corresponding error $e_n(x)$ of spline interpolation that

$$|e_n'''(x)| \to 0, \quad \text{uniformly on} \quad [0, 1], \tag{31}$$

if $f(x) \in C^4$. We now make the weaker assumption that $f'''(x)$ is continuous and of bounded variation on [0, 1], which is implied by the assumption made at the outset that $f'''(x)$ is absolutely continuous. Convergence can still be proved under this assumption by a more careful analysis of the integral (25).

Theorem 3. *Let $f'''(x)$ be absolutely continuous on* [0, 1]. *Let $\{\pi_n\}$ be a sequence of partitions of* [0, 1] *such that $|\pi_n| \to 0$ while $M_{\pi_n} \leqq M$ as $n \to \infty$. Let $e_n(x)$ be the error incurred when $f(x)$ is interpolated by a spline function on π_n. Then*

$$|e_n'''(x)| \to 0, \quad \textit{uniformly on} \quad [0, 1], \quad \textit{as} \quad n \to \infty. \tag{31}$$

Proof. Let $\epsilon > 0$ be given. Since $f'''(x)$ is absolutely continuous there exists $\delta > 0$ such that for all $I = [a, b] \subset [0, 1]$, with $b - a < \delta$

$$\int_I |df'''(y)| < \epsilon. \tag{32}$$

Since $|\pi_n| \to 0$, there exists N such that for $n \geqq N$, $|\pi_n| < \delta$. Let now $n \geqq N$, $\pi_n : 0 = x_0 < x_1 < \cdots < x_m = 1$, and $x \in (0, 1)$. By Corollary 2 to Lemma 4

$$G_3(x, y) = g(y) - g(x_j)C_j(y) - g(x_{j+1})C_{j+1}(y),$$

where $\|g\| \leqq 1$, $g(y) = 0$ for $y \notin (x_{j-1}, x_{j+2})$, for some $j \in [1, n-2]$. Hence

$$|e'''(x)| = \left|\int_0^1 G_3(x, y)\, df'''(y)\right| \leqq \left|\int_{x_{j-1}}^{x_{j+2}} g(y)\, df'''(y)\right| + \left|\int_0^1 C_j(y)\, df'''(y)\right| + \left|\int_0^1 C_{j+1}(y)\, df'''(y)\right|.$$

But

$$\left|\int_0^1 C_i(y)\, df'''(y)\right| \leqq \sum_{j=1}^{m} \left|\int_{x_{j-1}}^{x_j} C_i(y)\, df'''(y)\right| \leqq \sum_{j=1}^{m} \max_{[x_{j-1}, x_j]} |C_i(y)| \int_{x_{j-1}}^{x_j} |df'''(y)|.$$

Hence, by choice of n, and by the proof of Theorem 1,

$$\left| \int_0^1 C_i(y)\, df'''(y) \right| \leqq \epsilon \cdot K(M),$$

where $K(M)$ depends on M alone. Therefore

$$|e'''(x)| \leqq \epsilon(3 + 2K(M)),$$

and the Theorem follows now since $e'''(0) = e'''(0+)$, $e'''(1) = e'''(1-)$.

Remark. In view of Corollary 2 of Theorem 1, the preceding results remain true if the condition of (uniformly) bounded mesh ratio is replaced by the condition that the ratio of any two adjacent mesh-lengths is less than 2.

The preceding results were announced in Abstract 64T-296 of the Notices Am. Math. Soc., in which Mr. de Boor's name was omitted by mistake.

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